Functions of Random Variables

#Distribution Functions of Random Variables

Q1) If $X \sim Uniform(0,1)$, find the pdf of $Y = -2\ln X$. Name the distribution and its parameter values.

Solution : $\boldsymbol{Uniform(a,b)}$: $\boldsymbol{f_X(x) = \frac{1}{b-a}}$ $\boldsymbol{if\ a \le x \le b}$

$$Uniform(0,1):\ f_X(x) = 1\ ,0 \le x \le 1$$

$$Y = -2\ln X \implies X = e^{-\frac{Y}{2}} \implies \frac{d}{dy}X = -\frac{1}{2}e^{-\frac{y}{2}} \implies \left|\frac{d}{dy}X\right| = \frac{1}{2}e^{-\frac{y}{2}}$$

$$0 \le x \le 1 \Rightarrow \ln 0 \le \ln x \le \ln 1 \Rightarrow -\infty \le \ln x \le 0$$

$$\Rightarrow 0 \le -2\ln x \le \infty \Rightarrow 0 \le y \le \infty$$

$$\boldsymbol{f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = f_X\left(e^{-\frac{y}{2}}\right)\left|\frac{d}{dy}X\right|} = \frac{1}{2}e^{-\frac{y}{2}} \qquad \therefore Y \sim \exp\left(\lambda = \frac{1}{2}\right)$$

Q2) If $X \sim Uniform(a,b)$, find the constants (c) and (d) such that $Y = c + dX$ $Uniform(0,1)$.

Solution :

$$X \sim Uniform(a,b):\ f_X(x) = \frac{1}{b-a}\ ,a \le x \le b\ \ ;E(x) = \frac{b+a}{2}\ ,V(x) = \frac{(b-a)^2}{12}$$

$$Y \sim Uniform(0,1):\ f_Y(y) = 1\ ,0 \le x \le 1$$

$$Y = dX + c \Rightarrow X = \frac{Y-c}{d} \Rightarrow \left|\frac{d}{dy}X\right| = \left|\frac{1}{d}\right|$$

$$\boldsymbol{f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|} = f_X\left(\frac{y-c}{d}\right)\left|\frac{1}{d}\right| = \frac{1}{b-a}\frac{1}{|d|}$$

$$a \le x \le b \Rightarrow \begin{cases} for\ d > 0:\ \ a \le \frac{y-c}{d} \le b \Rightarrow ad + c \le y \le bd + c \rightarrow (1) \\ for\ d < 0:\ \ a \le \frac{y-c}{d} \le b \Rightarrow bd + c \le y \le ad + c \rightarrow (2) \end{cases}$$

1: $\begin{cases} (-) \quad ad + c = 0 \quad \rightarrow \quad (*) \\ \quad\quad\ \ bd + c = 1 \quad \rightarrow \quad (**) \end{cases}$

$$ad - bd = -1$$

$$d(a-b) = -1 \quad \Rightarrow \quad \boldsymbol{d} = -\frac{\mathbf{1}}{\boldsymbol{a-b}} = \frac{\mathbf{1}}{\boldsymbol{b-a}}$$

substitute (d) into equation (*) we get

$$a\left(\frac{1}{b-a}\right) + c = 0 \quad \Rightarrow \quad c = -\frac{a}{b-a}$$

$$\boldsymbol{c} = \frac{\boldsymbol{a}}{\boldsymbol{a-b}}$$

2: $\begin{cases} (-) \quad bd + c = 0 \quad \rightarrow \quad (i) \\ \quad\quad\ \ ad + c = 1 \quad \rightarrow \quad (ii) \end{cases}$

$$bd - ad = -1$$

$$d(b-a) = -1 \Rightarrow \quad \boldsymbol{d} = -\frac{\mathbf{1}}{\boldsymbol{b-a}} = \frac{\mathbf{1}}{\boldsymbol{a-b}}$$

substitute (d) into equation (i) we get

$$\frac{b}{a-b} + c = 0 \Rightarrow c = -\frac{b}{a-b}$$

$$\boldsymbol{c} = \frac{\boldsymbol{b}}{\boldsymbol{b-a}}$$

Therefore $(c,d) = \left\{ \left(\frac{\boldsymbol{a}}{\boldsymbol{a-b}}, \frac{\mathbf{1}}{\boldsymbol{b-a}}\right) ; \left(\frac{\boldsymbol{b}}{\boldsymbol{b-a}}, \frac{\mathbf{1}}{\boldsymbol{a-b}}\right) \right\}$.

Another method:

$$f_Y(y) = f_X\left(\frac{y-c}{d}\right)\left|\frac{1}{d}\right| = \frac{1}{b-a}\frac{1}{|d|} = \mathbf{1}$$

Thus, $|d| = \frac{1}{b-a} \Rightarrow \quad d = \mp\frac{1}{b-a}$

$\boldsymbol{for\ d > 0}$: $\quad \boldsymbol{d} = \frac{\mathbf{1}}{\boldsymbol{b-a}}$

$0 < y < 1 \Rightarrow \quad 0 < c + dx < 1 \quad \Rightarrow \quad -c < dx < 1 - c \quad \Rightarrow \quad -\frac{c}{d} < x < \frac{1-c}{d}$

Thus, $a = -\frac{c}{d} \quad \Rightarrow \quad a = -c\,(\,b-a) \Rightarrow \quad \boldsymbol{c} = \frac{\boldsymbol{a}}{\boldsymbol{a-b}}$

$\boldsymbol{for\ d < 0}$: $\quad \boldsymbol{d} = \frac{\mathbf{1}}{\boldsymbol{a-b}}$

$0 < y < 1 \Rightarrow \quad 0 < c + dx < 1 \quad \Rightarrow \quad -c < dx < 1 - c \quad \Rightarrow \quad \frac{1-c}{d} < x < -\frac{c}{d}$

Thus, $a = \frac{1-c}{d} \quad \Rightarrow \quad a = (1-c)(a-b) \Rightarrow \quad \frac{a}{a-b} = 1 - c \quad \Rightarrow \quad c = 1 - \frac{a}{a-b} \quad \Rightarrow \quad \boldsymbol{c} = \frac{\boldsymbol{b}}{\boldsymbol{b-a}}$

Another method:

$E(x) = \frac{a+b}{2}$, $V(x) = \frac{(b-a)^2}{12}$

$\frac{1}{2} = E(Y) = E(c + dX) = c + d\,E(X) = c + d\left(\frac{a+b}{2}\right) \quad \rightarrow (1)$

$\frac{1}{12} = V(Y) = V(c + dX) = d^2V(X) = d^2\,\frac{(b-a)^2}{12} \rightarrow (2)$

From 2: $d^2 = \frac{1}{(b-a)^2} \Rightarrow \quad d = \mp \frac{1}{b-a}$

From 1: $c = \frac{1}{2} - \frac{d\,(a+b)}{2} \quad \Rightarrow c = \frac{1}{2}[1 - d(a+b)]$

Thus, for $\boldsymbol{d = \frac{1}{b-a}} \quad \Rightarrow \quad c = \frac{1}{2}\left[1 - \frac{a+b}{b-a}\right] \Rightarrow c = \frac{1}{2}\left[\frac{b-a-a-b}{b-a}\right] \Rightarrow c = \frac{1}{2}\left[-\frac{2a}{b-a}\right]$

$$\Rightarrow c = \frac{a}{a-b}$$

Thus, for $\boldsymbol{d = \frac{1}{a-b}} \quad \Rightarrow \quad c = \frac{1}{2}\left[1 - \frac{a+b}{a-b}\right] \Rightarrow \quad c = \frac{1}{2}\left[\frac{a-b-a-b}{a-b}\right] \Rightarrow \quad c = \frac{1}{2}\left[\frac{-2b}{a-b}\right]$

$$\Rightarrow c = \frac{b}{b-a}$$

Therefore $(c,d) = \left\{ \left(\frac{a}{a-b}, \frac{1}{b-a}\right) ; \left(\frac{b}{b-a}, \frac{1}{a-b}\right) \right\}$.

Q3) If $X \sim Normal(\mu, \sigma^2)$, find the pdf of $Y = e^X$.

Solution :

$$f_X(x) = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , -\infty \le x \le \infty$$

$$Y = e^X \quad \Longrightarrow X = \ln Y \quad \Longrightarrow \frac{d}{dY}\,X = \frac{1}{Y} \quad \Longrightarrow \left|\frac{d}{dY}\,X\right| = \frac{1}{Y}$$

$$-\infty \le x \le \infty \quad \Longrightarrow \quad 0 \le e^x \le \infty \quad \Longrightarrow \quad 0 \le y \le \infty$$

$$f_Y(y) = f_X(\ln y)\left|\frac{d}{dy}\,X\right| = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2}\left(\frac{1}{y}\right)$$

Q4) If $X \sim Exponential(1)$, find the pdf of $Y = -\ln X$.

Solution :

$f_X(x) = e^{-x} \quad , x \ge 0$

$$Y = -\ln X \Rightarrow \quad e^Y = \frac{1}{X} \quad \Rightarrow \quad X = e^{-Y} \quad \Rightarrow \frac{d}{dY}\,X = -e^{-Y} \quad \Rightarrow \left|\frac{d}{dY}\,X\right| = e^{-Y}$$

$x \ge 0 \quad \Rightarrow \ln x \ge -\infty \quad \Rightarrow \quad -\infty < -\ln x < \infty \quad \Rightarrow \quad -\infty < y < \infty$

$$f_Y(y) = f_X(e^{-y})\left|\frac{d}{dy}\,X\right| = e^{-e^{-y}}\ e^{-y} = e^{-(y+e^{-y})}$$

Note:

- Derivative of the natural logarithm $\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$
- Derivative of the natural exponential function $\frac{d}{dx}(e^{g(x)}) = g'(x)\, e^{g(x)}$
- $\ln(0) = -\infty \;\; ; \;\; \ln(1) = 0 \;\; ; \ln(\infty) = \infty$
- $e^{-\infty} = 0 \;\; ; \;\; e^0 = 1$

Q5) If $X \sim Uniform(0,1)$, find the pdf of $Y = \sqrt{X}$.

Solution :

$$Uniform(0,1): \; f_X(x) = 1 \;, 0 \le x \le 1$$

$$Y = \sqrt{X} \;\Rightarrow\; X = Y^2 \;\Rightarrow\; \frac{d}{dY} X = 2Y \;\Rightarrow\; \left|\frac{d}{dY} X\right| = 2Y$$

$$0 \le x \le 1 \;\Rightarrow\; 0 \le \sqrt{x} \le 1 \;\Rightarrow\; 0 \le y \le 1$$

$$f_Y(y) = f_X(y^2)\left|\frac{d}{dy} x\right| = 2y$$

Q6) The pdf of X is given by $f_X(x) = \frac{1}{2}x \;\; ; 0 < x < 2$.

a. Find the pdf of $Y = X^3$.

b. Find $P\left(\frac{1}{2} < X < 1\right)$ and $P\left(\frac{1}{8} < Y < 1\right)$. Are they the same or different? Why?

Solution :

$$Y = X^3 \;\Rightarrow\; X = Y^{\frac{1}{3}} \;\Rightarrow\; \left|\frac{d}{dy} X\right| = \frac{1}{3} Y^{-\frac{2}{3}}$$

$$0 < x < 2 \;\Rightarrow\; 0 < x^3 < 8 \;\Rightarrow\; 0 < y < 8$$

$$f_Y(y) = f_X(y^{1/3})\left|\frac{d}{dy} x\right| = \frac{1}{2} y^{\frac{1}{3}} \; \frac{1}{3} Y^{-\frac{2}{3}} = \mathbf{\frac{1}{6}\, y^{-\frac{1}{3}}}$$

$$P\left(\frac{1}{2} < X < 1\right) = \frac{1}{2}\int_{1/2}^{1} x\, dx = \frac{1}{4}[\,x^2\,]_{1/2}^{1} = \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{1}{4}\left(\frac{3}{4}\right) = \frac{3}{16}$$

$$P\left(\frac{1}{8} < Y < 1\right) = \frac{1}{6}\int_{1/8}^{1} y^{-1/3}\, dy = \frac{1}{6}\;\frac{3}{2}\;\left[y^{\frac{2}{3}}\right]_{1/8}^{1} = \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{1}{4}\left(\frac{3}{4}\right) = \frac{3}{16}$$

We can see that are the same because

$\frac{1}{2} < x < 1 \;\Rightarrow\; \frac{1}{8} < x^3 < 1 \;\Rightarrow\; \frac{1}{8} < y < 1$

and so $P\left(\frac{1}{2} < X < 1\right) = P\left(\frac{1}{8} < Y < 1\right)$

Q7) If $X \sim \chi_4{}^2$, find P(X > 5). (delete)