5) Show that the following set of functions are orthogonal on the given intervals and find the norm of each function .

a) $\left\{\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \sin\frac{4\pi x}{L}, \sin\frac{5\pi x}{L}, ...\right\}; [0, L]$

b) $\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\right\}; m, n = 1, 2, 3, 4...; [0, L]$

c) $\{\sin(2n+1)\}; n = 0, 1, 2, 3, 4...; [0, \frac{\pi}{2}]$.

5(a):

For $m \neq n$

$$\int_0^p \sin\frac{n\pi}{p}x \sin\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^p \left(\cos\frac{(n-m)\pi}{p}x - \cos\frac{(n+m)\pi}{p}x\right)dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\,\Big|_0^p - \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\,\Big|_0^p$$

$$= 0.$$

For $m = n$

$$\int_0^p \sin^2\frac{n\pi}{p}x\,dx = \int_0^p \left[\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right]dx = \frac{1}{2}x\,\Big|_0^p - \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\,\Big|_0^p = \frac{p}{2}$$

so that

$$\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}.$$

3. a) Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi. \end{cases}$$

b) Use a) to show that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12}$$

c) Use b) to obtain a numerical series which represents the value $\frac{\pi^2}{8}$.

DEFINITION 11.2.1 Fourier Series

The Fourier series of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right), \tag{8}$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx \tag{9}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx \tag{10}$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx. \tag{11}$$

CONVERGENCE OF A FOURIER SERIES The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

THEOREM 11.2.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.*

For a proof of this theorem you are referred to the classic text by Churchill and Brown.†

*In other words, for x a point in the interval and $h > 0$,

$$f(x+) = \lim_{h \to 0} f(x + h), \quad f(x-) = \lim_{h \to 0} f(x - h).$$

†Ruel V. Churchill and James Ward Brown, *Fourier Series and Boundary Value Problems* (New York: McGraw-Hill).

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left(\frac{x^2}{\pi}\sin nx\,\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x^2}{n}\cos nx\,\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}[(-1)^n - 1]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin nx\right]$$

EXAMPLE 2 Convergence of a Point of Discontinuity

The function (12) in Example 1 satisfies the conditions of Theorem 11.2.1. Thus for every x in the interval $(-\pi, \pi)$, except at $x = 0$, the series (13) will converge to $f(x)$. At $x = 0$ the function is discontinuous, so the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$

PERIODIC EXTENSION Observe that each of the functions in the basic set (1) has a different fundamental period*—namely, $2p/n$, $n \geq 1$—but since a positive integer multiple of a period is also a period, we see that all of the functions have in common the period $2p$. (Verify.) Hence the right-hand side of (2) is $2p$-periodic; indeed, $2p$ is the **fundamental period** of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$, but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 11.2.1 to the periodic extension of f, or we may assume from the outset that the given function is periodic with period $2p$; that is, $f(x + 2p) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average

$$\frac{f(p-) + f(-p+)}{2}$$

at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on.

The Fourier series in (13) converges to the periodic extension of (12) on the entire x-axis. At $0, \pm 2\pi, \pm 4\pi, \ldots$ and at $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the series converges to the values

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi-) + f(-\pi+)}{2} = 0,$$

respectively. The solid dots in Figure 11.2.2 represent the value $\pi/2$.

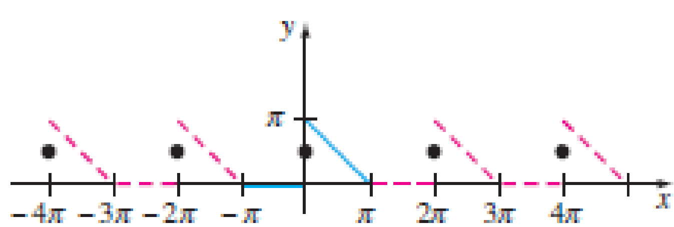

FIGURE 11.2.2 Periodic extension of function shown in Figure 11.2.1

The function in Problem 5 is discontinuous at $x = \pi$, so the corresponding Fourier series converges

to $\pi^2/2$ at $x = \pi$. That is,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin n\pi\right]$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

and

$$\frac{\pi^2}{6} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

At $x = 0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

From Problem 17

$$\frac{\pi^2}{8} = \frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{1}{2}\left(2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

$n=0$

4. Test whether each of the following given functions is odd or even, then expand it in a cosine or sine series

a) $f(x) = |\cos x|, \quad |x| < \pi$

b) $g(x) = x\cos x, \quad |x| < \pi$

c) $$h(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2. \end{cases}$$

d) $k(x) = x^2\,|x|, \quad |x| < 1$

e) $$M(x) = \begin{cases} x+1, & -3 < x < 0 \\ -x+1, & 0 \le x < 3 \end{cases}.$$

THEOREM 11.3.1 Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

DEFINITION 11.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function on the interval $(-p, p)$ is the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \tag{1}$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \tag{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx. \tag{3}$$

(*ii*) The Fourier series of an odd function on the interval $(-p, p)$ is the sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x, \tag{4}$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx. \tag{5}$$

4(c):

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x \cos \frac{n\pi}{2} x\,dx = \frac{4}{n^2\pi^2}\left(\cos \frac{n\pi}{2} - 1\right)$$

$$b_n = \int_0^1 x \sin \frac{n\pi}{2} x\,dx + \int_1^2 1 \cdot \sin \frac{n\pi}{2} x\,dx = \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos \frac{n\pi}{2} - 1\right) \cos \frac{n\pi}{2} x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}\right] \sin \frac{n\pi}{2} x$$

5. Find the half range sine series for the functions

a) $f(x) = \begin{cases} x, & 0 \le x \le \pi l2 \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$

b) $g(x) = x - x^2, \ 0 < x < 1$

c) $h(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2. \end{cases}$

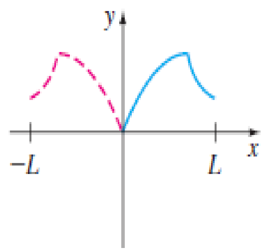

FIGURE 11.3.7 Even reflection

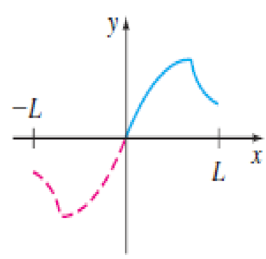

FIGURE 11.3.8 Odd reflection

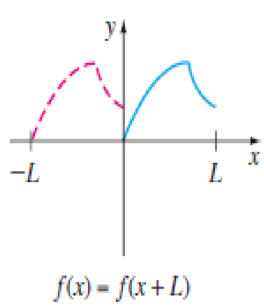

FIGURE 11.3.9 Identity reflection

HALF-RANGE EXPANSIONS Throughout the preceding discussion it was understood that a function f was defined on an interval with the origin as its midpoint—that is, $(-p, p)$. However, in many instances we are interested in representing a function that is defined only for $0 < x < L$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary *definition* of $f(x)$ for $-L < x < 0$. For brevity we consider the three most important cases. If $y = f(x)$ is defined on the interval $(0, L)$, then

(*i*) reflect the graph of f about the y-axis onto $(-L, 0)$; the function is now even on $(-L, L)$ (see Figure 11.3.7); or

(*ii*) reflect the graph of f through the origin onto $(-L, 0)$; the function is now odd on $(-L, L)$ (see Figure 11.3.8); or

(*iii*) define f on $(-L, 0)$ by $y = f(x + L)$ (see Figure 11.3.9).

Note that the coefficients of the series (1) and (4) utilize only the definition of the function on $(0, p)$ (that is, half of the interval $(-p, p)$). Hence in practice there is no actual need to make the reflections described in (*i*) and (*ii*). If f is defined for $0 < x < L$, we simply identify the half-period as the length of the interval $p = L$. The coefficient formulas (2), (3), and (5) and the corresponding series yield either an even or an odd periodic extension of period $2L$ of the original function. The cosine and sine series that are obtained in this manner are known as **half-range expansions.** Finally, in case (*iii*) we are defining the function values on the interval $(-L, 0)$ to be same as the values on $(0, L)$. As in the previous two cases there is no real need to do this. It can be shown that the set of functions in (1) of Section 11.2 is orthogonal on the interval $[a, a + 2p]$ for any real number a. Choosing $a = -p$, we obtain the limits of integration in (9), (10), and (11) of that section. But for $a = 0$ the limits of integration are from $x = 0$ to $x = 2p$. Thus if f is defined on the interval $(0, L)$, we identify $2p = L$ or $p = L/2$. The resulting Fourier series will give the periodic extension of f with period L. In this manner the values to which the series converges will be the same on $(-L, 0)$ as on $(0, L)$.

5(a):

$$a_0 = \frac{2}{\pi}\left(\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right) = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right) = \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)$$

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right) = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)\cos nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\,\sin nx$$

5(c):

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x \cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$b_n = \int_0^1 x \sin\frac{n\pi}{2}x\,dx + \int_1^2 1 \cdot \sin\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}\right]\sin\frac{n\pi}{2}x$$

6. Find the half range cosine series for the functions

a) $f(x) = \frac{1}{3}x(1-x)$, $0 < x < 1$

b) $g(x) = \begin{cases} -x, & 0 < x \leq \pi l2 \\ \pi - x, & \pi/2 \leq x < \pi. \end{cases}$

c) $h(x) = 1 - \cos x$, $0 < x < \pi$.

Exercises

1. Find the complex form of the Fourier series for:

a) $f(x) = e^x$, $-\pi < x < \pi$

b) $g(x) = e^{-x}$, $-1 < x < 1$