Bull. Malays. Math. Sci. Soc. (2020) 43:2297–2312
https://doi.org/10.1007/s40840-019-00805-w

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Finite Orders Which Are Reconstructible up to Duality by Their Comparability Graphs

Mohammad Alzohairi[1] · **Moncef Bouaziz**[1] · **Youssef Boudabbous**[1] · **Ahmad Sharary**[1]

Received: 29 November 2018 / Revised: 8 May 2019 / Published online: 15 July 2019

Abstract
A finite order P on a set V is reconstructible (respectively, reconstructible up to duality) by its comparability graph if each order on V which has the same comparability graph as P is isomorphic to P (respectively, is isomorphic to P or to its dual $P^{\star}$). In this paper, we describe the finite orders which are reconstructible up to duality by their comparability graphs. This result is motivated by the characterization, obtained by Gallai (Acta Math Acad Sci Hungar 18:25–66, 1967), of the pairs of finite orders having the same comparability graph. Notice that a characterization of the finite orders which are reconstructible by their comparability graphs is easily deduced from Gallai's result.

Keywords Ordered set · Comparability graph · Reconstruction · Isomorphism up to duality · Modular decomposition

Mathematics Subject Classification 05C60 · 05C20

Communicated by Xueliang Li.

✉ Youssef Boudabbous
youssef_boudabbous@yahoo.fr; yboudabbous@ksu.edu.sa

Mohammad Alzohairi
zohairi@ksu.edu.sa

Moncef Bouaziz
moncef.bouaziz@yahoo.fr; mbouaziz@ksu.edu.sa

Ahmad Sharary
asharary@ksu.edu.sa

[1] Department of Mathematics, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

1 Introduction

In various domains of mathematics, it is natural to investigate the relationships between the global properties and the local properties of a given structure. In the context of finite discrete structures, the idea of reconstruction is to describe a structure in terms of certain types of its substructures. The origins of reconstruction are: Ulam's conjecture for finite graphs [2,3,27], the problem of Das for finite orders [12,21], Fraïssé's problem for finite relations [16], and the problem of Pouzet for finite binary relations [2,3]. The origin of half-reconstruction is the problem of Hagendorf for binary relations [20]. Recently, the reconstruction of graphs up to complementation, which corresponds to the half-reconstruction for digraphs, is studied [10,11,24,25].

Throughout this paper, the word "order" will mean a finite order.

Let P be an order on a set of vertices V.

The elements x and y of V are *comparable* if either $x < y$ or $y < x$ in P; otherwise they are *incomparable* and we write $x \parallel y$. For disjoint subsets X and Y of V, $X < Y$ (resp. $X \parallel Y$) means that $x < y$ (resp. $x \parallel y$) for every $(x, y) \in X \times Y$. To simplify, we write $x < Y$ (resp. $X < y$) for $\{x\} < Y$ (resp. $X < \{y\}$), and $x \parallel Y$ for $\{x\} \parallel Y$.

The *comparability graph* of P is the symmetric graph $Comp(P) = (V, E(Comp(P))$, where $\{x, y\} \in E(Comp(P))$ whenever $x \neq y$ and x and y are comparable. The order P is *connected* if $Comp(P)$ is connected, and the *connected components* of P are those of $Comp(P)$. Let X be a subset of V. We denote by $P_{\restriction X}$ the *order induced by* P on X. The subset X is a *chain* if $P_{\restriction X}$ is linear, and it is an *antichain* if its elements are pairwise incomparable. We say that P is a chain (resp. antichain) whenever V is a chain (resp. an antichain).

An *isomorphism* f from P onto an order P' on a set V' is a bijection from V onto V' such that $x \leq y$ in P if and only if $f(x) \leq f(y)$ in P', for any $x, y \in V$. The orders P and P' are *isomorphic*, in which case we write $P \cong P'$, if there exists an isomorphism from P onto P'.

The *dual* of P is the order denoted by $P^\star$ and defined on the set V as follows: $x \leq y$ in $P^\star$ if and only if $y \leq x$ in P. The order P is *self-dual* if it is isomorphic to its dual.

An order is *isomorphic up to duality* to P if it is isomorphic to P or to its dual $P^\star$.

The order P is reconstructible (respectively, reconstructible up to duality) by its comparability graph if each order on V which has the same comparability graph as P is isomorphic to P (respectively, is isomorphic up to duality to P).

The following remark is trivial.

Remark 1.1 Let P be an order.

(1) If P is reconstructible by its comparability graph, then P is reconstructible up to duality by its comparability graph.
(2) If P is reconstructible up to duality by its comparability graph and P is self-dual, then P is reconstructible by its comparability graph.

Let $P_i = (V_i, \leq_i), i = 1, 2, ..., n$, be orders such that the V_i's are pairwise disjoint. The *disjoint sum* $P_1 + \cdots + P_n$ (respectively, *direct sum* (*ordinal sum* or *linear sum*) $P_1 \oplus \cdots \oplus P_n$) is the order $(V, \leq)$, where $V = V_1 \cup \cdots \cup V_n$ and $a \leq b$ if $a \leq_i b$ in P_i for some i (respectively, if $a \leq_i b$ in P_i for some i or $a \in V_i$ and $b \in V_j$ for some $i < j$).

Let P be an order on a set of vertices V.

A subset M of V is a *module* [26] of P if for each element x of $V \setminus M$, either $x < M$ or $M < x$ or $x \parallel M$. This concept is also named *interval* in [17] and autonomous set in [18]. The sets V, ϕ, and the singletons $\{x\}$, where $x \in V$, are modules of P which are called *trivial*. The order P is *indecomposable* if all its modules are trivial; otherwise it is *decomposable*. Observing that all orders with three vertices are decomposable, we say that an order is *prime* if it is indecomposable with at least four vertices.

Given a subset M of V, we say that M is self-dual, respectively is isomorphic, respectively is isomorphic up to duality, to an order Q, if $P_{\restriction M}$ is self-dual, respectively is isomorphic, respectively is isomorphic up to duality, to Q.

A *modular partition* of P is a partition $\overline{V}$ of V such that all its elements are modules of P. The *quotient* $P/\overline{V}$ of P by $\overline{V}$ is the order on $\overline{V}$ defined as follows: $X < Y$ in $P/\overline{V}$ if and only if $x < y$ in P for each $x \in X$ and each $y \in Y$.

A module M of P is *strong* if either $M' \subseteq M$ or $M \subseteq M'$ for every module M' of P such that $M \cap M' \neq \phi$. The trivial modules are strong.

For $\mid V \mid \geq 2$, the set $G(P)$ of maximal (with respect to inclusion) strong proper modules of P is a modular partition of P, called the *canonical partition* of P, and the elements of $G(P)$ are the *modular components* of P. For $\mid V \mid \leq 1$, the unique partition of the set V is called the *canonical partition* of P. We consider $\emptyset$ as the unique partition of $\emptyset$.

The *frame* of P is its quotient by its canonical partition.

The following Gallai decomposition theorem of orders plays an essential role in our study.

Theorem 1.2 [18,23] (Gallai's decomposition) *Given an order P on at least two elements, the frame $P/G(P)$ of P is either prime, a chain, or an antichain.*

Let M be a module of P, and Q be an order on M with $Comp(Q) = Comp(P_{\restriction M})$. The order obtained from P by *replacing* $P_{\restriction M}$ by Q is the order, denoted by ${}_M P_Q$, defined on V as follows: $({}_M P_Q)_{\restriction M} = Q$, and for $x,\ y \in V$ such that $\{x, y\} \setminus M \neq \emptyset$, $x \leq y$ in ${}_M P_Q$ if and only if $x \leq y$ in P. Clearly, M is a module of the order ${}_M P_Q$ and $Comp({}_M P_Q) = Comp(P)$. In case $Q = P^\star_{\restriction M}$, we denote ${}_M P_Q$ by $Inv(M, P)$ and say that ${}_M P_Q$ is obtained from P by *module inversion*.

Let $S = \{M_1,\ M_2,\ ..., M_k\}$ be a set of disjoint modules of P, and let $T = \{Q_1,\ Q_2,\ ..., Q_k\}$ be a set of orders such that $Q_1,\ Q_2,\ ..., Q_k$ are defined on $M_1,\ M_2,\ ..., M_k$, respectively. We denote by ${}_S P_T$ the order obtained from P by successive iterations of replacement of the orders $P_{\restriction M_1},\ P_{\restriction M_2},\ ..., P_{\restriction M_k}$ as follows: ${}_S P_T = P_k$, where $P_{i+1} = {}_{M_{i+1}} (P_i)_{Q_{i+1}}$ for $0 \leq i \leq k-1$ and $P_0 = P$.

In case $Q_i = P^*_{\restriction M_i}$ for $i = 1,\ 2,\ ..., k$, we use $Inv(S, P)$ instead of ${}_S P_T$ as follows: $Inv(S, P) = P_k$, where $P_{i+1} = Inv(M_{i+1}, P_i)$ for $0 \leq i \leq k-1$, and $P_0 = P$.

Given an order P' on V, $P \mathcal{I} P'$ signifies that there are orders $P_0 = P,\ ...,\ P_n = P'$ such that for $0 \leq i \leq n-1$, $P_{i+1} = Inv(M_i, P_i)$, where M_i is a module of P_i.

The Gallai inversion theorem is the following.

Theorem 1.3 [18] (Gallai's inversion) *Given two orders P and Q with the same vertex set, $Comp(P) = Comp(Q)$ if and only if $P \mathcal{I} Q$.*

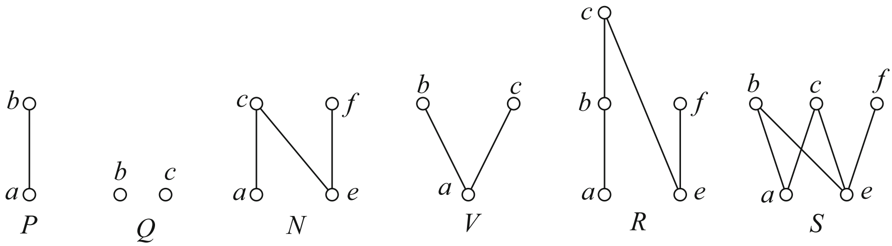

Fig. 1 .

Notice that a generalization of the Gallai inversion theorem for digraphs was obtained in [8].

Given two orders H and K with a unique common vertex v, we denote by $H(v, K)$ the order obtained from H by dilating H on the vertex v by K as follows. The set of vertices of $H(v, K)$ is $V(H) \cup V(K)$, $H(v, K)_{\restriction V(H)} = H$, $H(v, K)_{\restriction V(K)} = K$, and $V(K)$ is a module of $H(v, K)$.

Example 1.4 (See Fig. 1). Consider the orders P, Q, N, V, R and S represented by their Hasse diagrams in Figure 1.

Since each of the orders P and Q has exactly two vertices, these orders are indecomposable. It is easy to verify that the order N is prime. Therefore, for each element H of $\{P, Q, N\}$, $G(H) = \{\{x\} : x \in V(H)\}$ and the frame of H is isomorphic to H.

The order V is obtained by dilating P on the vertex b by Q. Thus, $V = P(b, Q)$. Moreover, V is decomposable with $G(V) = \{\{a\}, \{b, c\}\}$, and its frame is a chain isomorphic to P.

The order R is obtained by dilating N on the vertex a by P. Thus, $R = N(a, P)$. Moreover, R is decomposable with $G(R) = \{\{a, b\}, \{c\}, \{e\}, \{f\}\}$, and its frame is a prime order isomorphic to N.

The order S is obtained by dilating N on the vertex c by Q. Thus, $S = N(c, Q)$. Moreover, S is decomposable with $G(S) = \{\{b, c\}, \{a\}, \{e\}, \{f\}\}$, and its frame is a prime order isomorphic to N.

2 Presentation of the Results

In order to state our results, we introduce the following notations and definition.

Notation 2.1 We consider the following classes of orders.

- $\mathcal{H} = \{P : P$ is an order which is reconstructible up to duality by its comparability graph$\}$.
- For every integer $n \geq 0$, $\mathcal{H}_n = \{P \in \mathcal{H} : P$ has exactly n non-self-dual modules$\}$.

Notice that $\mathcal{H} = \underset{n \geq 0}{\cup} \mathcal{H}_n$.

Notation 2.2 Consider the following subclasses of the class $\mathcal{H}_o$.

- $\mathcal{H}_{0,p} = \{P \in \mathcal{H}_0 : \ the\ frame\ of\ P\ is\ prime\}$.
- $\mathcal{H}_{0,c} = \{P \in \mathcal{H}_0 : \ the\ frame\ of\ P\ is\ a\ chain\}$.
- $\mathcal{H}_{0,a} = \{P \in \mathcal{H}_0 : \ the\ frame\ of\ P\ \ is\ an\ antichain\}$.

Clearly, $\mathcal{H}_{0,p} \cap \mathcal{H}_{0,c} = \mathcal{H}_{0,p} \cap \mathcal{H}_{0,a} = \emptyset$,
$\mathcal{H}_{0,c} \cap \mathcal{H}_{0,a} = \{P :\ \ P\ is\ an\ order\ with\ at\ most\ one\ vertex\}$. Moreover, by Theorem 1.2, $\mathcal{H}_o = \mathcal{H}_{0,p} \cup \mathcal{H}_{0,c} \cup \mathcal{H}_{0,a}$, and $\mathcal{H}_{0,c} \cup \mathcal{H}_{0,p}$ is the class of connected members of $\mathcal{H}_0$.

Definition 2.3 Let P be an order on a set V, with $\mid V \mid \geq 4$, satisfying the following conditions:

- The frame $Q = P/G(P)$ of P is prime;
- There exist a vertex x with $\{x\} \in G(P)$ and an isomorphism f_x from Q onto $Q^\star$ such that $f_x(\{x\}) = \{x\}$, and for every $M \in G(P) \setminus \{\{x\}\}$, the induced orders $P_{\restriction M}$ and $P_{\restriction f_x(M)}$ are isomorphic members of $\mathcal{H}_0$.

Clearly, P is self-dual. Such an element x is called a *good vertex*, such an isomorphism f_x is called a *good isomorphism* associated with x, and P is called a *good self-dual order*.

Our first result is the following characterization of the orders which are reconstructible by their comparability graphs, which is easily deduced from the Gallai inversion theorem.

Proposition 2.4 *An order P is reconstructible by its comparability graph if and only if every module of P is self-dual.*

Our main result is the following description of the orders which are reconstructible up to duality by their comparability graphs.

Theorem 2.5 (1) *An order P belongs to $\mathcal{H}_0$ if and only if all its modules are self-dual.*
(2) *An order P belongs to $\mathcal{H}_1$ if and only if either P is non-self-dual with a prime frame and $P_{\restriction X} \in \mathcal{H}_0$ for every $X \in G(P)$, or P is the direct sum of two non-isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$.*
(3) *For $k \geq 2$, an order P belongs to $\mathcal{H}_k$ if and only if it satisfies one of the following two assertions.*

  (a) *P is obtained from a good self-dual order by dilating a good vertex by a member of $\mathcal{H}_{k-1}$.*
  (b) *P is the disjoint sum of $q-1$ members of $H_{0,c} \cup H_{0,p}$ and one connected member of $\mathcal{H}_{k-2^{q-1}+1}$, where $2 \leq q \leq log_2(k) + 1$.*

Thus, by Theorem 2.5, the class of the orders which are reconstructible up to duality by their comparability graphs is described by means of a recursive procedure and makes use of Gallai's decomposition.

Our work is also motivated by the problem of ($\leq k$)-reconstruction (respectively, ($\leq\ k$)-half-reconstruction) of binary relations which was introduced by Fraïssé's [16] (respectively, Hagendorf [20]). Indeed, Proposition 2.4 characterizes the (≤ 2)-reconstructible orders, and Theorem 2.5 describes the ($\leq\ 2$)-half-reconstructible orders (see Sect. 6).

Finally, notice that our method of the description of the orders which are reconstructible up to duality by their comparability graphs is similar to the method of the description of the (≤ 3)-half-reconstructible finite tournaments obtained by Boudabbous et al [5].

3 Preliminaries

In this section, we recall and prove some results which are needed as major tools in our proofs.

The following lemma lists some basic properties of modules.

Lemma 3.1 [13] *Let P be an order.*

(1) *The sets $\emptyset$, V and $\{x\}$, where $x \in V$, are modules of P.*
(2) *If M, N are modules of P, then $M \cap N$ is a module of P.*
(3) *If M, N are nondisjoint modules of P, then $M \cup N$ is a module of P.*
(4) *If M, N are modules of P such that $M \setminus N \neq \emptyset$, then $N \setminus M$ is a module of P.*
(5) *If M, N are disjoint modules of P, then either $M < N$ or $N < M$ or $M \parallel N$.*
(6) *Given a module M of P and a subset W of V, the trace $M \cap W$ is a module of the induced order $P_{\restriction W}$.*
(7) *If M is a module of P and N is a module of $P_{\restriction M}$, then N is a module of P.*

Notice that (5) justifies the definition of the quotient order.

The following result is the key of the Gallai inversion theorem.

Corollary 3.2 [18] *Let P be a prime order on a set V. If P' is an order on V such that $Comp(P) = Comp(P')$, then either $P' = P$ or $P' = P^{\star}$.*

The following corollary is easily deduced from Corollary 3.2.

Corollary 3.3 *Let P and P' be two orders with the same vertex set V with $\mid V \mid \geq 2$. If $Comp(P) = Comp(P')$, then the following assertions hold.*

(1) *$G(P') = G(P)$.*
(2) *If $P/G(P)$ is prime, then either $P'/G(P) = P/G(P)$ or $P'/G(P) = P^{\star}/G(P)$.*
(3) *If $P/G(P)$ is an antichain, then $P'/G(P)$ is an antichain.*
(4) *If $P/G(P)$ is a chain, then $P'/G(P)$ is a chain.*

Notice that (3) and (4) are trivial consequences of (1) and (2).

Remark 3.4 Let $P = (V, \leq)$ be an order with $\mid V \mid \geq 2$.

(1) If the frame $P/G(P)$ of P is prime, then the Gallai partition $G(P)$ of P is the set of the maximal proper modules of P.
(2) If Q is a modular partition of P for which the corresponding quotient P/Q is prime, then $Q = G(P)$.
(3) If $P/G(P)$ is a chain $V_1 < V_2 < \cdots < V_m$, then a subset M of V is a module of P if and only if M is a module of some $P_{\restriction V_i}$ or there are $1 \leq i_1 \leq i_2 \leq m$ such that $M = \bigcup\limits_{i_1 \leq i \leq i_2} V_i$.

(4) If $P/G(P)$ is an antichain, then a subset M of V is a module of P if and only if there is a member A of $G(P)$ such that M is a module of $P_{\restriction A}$ or M is the union of some members of $G(P)$.
(5) If P has a modular partition Q with $\mid Q \mid \geq 2$ such that the quotient P/Q is a chain (respectively, an antichain), then the frame of P is a chain (resp. an antichain) and $G(P)$ is the largest modular partition of P for which the corresponding quotient is a chain (respectively, an antichain).
(6) If the frame $P/G(P)$ of P is prime, then a proper subset M of V is a module of P if and only if there is a member A of $G(P)$ such that M is a module of $P_{\restriction A}$.
(7) The order P is not connected if and only if $P/G(P)$ is an antichain. Moreover, if P is not connected, then $G(P)$ is the set of connected components of P.

Remark 3.5 Let P be an order, on at least two elements, with a chain (respectively, an antichain) frame. Then, for every $M \in G(P)$ with $\mid M \mid \geq 2$, the frame of $P_{\restriction M}$ is not a chain (respectively, is not an antichain).

The next result is “the balanced lemma.”

Lemma 3.6 [7] *Let P, Q, and Q' be orders with vertex sets $V(P) = \{1, 2, ..., n\}$, where $n \geq 2$, $V(Q)$, $V(Q')$, respectively, such that $V(P) \cap V(Q) = V(P) \cap V(Q') = \{k\}$. Then $P(k, Q) \cong P(k, Q')$ if and only if $Q \cong Q'$.*

Notice that Lemma 3.6 was firstly communicated by A. Boussaïri, and a detailed proof of this lemma is presented by J. Dammak in [9].

Finally, let $P = (V, \leq)$ be an order, with $\mid V \mid \geq 2$, such that its frame $P/G(P)$ is a chain. The modular partition $\widetilde{G}(P)$ is defined as follows. For $A \subseteq V$, $A \in \widetilde{G}(P)$ if and only if $A \in G(P)$ with $\mid A \mid \geq 2$, or A is a maximal union of consecutive vertices of $P/G(P)$ which are singletons. Notice that P is a chain if and only if $\mid \widetilde{G}(P) \mid = 1$, and if P is not a chain, then $P/\widetilde{G}(P)$ is a chain $A_1 < \cdots < A_k$ with $k \geq 2$.

4 Proof of Proposition 2.4

If M is a non-self-dual module of an order P, we consider the order $P' = Inv(M, P)$. By Theorem 1.3, $Comp(P') = Comp(P)$.

No, we apply Lemma 3.6 to show that $P \ncong P'$. Select a vertex $m_0 \in M$ and let P_0 be the induced order on the set of vertices $(V(P) \setminus M) \cup \{m_0\}$. Since $P_{\restriction M} \ncong P^*_{\restriction M}$, $P = P_0(m_0, P_{\restriction M}) \ncong P_0(m_0, P^*_{\restriction M}) = P'$. Hence, P is not reconstructible by its comparability graph.

Conversely, let $P = (V, \leq)$ be an order such that every module of P is self-dual. Let P' be an order on V such that $Comp(P') = Comp(P)$. By Theorem 1.3, there is a sequence of orders $P_0 = P, ..., P_n = P'$ such that for $0 \leq i \leq n-1$, $P_{i+1} = Inv(M_i, P_i)$, where M_i is a module of P_i. Since P_1 is obtained from P by module inversion, and every module of P is self-dual, $P_1 \cong P_0$ and hence every module of P_1 is self-dual. By applying this argument to P_1 and P_2, P_2 and P_3,..., P_{n-1} and P_n, we obtain $P' \cong P$. □

5 Proof of Theorem 2.5

We proceed by establishing some lemmas which are needed for the proof of Theorem 2.5.

The following lemma is an immediate consequence of Remark 1.1 and Proposition 2.4.

Lemma 5.1 *Let P be an order. If P is reconstructible up to duality by its comparability graph and P is self-dual, then $P \in \mathcal{H}_0$.*

The following corollary follows immediately from Remark 1.1, Proposition 2.4, and Lemma 5.1.

Corollary 5.2 *Let P be an order.*

(1) *$P \in \mathcal{H}_0$ if and only if every module of P is self-dual.*
(2) *If $P \in \mathcal{H}_k$ with $k \geq 1$, then P is non-self-dual.*

We need the following notation.

Notation 5.3 Given an order P on at least two elements, and an order Q, we define the following two sets.

- $G(P, Q) = \{M \in G(P) : P_{\restriction M} \ is\ isomorphic\ to\ Q\}$.
- $G(P, \{Q, Q^{\star}\}) = \{M \in G(P) : P_{\restriction M} \ is\ isomorphic\ up\ to\ duality\ to\ Q\}$.

The following remark is trivial.

Remark 5.4 Let P and P' be two orders on at least two elements.

(1) If f is an isomorphism from P onto P', then $f(G(P)) = G(P')$. Moreover, for each member M of $G(P)$, $f(G(P, P_{\restriction M})) = G(P', P_{\restriction M})$.
(2) If f is an isomorphism from P onto P' or $(P')^{\star}$, then $f(G(P)) = G(P')$. Moreover, for each member M of $G(P)$, $f(G(P, \{P_{\restriction M}, P^{\star}_{\restriction M}\})) = G(P', \{P_{\restriction M}, P^{\star}_{\restriction M}\})$.
(3) If the frame of P is a chain with first element M and last element N, the frame of P' is a chain with first element M' and last element N', and f is an isomorphism from P onto P' or $(P')^{\star}$, then $f(\{M, N\}) = \{M', N'\}$.

Lemma 5.5 *Let P be an order on at least two elements. If P is reconstructible up to duality by its comparability graph, then for each element X of $G(P)$, the suborder $P_{\restriction X}$ is reconstructible up to duality by its comparability graph.*

Proof We prove by contraposition. Assume that there is an element A of $G(P)$ such that $P_{\restriction A}$ is not reconstructible up to duality by its comparability graph. Thus, for every $Y \in G(P, \{P_{\restriction A}, P^{\star}_{\restriction A}\})$, there is an order Q_Y on the set Y such that $Comp(Q_Y) = Comp(P_{\restriction Y})$ and Q_Y is not isomorphic up to duality to $P_{\restriction Y}$. Let P' be the order obtained from P by replacing $P_{\restriction Y}$ by Q_Y for each $Y \in G(P, \{P_{\restriction A}, P^{\star}_{\restriction A}\})$. Clearly, $Comp(P') = Comp(P)$. Thus, by Corollary 3.3, $G(P') = G(P)$. Since $G(P', \{P_{\restriction A}, P^{\star}_{\restriction A}\}) = \emptyset$ and $G(P, \{P_{\restriction A}, P^{\star}_{\restriction A}\}) \neq \emptyset$, Remark 5.4 (2) implies that P' is not isomorphic up to duality to P. Thus, P is not reconstructible up to duality by its comparability graph. $\square$

Lemma 5.6 *Let P be an order on a set V with $\mid V \mid \geq 2$. If P is reconstructible up to duality by its comparability graph, then there is at most one member M of $G(P)$ such that the order $P_{\restriction M}$ is non-self-dual.*

Proof To the contrary, suppose that the set $\mathcal{F}$ of non-self-dual members of $G(P)$ has at least two elements. We consider the following two cases.

Case 1. There are two elements M_1, M_2 of $\mathcal{F}$ such that the orders $P_{\restriction M_1}$ and $P_{\restriction M_2}$ are not isomorphic up to duality.

We consider the sets $\mathcal{F}_1 = G(P, P_{\restriction M_1})$, $\mathcal{F}_1^\star = G(P, P^\star_{\restriction M_1})$, $\mathcal{F}_2 = G(P, P_{\restriction M_2})$ and $\mathcal{F}_2^\star = G(P, P^\star_{\restriction M_2})$, which are subsets of $\mathcal{F}$, and the orders $P_1 = Inv(\mathcal{F}_1^\star \cup \mathcal{F}_2^\star, P)$ and $P_2 = Inv(\mathcal{F}_1 \cup \mathcal{F}_2^\star, P)$. By Theorem 1.3, $Comp(P_1) = Comp(P_2) = Comp(P)$. Hence, $G(P_1) = G(P_2) = G(P)$, by Corollary 3.3.

On the other hand, it is easy to verify that:

$G(P_1, P^\star_{\restriction M_1}\}) = G(P_1, P^\star_{\restriction M_2}\}) = \emptyset$, $G(P_2, P_{\restriction M_1}\}) = G(P_2, P^\star_{\restriction M_2}\}) = \emptyset$, $G(P_1, \{P_{\restriction M_1}, P^\star_{\restriction M_1}\}) = G(P_1, P_{\restriction M_1}) = \mathcal{F}_1 \cup \mathcal{F}_1^\star$, $G(P_1, \{P_{\restriction M_2}, P^\star_{\restriction M_2}\}) = G(P_1, P_{\restriction M_2}) = G((P_1)^\star, P^\star_{\restriction M_2}) = \mathcal{F}_2 \cup \mathcal{F}_2^\star$, $G(P_2, \{P_{\restriction M_1}, P^\star_{\restriction M_1}\}) = G(P_2, P^\star_{\restriction M_1}) = \mathcal{F}_1 \cup \mathcal{F}_1^\star$, and $G(P_2, \{P_{\restriction M_2}, P^\star_{\restriction M_2}\}) = G(P_2, P_{\restriction M_2}) = \mathcal{F}_2 \cup \mathcal{F}_2^\star$.

Thus, $M_1 \in G(P_1, P_{\restriction M_1})$, $G(P_2, P_{\restriction M_1}) = \varnothing$, $M_2 \in G((P_1)^\star, P^\star_{\restriction M_2})$ and $G(P_2, P^\star_{\restriction M_2}) = \varnothing$.

Hence, there is no permutation f of V such that $f(G(P_1, P_{\restriction M_1})) = G(P_2, P_{\restriction M_1})$ or $f(G((P_1)^\star, P^\star_{\restriction M_2})) = G(P_2, P^\star_{\restriction M_2})$. Thus, Remark 5.4 (1) implies that the order P_2 is neither isomorphic to P_1 nor to $P_1^\star$. Hence, P_1 and P_2 are not isomorphic up to duality. Therefore, at least one of the orders P_1 and P_2 is not isomorphic up to duality to P. This contradicts the fact that P is reconstructible up to duality by its comparability graph.

Case 2. For every pair $\{Y, Z\}$ of elements of $\mathcal{F}$, the orders $P_{\restriction Y}$ and $P_{\restriction Z}$ are isomorphic up to duality.

Let M_1 be an element of $\mathcal{F}$, and consider the sets $\mathcal{F}_1 = G(P, P_{\restriction M_1})$ and $\mathcal{F}_1^\star = G(P, P^\star_{\restriction M_1})$, which are subsets of $\mathcal{F}$.

Clearly, $\mathcal{F}$ is a disjoint union of $\mathcal{F}_1$ and $\mathcal{F}_1^\star$.

We consider the orders $P_1 = Inv(\mathcal{F}_1^\star, P)$ and $P_2 = Inv(\mathcal{F}_1 \setminus \{M_1\}, P)$. By Theorem 1.3, $Comp(P_1) = Comp(P_2) = Comp(P)$. Hence, $G(P_1) = G(P_2) = G(P)$, by Corollary 3.3.

On the other hand, it is easy to verify that:

$G(P_1, P_{\restriction M_1}) = G((P_1)^\star, P^\star_{\restriction M_1}) = \mathcal{F}$, $G(P_2, P_{\restriction M_1}) = \{M_1\}$, and $G(P_2, P^\star_{\restriction M_1}) = \mathcal{F} \setminus \{M_1\}$.

We observe that $|G(P_1, P_{\restriction M_1})| \neq |G(P_2, P_{\restriction M_1})|$, since $|\mathcal{F}| \geq 2$.

Moreover, $\mid G((P_1)^\star, P^\star_{\restriction M_1}) \mid \neq \mid G(P_2, P^\star_{\restriction M_1}) \mid$ because $|\mathcal{F}| \neq |\mathcal{F} \setminus \{M_1\}|$. Therefore, Remark 5.4 (1) implies that the order P_2 is neither isomorphic to P_1 nor isomorphic to $P_1^\star$. Thus, P_2 and P_1 are not isomorphic up to duality, and hence, at least one of them is not isomorphic up to duality to P. This contradicts the fact that P is reconstructible up to duality by its comparability graph. ◻

Lemma 5.7 *Given an order P on a set V with $|V| \geq 2$, the following assertions are equivalent.*

(1) *The order P belongs to $\mathcal{H}_1$.*

(2) *The module* V *is the only non-self-dual module of* P.
(3) *The order* P *satisfies only one of the following two conditions:*

   (a) *The order* P *is non-self-dual with a prime frame, and for each element* M *of* $G(P)$, *the order* $P_{\restriction M}$ *is a member of* $\mathcal{H}_0$.
   (b) *The order* P *is the direct sum of two non-isomorphic members of* $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$.

Proof (1) $\Rightarrow$ (2). This follows from Corollary 5.2 (2).

(2) $\Rightarrow$ (3). Assume that V is the only non-self-dual module of P. Then Lemma 3.1 (7) implies that every module of $P_{\restriction M}$ is self-dual, for each member M of $G(P)$. Thus, Corollary 5.2 (1) implies that $P_{\restriction M} \in \mathcal{H}_0$, for each member M of $G(P)$. Since a disjoint sum of self-dual orders is self-dual, the frame $P/G(P)$ is not an antichain. If this frame is prime, then P satisfies condition (a). In the sequel, we assume that the frame $P/G(P)$ of P is a chain. If $G(P)$ is a pair $\{V_1, V_2\}$, then, by Remark 3.5, the orders $P_{\restriction V_1}$ and $P_{\restriction V_2}$ are elements of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$, and they are non-isomorphic because P is non-self-dual. Therefore, if $\mid G(P) \mid= 2$, then P satisfies condition (b).

Now, we proceed by contradiction to show that $\mid G(P) \mid= 2$. Suppose that $G(P) = \{V_1, V_2, ..., V_k\}$, where $k \geq 3$, and the frame $P/G(P)$ is the chain.

$V_1 < V_2 < \cdots < V_k$. If there is an $i \in \{2, ..., k\}$ such that $P_{\restriction V_i}$ is not isomorphic to $P_{\restriction V_1}$, then let i_0 be the smallest such integer. By Remark 3.4 (3), $V_{i_0-1} \cup V_{i_0}$ is a proper module of P, and $P_{\restriction V_{i_0-1} \cup V_{i_0}}$ is non-self-dual because $P_{\restriction V_{i_0-1}} \cong P^{\star}_{\restriction V_{i_0-1}} \cong P_{\restriction V_1}$ and $P_{\restriction V_{i_0}} \cong P^{\star}_{\restriction V_{i_0}} \not\cong P_{\restriction V_1}$. This contradicts that V is the only non-self-dual module of P. Hence, $P_{\restriction V_j} \cong P_{\restriction V_1}$ for each $j \in \{2, ..., k\}$. This contradicts the fact that P is not self-dual.

(3) $\Rightarrow$ (1). Assume that P satisfies condition (a) or condition (b). Using Remark 3.4, we can easily see that V is the only non-self-dual module of P.

Now, let P' be an order on V such that $Comp(P') = Comp(P)$.

By Corollary 3.3 (1), $G(P') = G(P)$. Thus, $P'/G(P) = P/G(P)$ or $P'/G(P) = P^{\star}/G(P)$, by Corollary 3.3 (2), when P satisfies condition (a), and because $\mid G(P) \mid= 2$ when P satisfies condition (b). On the other hand, $P'_{\restriction M} \cong P_{\restriction M} \cong P^{\star}_{\restriction M}$ for each $M \in G(P)$, since $P_{\restriction M} \in \mathcal{H}_0$. It follows that $P' \cong P$ when $P'/G(P) = P/G(P)$ and $P' \cong P^{\star}$ when $P'/G(P) = P^{\star}/G(P)$. Thus, P' is isomorphic up to duality to P. Hence, P is reconstructible up to duality by its comparability graph, and therefore, $P \in \mathcal{H}_1$. $\square$

Lemma 5.8 *Let* P *be an order on a set* V, *with* $|V| \geq 2$, *such that the frame of* P *is a chain. If* P *is reconstructible up to duality by its comparability graph, then the following assertions hold.*

(1) $P_{\restriction M} \in \mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$, *for every member* M *of* $G(P)$.
(2) *If* $\mid G(P) \mid\geq 3$, *then* $P_{\restriction M} \cong P_{\restriction N}$, *for any members* M, N *of* $G(P)$.
(3) $P \in \mathcal{H}_0 \cup \mathcal{H}_1$.

Proof (1) Consider a member M of $G(P)$.

First, we prove that the order $P_{\restriction M}$ is self-dual. To the contrary, suppose that there is a non-self-dual member M of $G(P)$. Then, by lemmas 5.5, 5.6 and 5.1 , the order $P_{\restriction Y}$ is a member of $\mathcal{H}_0$, for every member Y of $G(P) \setminus \{M\}$. Consider two orders

P_1 and P_2 on the set V such that $G(P_1) = G(P_2) = G(P)$, the frames of P_1 and P_2 are chains, M is the first element of the frame $P_1/G(P)$ and is the last element of the frame $P_2/G(P)$, and $P_{1_{\restriction N}} = P_{2_{\restriction N}} = P_{\restriction N}$, for each member N of $G(P)$. Clearly, $Comp(P_1) = Comp(P_2) = Comp(P)$. The order P_2 is not isomorphic to P_1 because the first element of $P_2/G(P)$ is self-dual while the first element of $P_1/G(P)$ is non-self-dual. Moreover, M is the first element of $P_1/G(P)$ and of $P_2^\star/G(P)$, and $P^\star_{2_{\restriction M}} \ncong P_{1_{\restriction M}}$ because $P^\star_{2_{\restriction M}} = P^\star_{1_{\restriction M}} = P^\star_{\restriction M}$ and $P_{\restriction M}$ is non-self-dual. So $P_2^\star$ is not isomorphic to P_1. Hence, P_1 and P_2 are not isomorphic up to duality, and therefore, at least one of them is not isomorphic up to duality to P. This contradicts the fact that P is reconstructible up to duality by its comparability graph.

Second, since the order $P_{\restriction M}$ is self-dual, Lemma 5.5 and Lemma 5.1 imply that $P_{\restriction M} \in \mathcal{H}_0$, and hence, $P_{\restriction M} \in \mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$ by Remark 3.5.

(2) Assume that $| G(P) | \geq 3$. To the contrary, suppose that there are two members M_1, M_2 of $G(P)$ such that the suborders $P_{\restriction M_1}$ and $P_{\restriction M_2}$ are non-isomorphic. Consider a member M_3 of $G(P) \setminus \{M_1, M_2\}$. Notice that the orders $P_{\restriction M_1}$, $P_{\restriction M_2}$, and $P_{\restriction M_3}$ are self-dual because they are members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$ by the first assertion.

Consider three orders P_1, P_2, and P_3 on the set V such that $G(P_1) = G(P_2) = G(P_3) = G(P)$, the frames of P_1, P_2, and P_3 are chains, M_1 is the first element of each of the frames $P_1/G(P)$ and $P_2/G(P)$, M_2 is the last element of the frame $P_1/G(P)$ and is the first element of the frame $P_3/G(P)$, M_3 is the last element of each of the frames $P_2/G(P)$ and $P_3/G(P)$, and $P_{1_{\restriction N}} = P_{2_{\restriction N}} = P_{3_{\restriction N}} = P_{\restriction N}$, for each member N of $G(P)$. Clearly, $Comp(P_1) = Comp(P_2) = Comp(P_3) = Comp(P)$.

Case 1. $P_{\restriction M_3}$ is neither isomorphic to $P_{\restriction M_1}$ nor isomorphic to $P_{\restriction M_2}$. Then, using Remark 5.4 (3), we obtain that the orders P_1 and P_2 are not isomorphic up to duality. Therefore, at least one of them is not isomorphic up to duality to P.

Case 2. $P_{\restriction M_3} \cong P_{\restriction M_1}$. Again by Remark 5.4 (3), we obtain that the orders P_1 and P_2 are not isomorphic up to duality. Therefore, at least one of them is not isomorphic up to duality to P.

Case 3. $P_{\restriction M_3} \cong P_{\restriction M_2}$. Thus, Remark 5.4 (3) implies that the orders P_1 and P_3 are not isomorphic up to duality. Therefore, at least one of them is not isomorphic up to duality to P.

In the three cases, we get a contradiction with the fact that P is reconstructible up to duality by its comparability graph.

(3) First, assume that $P_{\restriction M} \cong P_{\restriction N}$, for any members M, N of $G(P)$. Since a direct sum of isomorphic self-dual orders is self-dual, the order P is self-dual. Hence, $P \in \mathcal{H}_0$ by Lemma 5.1.

Second, assume that there are two members M_1, M_2 of $G(P)$ such that the suborders $P_{\restriction M_1}$ and $P_{\restriction M_2}$ are non-isomorphic. The second assertion implies that $| G(P) |= 2$, and hence, the order P is the direct sum of two non-isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$. Thus, $P \in \mathcal{H}_1$ by Lemma 5.7. □

Lemma 5.9 *Given an order P on a set V, with $|V| \geq 2$, such that the frame of P is a chain, the following assertions hold.*

(1) *$P \in \mathcal{H}_0$ if and only if P is the direct sum of at least two isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$.*

(2) *$P \in \mathcal{H}_1$ if and only if the order P is the direct sum of two non-isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$.*

Proof (1) First, we prove that if $P \in \mathcal{H}_0$, then P is the direct sum of at least two isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$. To the contrary, suppose that $P \in \mathcal{H}_0$ and $G(P)$ has two non-isomorphic members. Then there are two consecutive vertices M and N of the chain $P/G(P)$ such that the orders $P_{\restriction M}$ and $P_{\restriction N}$ are not isomorphic. Thus, $M \cup N$ is a non-self-dual module of P; which contradicts the fact that $P \in \mathcal{H}_0$.

Second, assume that P is the direct sum of at least two isomorphic members of $\mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$. Thus, P has a modular partition $\mathcal{Q} = \{V_1, ..., V_k\}$, with $k \geq 2$, such that $P_{\restriction V_1} \in \mathcal{H}_{0,p} \cup \mathcal{H}_{0,a}$, $P_{\restriction V_i} \cong P_{\restriction V_1}$ for each $i \in \{2, ..., k\}$, and the quotient $P/\mathcal{Q}$ is the chain: $V_1 < \cdots < V_k$. Since $P_{\restriction V_i} \cong P_{\restriction V_1}$ for each $i \in \{2, ..., k\}$, using Remark 3.5, we see that $\mathcal{Q}$ is the largest modular partition of P for which the corresponding quotient is a chain. Therefore, Remark 3.4 (5) implies that $\mathcal{Q} = G(P)$.

Since $P_{\restriction V_i} \cong P_{\restriction V_1}$ for each $i \in \{2, ..., k\}$, the order P is self-dual. Now, consider a proper module M of P. By Remark 3.4 (3), either M is a module of some $P_{\restriction V_i}$ or there are $1 \leq i_1 \leq i_2 \leq k$ such that $M = \bigcup\limits_{i_1 \leq i \leq i_2} V_i$. In both cases, the order $P_{\restriction M}$ is self-dual.

Therefore, every module of P is self-dual, and hence, $P \in \mathcal{H}_0$.

(2) Since the frame of P is a chain, this second assertion follows immediately from Lemma 5.7. □

We are now ready to present a proof of the main result.

Proof of Theorem 2.5 The first assertion is the first one of Corollary 5.2. The second assertion is an immediate consequence of Lemma 5.7.

We now proceed to prove the last assertion. Let k be an integer with $k \geq 2$.

Denote by $\mathcal{C}_1$ the set of orders obtained from some good self-dual order by dilating a good vertex by a member of $\mathcal{H}_{k-1}$.

For each integer q with $2 \leq q \leq log_2(k) + 1$, denote by $\mathcal{C}_q$ the set of orders which are disjoint sums of $q - 1$ members of $\mathcal{H}_{0,c} \cup \mathcal{H}_{0,p}$ and one connected member of $\mathcal{H}_{k-2^{q-1}+1}$.

We will show that $\mathcal{H}_k = \bigcup\limits_{1 \leq q \leq log_2(k)+1} \mathcal{C}_q$.

First, we consider a member P of $\mathcal{C}_1$, and let P_1 be a good self-dual order, x be a good vertex of P_1, and f_x be a good isomorphism associated with x, and assume that P is obtained from P_1 by dilating the vertex x by a member R of $\mathcal{H}_{k-1}$. Put $Q = (G(P_1) \setminus \{\{x\}\}) \cup \{V(R)\}$, where $V(R)$ denotes the set of vertices of R. Clearly, Q is a modular partition of the order P for which the quotient $P/\ Q$ is prime because it is isomorphic to the frame $P_1/G(P_1)$ of P_1. Thus, by Remark 3.4 (2), $Q = G(P)$. Let P' be an order on the set $V(P)$ of vertices of P such that $Comp(P') = Comp(P)$. Then, by Corollary 3.3, $G(P') = G(P)$, and either $P'/G(P) = P/G(P)$ or $P'/G(P) = P^\star/G(P)$. For every element M of $G(P_1) \setminus \{\{x\}\} = G(P) \setminus \{V(R)\}$, the orders $P'_{\restriction M}$, $P_{\restriction M}$, $P^\star_{\restriction M}$, $P^\star_{\restriction f_x(M)}$ and $P'_{\restriction f_x(M)}$ are isomorphic because $P_{\restriction M} \in \mathcal{H}_0$. Moreover, since $R \in \mathcal{H}_{k-1}$, R is reconstructible up to duality by its comparability graph and in particular, $P'_{\restriction V(R)}$ is isomorphic up to duality to R. By interchanging P' and $(P')^\star$, if necessary, we may assume that $P'_{\restriction V(R)}$

is isomorphic to R. Let φ be an isomorphism from $P_{\restriction V(R)}$ onto $P'_{\restriction V(R)}$. Clearly, the orders P' and P are isomorphic when $P'/G(P) = P/G(P)$. We now assume that $P'/G(P) = P^{\star}/G(P)$. For each element M of $G(P) \setminus \{V(R)\} = G(P_1) \setminus \{\{x\}\}$, we consider an isomorphism φ_M from $P_{\restriction M}$ onto $P^{\star}_{\restriction f_x(M)}$. We define a permutation g of $V(P)$ as follows: $g(t) = \varphi(t)$ if $t \in V(R)$, and $g(t) = \varphi_M(t)$ if $t \in V(P_1) \setminus \{x\}$, where M is the element of $G(P_1)$ containing t. It is easy to verify that g is an isomorphism from P onto P'. Hence, P is reconstructible up to duality by its comparability graph. Moreover, the second assertion of Corollary 5.2 implies that $V(R)$ is a non-self-dual module of P because $R \in \mathcal{H}_{k-1}$ and $k \geq 2$. It follows that $P \notin \mathcal{H}_0$, and hence, Lemma 5.1 implies that P is not self-dual. By Remark 3.4 (6), the set of non-self-dual modules of P is the union of $\{V(P)\}$ and the set of non-self-dual modules of R. Since $R \in \mathcal{H}_{k-1}$, P has exactly k non-self-dual modules. So $P \in \mathcal{H}_k$, and hence, $\mathcal{C}_1 \subseteq \mathcal{H}_k$.

Next, we consider an integer q with $2 \leq q \leq log_2(k) + 1$, and a member P of $\mathcal{C}_q$, which is a disjoint sum of q orders $R_1, R_2, ..., R_q$, where R_1 is a connected member of $\mathcal{H}_{k-2^{q-1}+1}$, and for each $2 \leq i \leq q$, R_i is a member of $\mathcal{H}_{0,c} \cup \mathcal{H}_{0,p}$.

By Remark 3.4 (7), the orders $R_2, ..., R_q$ are connected, and $G(P) = \{V(R_i) : 1 \leq i \leq q\}$, where $V(R_i)$ is the set of vertices of R_i. By Remark 3.4 (4), the set of non-self-dual modules of P is the union of the set of the unions $\underset{i \in A}{\cup} V(R_i)$, where A is a subset of $\{1, ..., q\}$ containing 1 with $|A| \geq 2$, and the set of non-self-dual modules of R_1. Thus, P has exactly $(k - 2^{q-1} + 1) + (-1 + 2^{q-1}) = k$ non-self-dual modules. On the other hand, given an order P' on $V(P)$ such that $Comp(P') = Comp(P)$, it follows from Corollary 3.3 (1) that $G(P') = G(P) = \{V(R_i) : 1 \leq i \leq q\}$. Since R_1 is reconstructible up to duality by its comparability graph and $R_i \in \mathcal{H}_0$ for each $2 \leq i \leq q$, $P'_{\restriction V(R_1)}$ is isomorphic up to duality to $P_{\restriction V(R_1)} = R_1$, and $P'_{\restriction V(R_i)} \cong P_{\restriction V(R_i)}$ for each $2 \leq i \leq q$. Thus, $P' \cong P$ when $P'_{\restriction V(R_1)} \cong P_{\restriction V(R_1)}$, and $P' \cong P^{\star}$ otherwise. Thus, P' is isomorphic up to duality to P. Hence, P is reconstructible up to duality by its comparability graph. Since P has exactly k non-self-dual modules, $P \in \mathcal{H}_k$. Therefore, $\underset{2 \leq q \leq log_2(k)+1}{\cup} \mathcal{C}_q \subseteq \mathcal{H}_k$.

Conversely, we now prove that $\mathcal{H}_k \subseteq \underset{1 \leq q \leq log_2(k)+1}{\cup} \mathcal{C}_q$. Let P be a member of $\mathcal{H}_k$. By Corollary 5.2 (2), the order P is not self-dual. By Theorem 1.2 and Lemma 5.8, we discuss the following two cases according to the frame Q of P. The frame Q of P is a prime order or an antichain.

First, we assume that the frame $Q = P/G(P)$ is a prime order. By Lemma 5.6, there is at most one non-self-dual member of $G(P)$. If $M \in G(P)$ is self-dual, then by Lemma 5.5 and Lemma 5.1, $P_{\restriction M} \in \mathcal{H}_0$. So if $G(P)$ has no non-self-dual member, then $V(P)$ is the unique non-self-dual module of P; which contradicts the fact that $P \in \mathcal{H}_k$ and $k \geq 2$. So, $G(P)$ has a unique non-self-dual member X. Let $x \in X$, and let P_1 be the order $P_{\restriction((V(P)\setminus X)\cup\{x\})}$. Clearly, $G(P_1) = (G(P) \setminus \{X\}) \cup \{\{x\}\}$, and the frame $P_1/G(P_1)$ of P_1 is a prime order because it is isomorphic to Q. Thus, P_1 is an order with a prime frame such that $\{x\} \in G(P_1)$ and the order $P_{\restriction M}$ is a member of $\mathcal{H}_0$, for each $M \in G(P_1) \setminus \{\{x\}\}$. We consider the order $P_2 = Inv(X, P)$. By Theorem 1.3, $Comp(P_2) = Comp(P)$. Thus, P_2 is isomorphic up to duality to P because P is reconstructible up to duality by its comparability graph. On the other hand, X is the unique non-self-dual member of $G(P_2) = G(P)$, and $P_{2_{\restriction X}} \ncong P_{\restriction X}$ because

$P_{\restriction X}$ is non-self-dual. It follows that $P_2 \ncong P$, and hence, there is an isomorphism g from P onto $P_2^\star$ such that $g(X) = X$. Moreover, for each $M \in G(P) \setminus \{X\}$, the orders $P_{\restriction M}$, $P^\star_{\restriction M}$, $P_{2_{\restriction g(M)}}$, and $P^\star_{2_{\restriction g(M)}}$ are isomorphic because $P_{\restriction M} \in \mathcal{H}_0$. Clearly, g induces an isomorphism f from the frame $Q_1 = P_1/G(P_1)$ of P_1 onto $Q_1^\star$ such that $f(\{x\}) = \{x\}$, and for every $M \in G(P_1) \setminus \{\{x\}\}$, the orders $P_{1_{\restriction M}}$ and $P_{1_{\restriction f(M)}}$ are two isomorphic members of $\mathcal{H}_0$ because $P_{1_{\restriction M}} = P_{\restriction M}$ and $P_{1_{\restriction f(M)}} = P_{2_{\restriction g(M)}}$. Thus, the order P_1 is a good self-dual order, and P is obtained from P_1 by dilating its good vertex x by $P_{\restriction X}$. On the other hand, $V(P)$ is a non-self-dual module of P, and for every $M \in G(P) \setminus \{X\}$, $P_{\restriction M} \in \mathcal{H}_0$. Thus, the non-self-dual modules of P are $V(P)$ and the non-self-dual modules of $P_{\restriction X}$. Moreover, $P \in \mathcal{H}_k$, hence $P_{\restriction X} \in \mathcal{H}_{k-1}$. Therefore, the order P is obtained from a good self-dual order by dilating a good vertex x by a member of $\mathcal{H}_{k-1}$ when the frame $Q = P/G(P)$ of P is a prime order.

Next, we assume that the frame Q is an antichain. By Lemma 5.6, there is at most one non-self-dual member of $G(P)$. The module $V(P)$ is a non-self-dual module of P, and a disjoint sum of self-dual orders is a self-dual order. Hence, $G(P)$ has exactly one element X such that $P_{\restriction X}$ is not self-dual.

Let $G(P) = \{X_1, ..., X_q\}$, where $X_1 = X$. By Remark 3.4 (4), the modules of P are the modules of the P_{X_i}'s and the unions of some X_i's. By Lemma 5.1 and Lemma 5.5, for each $i \in \{2, ..., q\}$, $P_{\restriction X_i} \in \mathcal{H}_0$. Since a disjoint sum of a finite set of orders, which contains exactly one non-self-dual member, is a non-self-dual order, the non-self-dual modules of P are those of $P_{\restriction X}$ and the unions $\underset{i \in A}{\cup} X_i$, where A is a subset of $\{1, ..., q\}$ containing 1 with $| A | \geq 2$. Hence, the number of non-self-dual modules of P is $n_1 - 1 + 2^{q-1}$, where n_1 is the number of non-self-dual modules of the order $P_{\restriction X}$. Thus, $n_1 - 1 + 2^{q-1} = k$, and hence, $P_{\restriction X} \in \mathcal{H}_{k-2^{q-1}+1}$, because it follows from Lemma 5.5 that $P_{\restriction X}$ is reconstructible up to duality by its comparability graph. Moreover, $n_1 \geq 1$, $q \geq 2$, and $n_1 - 1 + 2^{q-1} = k$ imply that $2 \leq q \leq log_2(k) + 1$. By Remark 3.5, for $i \in \{2, ..., q\}$, the fact that $P_{\restriction X_i} \in \mathcal{H}_0$ implies that $P_{\restriction X_i} \in \mathcal{H}_{0,c} \cup \mathcal{H}_{0,p}$. On the other hand, the order $P_{\restriction X_1}$ is connected by Remarks 3.4 and 3.5 . Therefore, the order P is a disjoint sum of $q - 1$ members of $\mathcal{H}_{0,c} \cup \mathcal{H}_{0,p}$ and one connected member of $\mathcal{H}_{k-2^{q-1}+1}$ when the frame of P is an antichain. $\square$

6 Relation With the Reconstruction Problems of Fraïssé and Hagendorf

Recall the following notions used in reconstruction.

Let P be an order on a set V.

An order is *hemimorphic* to P if it is isomorphic to P or to its dual $P^\star$.

For an order P' on V and a positive integer k, P' is $(\leq k)$-hemimorphic (respectively, $(\leq k)$-*isomorphic*) to P if for each subset X of V with at most k elements, the induced orders $P_{\restriction X}$ and $P'_{\restriction X}$ are hemimorphic (respectively, isomorphic). The order P is $(\leq k)$-*half-reconstructible* (respectively, $(\leq k)$-*reconstructible*) if each order which is $(\leq k)$-hemimorphic (respectively, $(\leq k)$-*isomorphic*) to P is hemimorphic (respectively, isomorphic) to P.

The problem of $(\leq k)$-reconstruction (respectively, $(\leq k)$-half-reconstruction) of binary relations was introduced by Fraïssé [16] (respectively, Hagendorf [20]). For the studies on these two problems, see [1,4–6,15,19,20,22].

Clearly, two orders on the same set of vertices are (≤ 2)-hemimorphic if and only if they are (≤ 2)-isomorphic if and only if they have the same comparability graph. Hence, an order P is reconstructible (respectively, reconstructible up to duality) by its comparability graph if and only if P is (≤ 2)-reconstructible (respectively, (≤ 2)-half-reconstructible).

Therefore, Proposition 2.4 characterizes the (≤ 2)-reconstructible orders, and Theorem 2.5 describes the (≤ 2)-half-reconstructible orders.

7 An Open Problem

Consider the following problem which is proposed by one of the referees.

Problem 7.1 Characterize the comparability graphs G with the property that all the transitive orientations of G are isomorphic up to duality.

For the study of this problem, the following lemma can be used.

Lemma 7.2 *Given a comparability graph G, the following assertions are equivalent.*

(1) *All the transitive orientations of G are isomorphic up to duality.*
(2) *Each transitive orientation of G is an element of the union $\underset{n\geq 0}{\cup} \mathcal{H}_n$.*
(3) *There is a transitive orientation P of G such that $P \in \underset{n\geq 0}{\cup} \mathcal{H}_n$.*

Proof (1) $\Rightarrow$ (2). Assume that all the transitive orientations of G are isomorphic up to duality, and consider a transitive orientation P of G. Thus, $Comp(P) = G$, and each order $P^{'}$ on $V(P)$ which has the same comparability graph as P is isomorphic up to duality to P because $P^{'}$ is a transitive orientation of G. Therefore, the order P is reconstructible up to duality by its comparability graph, and hence, $P \in \underset{n\geq 0}{\cup} \mathcal{H}_n$.

(2) $\Rightarrow$ (3). It is immediate.

(3) $\Rightarrow$ (1). Assume that there is a transitive orientation P of G such that $P \in \underset{n\geq 0}{\cup} \mathcal{H}_n$, and consider a transitive orientation $P^{'}$ of G. Thus, $Comp(P) = Comp(P^{'}) = G$. On the other hand, since $P \in \underset{n\geq 0}{\cup} \mathcal{H}_n$, the order P is reconstructible up to duality by its comparability graph. It follows that $P^{'}$ is isomorphic up to duality to P. Therefore, all the transitive orientations of G are isomorphic up to duality. □

By Lemma 7.2, the solutions of Problem 7.1 are the comparability graphs of the elements of the union $\underset{n\geq 0}{\cup} \mathcal{H}_n$. On the other hand, Theorem 2.5 does not give a simple description of these graphs. Thus, the description of the solutions of Problem 7.1 remains an open problem.

Acknowledgements The authors are pleased to thank the anonymous referees for their careful reading and helpful suggestions. The authors would like to extend their sincere appreciations to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project No. RGP-056.

References

1. Alzohairi, M., Bouaziz, M., Boudabbous, Y.: Orders and (≤ 4)-Hemimorphy. J. Mult.-Valued Logic Soft Comput. **21**(3–4), 355–371 (2013)
2. Bondy JA.: A graph reconstructor's manual. Surveys in Combinatorics, Keendwell éd., London. Math. Soc. Lecture Note Ser. Cambridge Univ. Press, pp. 221-252 (1991)
3. Bondy, J.A., Hemminger, R.L.: Graph reconstruction, a survey. J. Graph Theory **1**, 227–268 (1997)
4. Bouaziz, M., Boudabbous, Y., El Amri, N.: Hereditarily hemimorphy of $\{-k\}$-hemimorphic tournaments for $k \geq 5$. J. Korean Math. Soc. **48**(3), 599–626 (2011)
5. Boudabbous, Y., Boussaïri, A., Chaïchaâ, A., El Amri, N.: Self-duality of modules and reconstruction of tournaments up to duality. J. Mult.-Valued Logic Soft Comput. **27**, 501–530 (2016)
6. Boudabbous, Y., Lopez, G.: The minimal non-($\leq k$)-reconstructible relations. Discrete Math. **291**, 19–40 (2005)
7. Boudabbous, Y., Si Kaddour, H.: $\{-1, 2\}$-hypomorphy and hereditarily hypomorphy coincide for posets. Contrib. Discrete Math. **4**, 12–20 (2009)
8. Boussaïri, A., Ille, P., Lopez, G., Thomassé, S.: The C_3-structure of tournaments. Discrete Math. **277**, 29–43 (2004)
9. Dammak, J.: Isomorphy and dilatation in digraphs. Adv. Pure Appl. Math. **5**(3), 161–171 (2014)
10. Dammak, J., Lopez, G., Pouzet, M., Si Kaddour, H.: Hypomorphy of graphs up to complementation. J. Combin. Theory Ser. B **99**(1), 84–96 (2009)
11. Dammak, J., Lopez, G., Pouzet, M., Si Kaddour, H.: Boolean sum of graphs and reconstruction up to complementation. Adv. Pure Appl. Math. **4**(3), 315–349 (2013)
12. Das, S.: On the structure of finite $T_0 + T_5$ spaces. Can. J. Math. **XXV6**, 1148–1158 (1973)
13. Ehrenfeucht, A., Rozenberg, G.: Primitivity is hereditary for 2-structures. Theor. Comput. Sci. **3**(70), 343–358 (1990)
14. Ehrenfeucht, A., Harju, T., Rozenberg, G.: The Theory of 2-Structures. A Framework for Decomposition and Transformation of Graphs. World Scientific, Singapore (1999)
15. El Amri, N.: La ($\leq k$)-demi-reconstructibilité des graphes pour $7 \leq k \leq 12$. Ars Comb. **122**, 193–226 (2015)
16. Fraïssé, R.: Abritement entre relations et spécialement entre chaînes. Symposia Mathematica **5**, 203–251 (1970)
17. Fraïssé R.: L'intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d'une relation. M.Pouzet and D. Richard eds., Order, Description and RolesNorth-Holland Amsterdam, pp. 313-342 (1984)
18. Gallai, T.: Transitiv orientierbare Graphen. Acta Math. Acad. Sci. Hungar. **18**, 25–66 (1967)
19. Hagendorf, J.G.: Restriction respectueuse et reconstruction des chaînes et des relations infinies. Z. Math. Logik. Grundlagen Math. **38**, 457–490 (1992)
20. Hagendorf, J.G., Lopez, G.: La demi-reconstructibilité des relations binaires d'au moins 13 éléments. C. R. Acad. Sci. Paris, Série I **317**, 7–12 (1993)
21. Ille, P., Rampon, J.X.: Reconstruction of posets with the same comparability graph. J. Comb. Theory Ser. B **74**, 368–377 (1998)
22. Lopez, G.: L'indéformabilité des relations et multirelations binaires. Zeitschr. f. Math. Logik und Grundlagen Math. **24**, 303–317 (1978)
23. Maffray F, Preissmann M.: A translation of Tibor Gallai's paper: Transitiv orientierbare Graphen, In: M. Filder (ed.) Theory of Graphs and its Applications Prague, pp 47-52 (1964)
24. Pouzet, M., Si Kaddour, H.: Isomorphy up to complementation. J. Comb. **7**(2–3), 285–305 (2016)
25. Pouzet, M., Si Kaddour, H., Trotignon, N.: Claw-freeness, 3-homogeneous subsets of a graph and a reconstruction problem. Contrib. Discrete Math. **6**(1), 86–97 (2011)
26. Spinrad, J.: P4-trees and substitution decomposition. Discrete Appl. Math. **39**, 263–291 (1992)
27. Ulam, S.M.: A Collection of Mathematical Problems. Intersciences Publishers, New York (1960)