KING SAUD UNIVERSITY **COLLEGE OF SCIENCE**

M203 **DEPARTMENT OF MATHEMATICS** **TIME: 90 Minutes**

(SEMESTER 1, 1441)

First Mid-term Exam

Note: All questions carry equal Marks.

Q1. Determine whether the sequence $\{\sqrt{n^4+4n^2} - n^2\}$ converges or diverges and if it converges, find its limit.

Q2. Find the sum of the series: $\sum_{n=3}^{\infty}\left[\frac{2^{3n}}{3^{2n}}+\frac{1}{n^2-3n+2}\right]$.

Q3. Determine whether the following series is absolutely convergent, conditionally convergent or divergent: $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$.

Q4. Find the interval of convergence and the radius of convergence of the power series:

$$\sum_{n=1}^{\infty}\frac{2^n(x-e)^n}{n}.$$

Q5. Find the MacLaurin series for the function $f(x)=\tan^{-1}(x)$ up to three non-zero terms and approximate the value of the integral

$$\int_0^{0.1}\tan^{-1}(x^2)\ dx.$$

Q#3) Determine whether the following series is absolutely convergent, conditionally convergent or divergent:
$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln(n)}.$$ [Marks: 5]

Soln. $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. We apply

Integral test: $f(x) = \frac{1}{x\ln(x)}$ is decreasing + +ve & continuous on $[2,\infty)$

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln(x)}\,dx$$ Put $\ln x = u \Rightarrow \frac{1}{x}dx = du$

$$= \lim_{t\to\infty} \left[\ln \ln x\right] = \infty; \text{ Divg!}$$ $\int \frac{1}{u}du = \ln u$ (2)

Now, $(-1)^n \frac{1}{n\ln(n)}$ is an Alternating series and by AST, $\sum (-1)^n \frac{1}{n\ln(n)}$ is convergent as $\lim_{n\to\infty} \frac{1}{n\ln n} = 0$ (2) and it is decreasing

Hence $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln(n)}$ is a conditionally convergent series (1)

Q#4) Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n (x-e)^n}{n}$.

[Marks: 5]

Soln. $\lim_{n\to\infty} \left| \frac{2^{n+1}(x-e)^{n+1}}{(n+1)} \times \frac{n}{2^n (x-e)^n} \right| = 2|x-e|$

Abs conv $\Leftrightarrow -\frac{1}{2} < x-e < \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2}+e < x < \frac{1}{2}+e$ (2)

At $x = -\frac{1}{2}+e$, we have $\sum_{n=1}^{\infty} 2^n \frac{(-\frac{1}{2}+\cancel{e}-\cancel{e})^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which is conv by AST (1)

At $x = \frac{1}{2}+e$, we have $\sum_{n=1}^{\infty} 2^n \frac{(\frac{1}{2}+\cancel{e}-\cancel{e})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divg. (1)

Hence interval of conv: $[-\frac{1}{2}+e, \frac{1}{2}+e)$ and radius $r = \frac{1}{2}\left[\frac{1}{2}+e+\frac{1}{2}-e\right] = \frac{1}{2}$

Q#5) Find the Maclaurin Series for the function $f(x) = \tan^{-1}$ up to three non-zero terms and approximate the value of the integral $\int_0^{0.1} \tan^{-1}(x^2)\,dx$ [Marks: 5]

Soln: $\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt \quad$ if $|t|<1$

$= \int_0^x (1 - t^2 + t^4 - \dots)\,dt$

$= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \dots\right]_0^x$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ 3

$\therefore \int_0^{0.1} \tan^{-1}(x^2)\,dx = \int_0^{0.1} \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \dots\right)dx$

$= \left[\frac{x^3}{3} - \frac{x^7}{7(3)} + \frac{x^{11}}{11(5)} - \dots\right]_0^{0.1}$

$= \frac{(0.1)^3}{3} - \frac{(0.1)^7}{21} + \frac{(0.1)^{11}}{55} - \dots$

$= 0.000333 - \frac{0.0000001}{21} +$

2

M- 203

I Mid-term Exam. (I Semester 1440/1441)

Time: 90 Minutes — Max. Marks: 25

Q#1) Determine whether the sequence $\{\sqrt{n^4+4n^2}-n^2\}$ converges or diverges and, if it converges, find its limit. [Marks: 5]

Soln. $\lim\limits_{n\to\infty} \dfrac{(\sqrt{n^4+4n^2}-n^2)(\sqrt{n^4+4n^2}+n^2)}{(\sqrt{n^4+4n^2}+n^2)}$ 3

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^4}+4\cancel{n^2}-\cancel{n^4}}{\cancel{n^2}(\sqrt{1+\frac{4}{n^2}}+1)} = \lim\limits_{n\to\infty} \dfrac{4}{\sqrt{1+\frac{4}{n^2}}+1} = \dfrac{4}{2} = 2$ 2

Q#2) Find the Sum of the Series: $\sum\limits_{n=3}^{\infty}\left[\dfrac{2^{3n}}{3^{2n}}+\dfrac{1}{n^2-3n+2}\right]$ [Marks: 5]

Soln. $\sum\limits_{n=3}^{\infty} \dfrac{2^{3n}}{3^{2n}} = \sum\limits_{n=3}^{\infty} \dfrac{(2^3)^n}{(3^2)^n}$ which is a conv. Geom. series with $\frac{8}{9}$ as the common ratio. 1

Its sum $S_1 = \dfrac{\left(\frac{8}{9}\right)^3}{1-\frac{8}{9}} = \left(\frac{8}{9}\right)^3 \times 9$

$= \dfrac{8^3}{9^2} = \dfrac{512}{81} = S_1$ 1

Now, $\sum\limits_{n=3}^{\infty} \dfrac{1}{n^2-3n+2} = \sum\limits_{n=3}^{n} \left(\dfrac{1}{n-2} - \dfrac{1}{n-1}\right)$ 1

$= \left(1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \tfrac{1}{4} \cdots - \tfrac{1}{n-2} - \tfrac{1}{n-1}\right)$

$\sum\limits_{n=3}^{\infty} \dfrac{1}{n^2-3n+2} = \lim\limits_{n\to\infty}\left[1-\dfrac{1}{n-1}\right] = 1 = S_2$ 1

Hence Sum: $S = S_1 + S_2 = \dfrac{512}{81}+1 = \dfrac{512+81}{81} = \dfrac{593}{81}$ 1

KING SAUD UNIVERSITY COLLEGE OF SCIENCE

M203 DEPARTMENT OF MATHEMATICS TIME: 90 Minutes

(SEMESTER 1, 1441)

Second Mid-term Exam

Note: All questions carry equal Marks.

Q1. Evaluate the iterated integral

$$\int_1^e \int_{\ln(y)}^1 \frac{e^{x^2}}{y}\,dxdy.$$

Q2. Use polar coordinates to evaluate the integral

$$\int_0^4 \int_{-\sqrt{4x-x^2}}^0 \sqrt{x^2+y^2}\,dydx.$$

Q3. Find the surface area of the part of the solid cut off from the paraboloid $z = x^2 + y^2$ by the plane $z = 4$.

Q4. Find the centroid of the solid bounded by the graphs of the equations: $x^2 + y^2 = 1,\ z = \sqrt{x^2 + y^2},\ z = 0.$

Q5. Evaluate the integral:

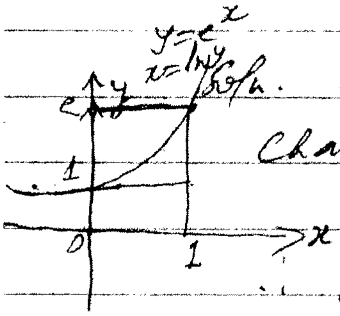
$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)\,dzdydx$$

M-203

II Mid-term Exam. (II-semester 1440/1441)

Max. Marks: 25

Q#1) Evaluate the iterated integral: $\displaystyle\int_1^e\int_{\ln(y)}^1 \frac{e^{x^2}}{y}\,dx\,dy$ [Marks: 5]



Soln. Given: $1\le y\le e$ and $\ln y\le x\le 1$ (~~Horizontal strip~~)

Changing, we get $0\le x\le 1$, $1\le y\le e^x$ } Vertical strip

$$\therefore \int_1^e\int_{\ln(y)}^1 \frac{e^{x^2}}{y}\,dx\,dy = \int_0^1\int_1^{e^x}\frac{e^{x^2}}{y}\,dy\,dx \quad (3)$$

$$=\int_0^1 e^{x^2}\left[\ln y\right]_1^{e^x}dx = \int_0^1 e^{x^2}(x)\,dx$$

$$\therefore \frac{1}{2}\int_0^1 e^u\,du \quad (1)$$

Put $x^2=u \Rightarrow 2x\,dx=du$ $\therefore x\,dx=\frac{1}{2}du$; If $x=0$, $u=0$ and if $x=1$, $u=1$

$$=\frac{1}{2}\left[e^u\right]_0^1=\frac{1}{2}(e-1) \quad (1)$$

Q#2) Use polar Coordinates to evaluate the integral

$$\int_0^4\int_{-\sqrt{4x-x^2}}^0 \sqrt{x^2+y^2}\,dy\,dx$$

[Marks: 5]

Soln. $-\sqrt{4x-x^2}\le y\le 0 \Rightarrow y^2=4x-x^2$ or, $x^2+y^2=4x$, $r^2=4r\cos\theta$

$$\therefore \int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\sqrt{r^2}\cdot r\,dr\,d\theta \quad (3) \quad =\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{4\cos\theta}d\theta$$

Also, we have $\displaystyle\int_{3\pi/2}^{2\pi}\int_0^{4\cos\theta} r\cdot r\,dr\,d\theta=0$

$$=\frac{64}{3}\int_{-\pi/2}^{\pi/2}\cos^3\theta\,d\theta = \frac{64}{3}\int_{-\pi/2}^{\pi/2}\cos^2\theta\cos\theta\,d\theta = \frac{64}{3}\int_{-\pi/2}^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta$$

Put $\sin\theta=t \Rightarrow$ (1) $\cos\theta\,d\theta=dt$

$$\frac{64}{3}\int_{-\pi/2}^{\pi/2}(1-t^2)\,dt = \frac{64}{3}\int_{-\pi/2}^{\pi/2}\left(t-\frac{t^3}{3}\right)dt = \frac{64}{3}\int_{-\pi/2}^{\pi/2}\left(\sin\theta-\frac{\sin^3\theta}{3}\right)d\theta = 0 \quad (1)$$

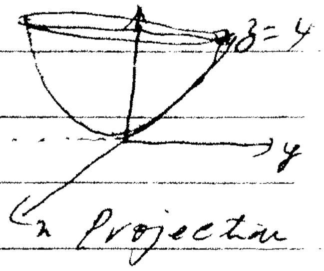
Q#3) Find the surface area of the part of the solid cut off from the paraboloid $z = x^2+y^2$ by the plane $z=4$

[Marks: 5]

Soln. we have $z = x^2+y^2 = f(x,y) \Rightarrow f_x(x,y) = 2x$; $f_y(x,y) = 2y$ ①

$\therefore S.A = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA = \iint_R \sqrt{1+4x^2+4y^2}\, dA$

$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$ ③



$\frac{1}{12}\int_0^{2\pi}\left[(1+4r^2)^{3/2}\right]_0^2 d\theta$ put $1+4r^2 = u \Rightarrow$

$8r\,dr = du$

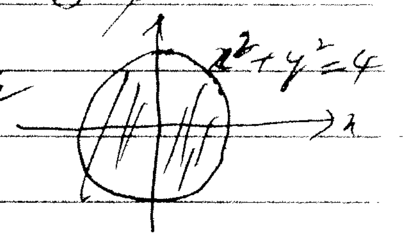
$r\,dr = \frac{1}{8}du$

$\frac{1}{8}\int \sqrt{u}\, du$

$= \frac{1\times 2}{8\times 3}u^{3/2} = \frac{1}{12}u^{3/2}$

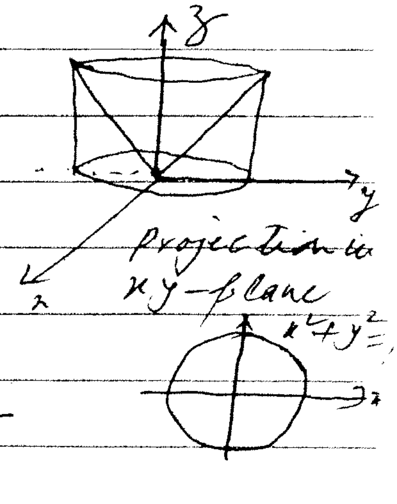
$= \frac{1}{12}\times 2\pi\left[17^{3/2}-1\right]$

$= \frac{\pi}{6}\left(17^{3/2}-1\right)$ ①



Q#4) Find the centroid of the solid bounded by the graphs of the equations $x^2+y^2=1$, $z=\sqrt{x^2+y^2}$, $z=0$ [Marks: 5]

Soln. Volume $V = \int_0^{2\pi}\int_0^1\int_0^r r\,dz\,dr\,d\theta = \frac{2\pi}{3}$ ③



$\bar{x} = \bar{y} = 0$ By symmetry of Figure.

we find $\bar{z} = \frac{M_{xy}}{V}$

$M_{xy} = \int_0^{2\pi}\int_0^1\int_0^r z\,dz\; r\,dr\,d\theta = \frac{1}{4}\pi$

$\therefore \bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4}\pi}{\frac{2\pi}{3}} = \frac{1}{4}\times\frac{3}{2} = \frac{3}{8}$ ①

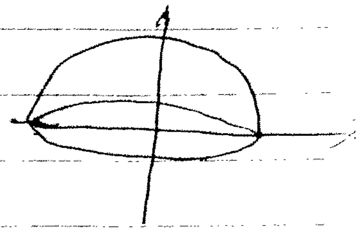
$\therefore (\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{3}{8}\right)$. ①

Q# 5) Evaluate the Integral: $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$

[Marks: 5]

Soln. We change it to Spherical Coordinates:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \quad (3)$$



$$= \int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^4 \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1024}{5}\int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1024}{5}\int_0^{\pi/2} \left[-\cos\varphi\right]_0^{\pi/2} d\theta \quad (1)$$

$$= \frac{1024}{5} \times \frac{\pi}{2}(0+1) = \frac{\cancel{1024}^{512}}{5\times \cancel{2}}\pi = \frac{512}{5}\pi \quad (1)$$

King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus)
(I-Semester 1441)

Max. Marks: 40 Time: 3 hrs

Marking Scheme: Q1[4+4+4]; Q2[4+4+4]; Q3[4+4+4+4].

Q. No: 1 (a) Determine whether the sequence $\left\{\left(\frac{n+2}{n+3}\right)^n\right\}$ converges or diverges and if it converges, find its limit.

(b) Find the interval of convergence and radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}.$$

(c) Find the MacLaurin series for $f(x) = \cos^2(x)$ and use its first three non-zero terms to approximate the integral $\int_0^1 \cos^2(\sqrt{x})\ dx$.

Q. No: 2 (a) Evaluate the integral

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_y^{\sqrt{\frac{\pi}{2}}} \cos(x^2)\, dxdy.$$

(b) Find the moment of inertia about the x-axis of the lamina with shape of the region bounded by $y = x^2$ and $y = 0$, and $x = 1$ with density $\delta = x + y$.

(c) Evaluate the triple integral:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} z\sqrt{x^2+y^2}\, dzdydx.$$

Q. No: 3 (a) Show that the following integral is independent of path and find its value:

$$\int_{(0,1)}^{(1,2)} (y + 2xy)dx + (x^2 + x)dy.$$

(b) Use Green's theorem to evaluate the line integral $\oint_C (e^x + x^3)\, dx + (yx^2 + y^3)dy$, where C is the path from (0,0) to (1,2) along the graph $y = 2x^2$ and from (1,2) to (0,0) along the graph $y = 2x$.

(c) Use Divergence theorem to evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n}\, dS$, where $\vec{F}(x, y, z) = (x^2 + \sin(yz))\,\vec{i} + (\cos(xz) - 2xy)\vec{j} + (e^y + z^2)\vec{k}$ and S is the surface of the region bounded by the cylinder $x^2 + y^2 = 1$, the xy-plane, and the paraboloid $z = 2 - x^2 - y^2$. (Provided S is oriented by the unit normal directed upward).

(d) Use Stoke's theorem to evaluate $\oint_C \vec{F} \cdot \vec{dr}$ for the vector field $\vec{F}(x, y, z) = 2z\vec{i} + 3x\vec{j} + y\vec{k}$, S is the surface of the paraboloid $z = 1 - x^2 - y^2$ and C is the trace of S in the xy-plane with counterclockwise direction.

Q1 (a) $u_n = \left(\frac{n+2}{n+3}\right)^n$. Let $y_n = \ln(u_n) = n\ln\left(\frac{n+2}{n+3}\right) = \dfrac{\ln(n+2) - \ln(n+3)}{\frac{1}{n}}$ (1)

$\lim_n y_n = \lim_n \dfrac{\frac{1}{n+2} - \frac{1}{n+3}}{-\frac{1}{n^2}} = \lim_n \dfrac{1}{(n+3)(n+2)} \times (-n^2) = \boxed{-1}$ (2). So $u_n = e^{y_n}$ and

$\lim_n u_n = e^{\lim y_n} = \boxed{e^{-1}}$ (1) [Marks: 4]

Q1 (b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$; $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-1)^n}\right| = \frac{1}{2}|x-1| < 1 \iff |x-1| < 2 \iff x \in (-1,3)$ (1)

$x=-1$; $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=1} \frac{(-2)^n}{2^n} = \sum (-1)^n$ Div. by nth term test (1)

$x=3$ $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=1} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1$; Div. (1) [Marks: 4]

Interval of Cov. $(-1;3)$

Radius of Cov. $\boxed{\rho = 2}$ (1)

Q1 (c) $f(x) = \cos^2(x) = \frac{1}{2}\left[\cos(2x) + 1\right]$. We know that $\cos(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}$ [Marks: 4]

So $\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \cdot 2^{2n} \cdot \frac{x^{2n}}{(2n)!}$

$\Rightarrow \frac{1}{2}\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \cdot 2^{2n-1} \cdot \frac{x^{2n}}{(2n)!} = \frac{1}{2} - 2 \cdot \frac{x^2}{2!} + 2^3 \cdot \frac{x^4}{4!} + \ldots$

$= \frac{1}{2} - x^2 + \frac{x^4}{3} + \ldots$

$\Rightarrow f(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$ (2)

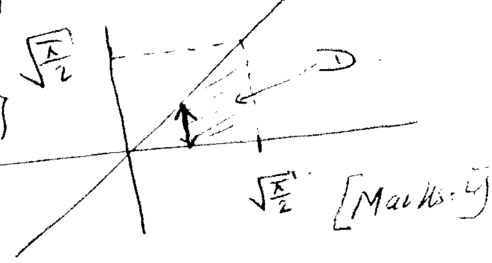
$= \boxed{1 - x^2 + \frac{x^4}{3} + \ldots + (-1)^n \cdot 2^{2n-1} \cdot \frac{x^{2n}}{(2n)!} + \ldots}$

$\int_0^1 \cos^2(\sqrt{x})\,dx \approx \int_0^1 f(\sqrt{x})\,dx \approx \int_0^1 1 - (\sqrt{x})^2 + \frac{(\sqrt{x})^4}{3}\,dx \approx \int_0^1 1 - x + \frac{x^2}{3}\,dx = \left[x - \frac{x^2}{2} + \frac{x^3}{9}\right]_0^1$ (2)

$\approx 1 - \frac{1}{2} + \frac{1}{9} \approx \frac{1}{2} + \frac{1}{9} = \boxed{\frac{11}{18}} = \boxed{0.611}$

Q2 (a) $\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos(x^2)\,dx\,dy =$

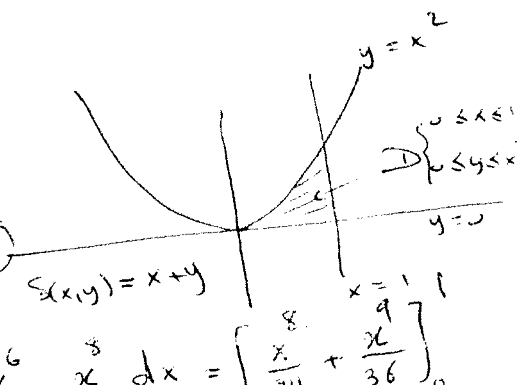
$D = \begin{cases} 0 \le y \le \sqrt{\frac{\pi}{2}} \\ y \le x \le \sqrt{\frac{\pi}{2}} \end{cases} = \begin{cases} 0 \le x \le \sqrt{\frac{\pi}{2}} \\ 0 \le y \le x \end{cases}$



[Marks: 4]

$\int_0^{\sqrt{\pi/2}} \int_0^x \cos(x^2)\,dy\,dx =$ (2) $\int_0^{\sqrt{\pi/2}} \left[y\cos(x^2)\right]_0^x dx$

$= \frac{1}{2}\int_0^{\sqrt{\pi/2}} 2x\cos(x^2)\,dx = \frac{1}{2}\left[\sin(x^2)\right]_0^{\sqrt{\pi/2}} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = \boxed{\frac{1}{2}}$ (2)



(2) $\delta(x,y) = x + y$

Q2 (b) $I_x = \iint_D y^2 \delta(x,y)\,dA = \int_0^1 \int_0^{x^2} y^2(x+y)\,dy\,dx$

$I_x = \int_0^1 \int_0^{x^2} y^2x + y^3\,dy\,dx = \int_0^1 \left[\frac{y^3x}{3} + \frac{y^4}{4}\right]_0^{x^2} dx = \int_0^1 \frac{x \cdot x^6}{3} + \frac{x^8}{4}\,dx = \left[\frac{x^8}{24} + \frac{x^9}{36}\right]_0^1$ (1)

[Marks: 4]

$\therefore I_x = \frac{1}{24} + \frac{1}{36} = \boxed{\frac{5}{72}} \approx \boxed{0.0694}$ (1)

3) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{x^2+y^2}} z\sqrt{x^2+y^2}\,dz\,dy\,dx; \quad E=\left\{\begin{array}{l}0\le\theta\le 2\pi\\ 0\le r\le 1\\ 0\le z\le r\end{array}\right\}.$ [Marks: 4]

$$=\int_0^{2\pi}\int_0^1\int_0^r z\cdot r\cdot dz\cdot r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\int_0^r z\,r^2\,dz\,dr\,d\theta=2\pi\int_0^1 r^2\left[\frac{z^2}{2}\right]_0^r dr=2\pi\int_0^1 r^2\cdot\frac{r^2}{2}\,dr \quad (3)$$

$$=\pi\left[\frac{r^5}{5}\right]_0^1=\boxed{\frac{\pi}{5}}. \quad (1)$$

Q3 ② $\displaystyle I=\int_{(0,1)}^{(1,2)}(y+2xy)\,dx+(x^2+x)\,dy.$ $f(x,y)=?,\ \left\{\begin{array}{l}f_x=y+2xy\\ f_y=x^2+x\end{array}\right.\Rightarrow f(x,y)=yx+x^2y$ [Marks: 4]

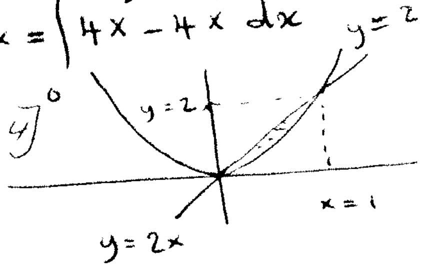
$$I=\Big[f(x,y)\Big]_{(0,1)}^{(1,2)}=\Big[yx+x^2y\Big]_{(0,1)}^{(1,2)}=(2+2)-(0)=\boxed{4} \quad (3)\ (1)$$

(b) Green's Th.

$$\oint_C (\underbrace{e^x+x^3}_{M})\,dx+(\underbrace{yx^2+y^3}_{N})\,dy=\iint_D\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA=\int_0^1\int_{2x^2}^{2x}(2xy)-0 \quad (3)$$

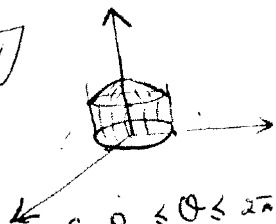
$$=\int_0^1\int_{2x^2}^{2x}2xy\,dy\,dx=\int_0^1\Big[xy^2\Big]_{2x^2}^{2x}dx=\int_0^1 x\cdot(2x)^2-x\cdot(2x^2)^2\,dx=\int_0^1 4x^3-4x^5\,dx$$

$$=\left[x^4-4\cdot\frac{x^6}{6}\right]_0^1=\left(1-\frac{4}{6}\right)-(0)=1-\frac{2}{3}=\boxed{\frac{1}{3}}. \quad (1)$$ [Marks: 4]



(c) Divergence Th.

$$\iint_S \vec{F}\cdot\vec{n}\,ds=\iiint_E \mathrm{div}(F)\,dV.$$ [Marks: 4]



$\mathrm{div}(F)=2x-2x+2z=2z.$

$E=\left\{\begin{array}{l}0\le\theta\le 2\pi\\ 0\le r\le 1\\ 0\le z\le 2-r^2\end{array}\right.$

$$\iint_S\vec{F}\cdot\vec{n}\,ds=\int_0^{2\pi}\int_0^1\int_0^{2-r^2}2z\cdot r\,dz\,dr\,d\theta=2\pi\int_0^1\Big[z^2\Big]_0^{2-r^2}\cdot r\,dr \quad (3)$$

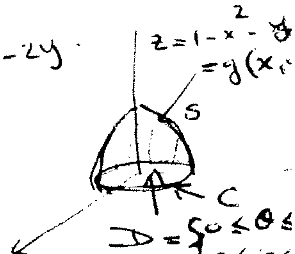
$$=2\pi\int_0^1(2-r^2)^2\cdot r\,dr=2\pi\int_0^1 4r-4r^3+r^5\,dr=2\pi\left[2r^2-r^4+\frac{r^6}{6}\right]_0^1$$

$$=2\pi\left[2-1+\frac{1}{6}\right]=\frac{14}{6}\pi=\boxed{\frac{7\pi}{3}}. \quad (1)$$

(d) $\displaystyle\int_C\vec{F}\cdot dr=\iint_S \overrightarrow{\mathrm{curl}}(F)\,ds$ (Stoke's Th). (1) [Marks: 4]

$\vec{F}=(2z;3x;y);\quad \overrightarrow{\mathrm{curl}}(F)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 2z&3x&y\end{vmatrix}=(1,2,3)$ (1)

$g(x,y)=1-x^2-y^2$
$g_x=-2x;\ g_y=-2y.$



$D=\left\{\begin{array}{l}0\le\theta\le 2\pi\\ 0\le r\le 1\end{array}\right.$

$$=\iint_D(-1)(-2x)-2\cdot(-2y)+3\,dA=\iint_D 2x+4y+3\,dA \quad (1)$$

$$=\int_0^{2\pi}\int_0^1\Big[2r\cos\theta+4r\sin\theta+3\Big]\cdot r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2\frac{r^3}{3}\cos\theta+4\frac{r^3}{3}\sin(\theta)+3\cdot\frac{r^2}{2}\right]_0^1 d\theta=\int_0^{2\pi}\frac{2}{3}\cos\theta+\frac{4}{3}\sin(\theta)+\frac{3}{2}\,d\theta=\left[\frac{2}{3}\sin\theta-\frac{4}{3}\cos\theta+\frac{3}{2}\theta\right]_0^{2\pi} \quad (1)$$

$$=\frac{3}{2}\cdot 2\pi=\boxed{3\pi}.$$

Page (12)