M-203

Home Assignment #1 (II-Sem. 1440/1441)

Solution

Max. Marks: 25

Q#1) Determine whether the sequence $\left\{\frac{(n+1)!}{n!-(n+1)!}\right\}_{n=1}^{\infty}$ converges or diverges and, if it converges, find the limit.

[Marks: 5]

Soln. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!-(n+1)!} = \lim\limits_{n\to\infty} \frac{1}{\frac{n!}{(n+1)!}-1}$ 3

$= \lim\limits_{n\to\infty} \frac{1}{\frac{1}{1+n}-1} = -1$ 2

Q#2) Find the sum of the following series: $\sum\limits_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$

[Marks: 5]

Soln. $\frac{2n+1}{(n^2+n)^2} = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ 2

$$\left.\begin{aligned} a_1 &= 1 - \frac{1}{2^2} \\ a_2 &= \frac{1}{2^2} - \frac{1}{3^2} \\ &\vdots \\ a_n &= \frac{1}{n^2} - \frac{1}{(n+1)^2} \end{aligned}\right\} \Rightarrow S_n = 1 - \frac{1}{(n+1)^2} \Rightarrow \lim_{n} S_n = 1$$

2 1

Q#3) Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n+1}}$ is absolutely convergent or conditionally convergent or divergent.

[Marks: 5]

Soln. Take $a_n = \frac{1}{\sqrt{2n+1}}$ and we see that $\lim\limits_{n\to\infty} a_n = 0$ and

a_n is decreasing because: $a' = -\frac{1}{(2n+1)^{3/2}} < 0$

Continue.....

Hence by AST, it is convergent. On the other hand the series $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$ diverges by LCT 2

with the divergent series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. 2

Hence the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{\sqrt{2n+1}}$ is conditionally convergent. 2

Q#4) Find the radius of convergence and the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n 5^n} (x-5)^n$

[Marks: 5]

Soln. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n \cdot 5^n}{(-1)^n (x-5)^n} \right|$

$$= \left|\frac{x-5}{5}\right|$$ 2

For absolute convergence $\left|\frac{x-5}{5}\right| < 1 \Leftrightarrow -5 < x-5 < 5$

$\Leftrightarrow 0 < x < 10$ 9

At $x=0$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n 5^n} = \sum \frac{1}{n}$

which is divergent 1

At $x = 10$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent by AST. 1 Hence interval of conv. $(0, 10]$

Radius of convergence $r = \frac{10}{2} = 5$. 1

continue...

Q#5) Find the Maclaurin Series of $\tan^{-1}x$ and use its first two non-zero terms to approximate the integral $\int_0^{0.5} x\tan^{-1}x\,dx$ and estimate the error of approximation.

[Marks: 5]

Soln. We know that $\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$ if $|t|<1$ (1)

$$= \int_0^x 1 - t^2 + (t^2)^2 - \cdots + (-1)^n (t^2)^n \,dt$$

$\vdots$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

(1)

Now, $\int_0^{0.5} x\tan^{-1}x\,dx = \int_0^{0.5} \left(x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots\right)dx$

$$= \left[\frac{x^3}{3} - \frac{x^5}{3(5)} + \frac{x^7}{7(5)} \cdots\right]_0^{0.5}$$

$$= \frac{1}{3(2)^3} - \frac{1}{3(5)\cdot(2)^5} \approx 0.00395$$

(2)

$$\text{Error} \le \left|\frac{1}{35(2^7)}\right| \approx 0.00024.$$

(1)