

1. Find the Fourier integral for the following functions:

$$\text{a) } f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$$

$$\text{b) } g(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

$$\text{c) } h(x) = e^{-|x|} \cos x,$$

$$\text{d) } k(x) = e^{-|x|} \sin x$$

$$\text{e) } M(x) = \begin{cases} 0, & -\infty < x < -1 \\ 2x, & -1 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$$

$$\text{f) } N(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$$

DEFINITION 14.3.1 Fourier Integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] d\alpha, \quad (4)$$

where $A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx$ (5)

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x dx. \quad (6)$$

CONVERGENCE OF A FOURIER INTEGRAL Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but slightly more restrictive than, the conditions for a Fourier series.

THEOREM 14.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval and let f be absolutely integrable on $(-\infty, \infty)$.^{*} Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

1(a):

5. From formula (5) in the text,

$$A(\alpha) = \int_0^{\infty} e^{-x} \cos \alpha x \, dx.$$

Recall $\mathcal{L}\{\cos kt\} = s/(s^2 + k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{1}{1 + \alpha^2}.$$

Now

$$B(\alpha) = \int_0^{\infty} e^{-x} \sin \alpha x \, dx.$$

Recall $\mathcal{L}\{\sin kt\} = k/(s^2 + k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{\alpha}{1 + \alpha^2}.$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha.$$

b) Find the Fourier cosine of the functions:

i) $f(x) = e^{-x} \cos x, x > 0,$ ii) $g(x) = xe^{-2x},$

ii) $h(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}.$

DEFINITION 14.3.2 Fourier Cosine and Sine Integrals

(i) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the cosine integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \cos \alpha x \, d\alpha, \quad (8)$$

where
$$A(\alpha) = \int_0^{\infty} f(x) \cos \alpha x \, dx. \quad (9)$$

(ii) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the sine integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} B(\alpha) \sin \alpha x \, d\alpha, \quad (10)$$

where
$$B(\alpha) = \int_0^{\infty} f(x) \sin \alpha x \, dx. \quad (11)$$

b(ii):

5. For the cosine integral,

$$A(\alpha) = \int_0^{\infty} x e^{-2x} \cos \alpha x \, dx.$$

But we know

$$\mathcal{L}\{t \cos kt\} = -\frac{d}{ds} \frac{s}{(s^2 + k^2)} = \frac{(s^2 - k^2)}{(s^2 + k^2)^2}.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{4 - \alpha^2}{(4 + \alpha^2)^2}.$$

Hence

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{(4 - \alpha^2) \cos \alpha x}{(4 + \alpha^2)^2} \, d\alpha.$$

For the sine integral,

$$B(\alpha) = \int_0^{\infty} x e^{-2x} \sin \alpha x \, dx.$$

From Problem 12, we know

$$\mathcal{L}\{t \sin kt\} = \frac{2ks}{(s^2 + k^2)^2}.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{4\alpha}{(4 + \alpha^2)^2}.$$

Hence

$$f(x) = \frac{8}{\pi} \int_0^{\infty} \frac{\alpha \sin \alpha x}{(4 + \alpha^2)^2} \, d\alpha.$$

4. Use Fourier integral to show that

$$\int_0^{\infty} \frac{\sin \pi \lambda}{1 - \lambda^2} \sin x \lambda d\lambda = \begin{cases} \frac{\pi}{2} \sin x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases} \quad (\text{a})$$

$$\int_0^{\infty} \frac{\cos \lambda x}{1 + \lambda^2} d\lambda = \frac{\pi}{2} e^{-x}, \quad x \geq 0. \quad (\text{b})$$

$$\int_0^{\infty} \frac{\lambda \sin \lambda x}{\beta^2 + \lambda^2} d\lambda = \frac{\pi}{2} e^{-\beta x}, \quad x > 0, \beta > 0. \quad (\text{c})$$
