COMPARING VARIABLES OF ORDINAL OR DICHOTOMOUS SCALES:

SPEARMAN RANK- ORDER, POINT-BISERIAL, AND BISERIAL CORRELATIONS

OBJECTIVE

In this lecture, you will learn the following items:

- How to compute the Spearman rank-order correlation coefficient.
- How to compute the point-biserial correlation coefficient.

INTRODUCTION

The statistical procedures in this chapter are quite different from those in the last several chapters. Unlike this chapter, we had compared samples of data. This lecture, however, examines the relationship between two variables.

In other words, this lecture will address how one variable changes with respect to another.

The relationship between two variables can be compared with a correlation analysis. If any of the variables are ordinal or dichotomous, we can use a nonparametric correlation.

The Spearman rank-order correlation, also called the Spearman's ρ, is used to compare the relationship between ordinal, or rank-ordered, variables.

The point-biserial and biserial correlations are used to compare the relationship between two variables if one of the variables is dichotomous. The parametric equivalent to these correlations is the Pearson product-moment correlation.

In this lecture, we will describe how to perform and interpret a Spearman rank-order, point-biserial, and biserial correlations.

THE CORRELATION COEFFICIENT

When comparing two variables, we use an obtained value called a correlation coefficient. A population's correlation coefficient is represented by the Greek letter rho, ρ. A sample's correlation coefficient is represented by the letter r.

We will describe two types of relationships between variables. A direct relationship is a positive correlation with an obtained value ranging from 0 to 1.0.

As one variable increases, the other variable also increases. An indirect, or inverse, relationship is a negative correlation with an obtained value ranging from 0 to −1.0. In this case, one variable increases as the other variable decreases.

In general, a significant correlation coefficient also communicates the relative strength of a relationship between the two variables. A value close to 1.0 or −1.0 indicates a nearly perfect relationship, while a value close to 0 indicates an especially weak or trivial relationship. A more detailed description of a correlation coefficient’s relative strength is presented.

Table 1 summarizes his findings.

TABLE 1

Correlation coefficient for a direct relationship	Correlation coefficient for an indirect relationship	Relationship strength of the variables
0.0	0.0	None/trivial
0.1	−0.1	Weak/small
0.3	−0.3	Moderate/medium
0.5	−0.5	Strong/large
1.0	−1.0	Perfect

There are three important caveats to consider when assigning relative strength to correlation coefficients, however.

First, Cohen’s work was largely based on behavioral science research. Therefore, these values may be inappropriate in fields such as engineering or the natural sciences.

Second, the correlation strength assignments vary for different types of statistical tests.

Third, r-values are not based on a linear scale. For example, $r = 0.6$ is not twice as strong as $r = 0.3$.

COMPUTING THE SPEARMAN RANK-ORDER CORRELATION COEFFICIENT

The Spearman rank-order correlation is a statistical procedure that is designed to measure the relationship between two variables on an ordinal scale of measurement if the sample size is $n \geq 4$. Use Formula 1 to determine a Spearman rank-order correlation coefficient r_s if none of the ranked values are ties. Sometimes, the symbol r_s is represented by the Greek symbol rho, or ρ:

$$r_s = 1 - \frac{6 \sum D_i^2}{n(n^2 - 1)} \qquad (1)$$

where n is the number of rank pairs and D_i is the difference between a ranked pair.

If ties are present in the values, use Formula 2, Formula 3, and Formula 4 to determine r_s:

$$r_s = \frac{(n^3 - n) - 6\sum D_i^2 - (T_x + T_y)/2}{\sqrt{(n^3 - n)^2 - (T_x + T_y)(n^3 - n) + T_x T_y}} \qquad (2)$$

where

$$T_x = \sum_{i=1}^{g} (t_i^3 - t_i) \qquad (3)$$

and

$$T_y = \sum_{i=1}^{g} (t_i^3 - t_i) \qquad (4)$$

g is the number of tied groups in that variable and t_i is the number of tied values in a tie group.

If there are no ties in a variable, then T = 0.

Use Formula 5 to determine the degrees of freedom for the correlation:

$$\mathbf{df = n - 2} \tag{5}$$

where n is the number of paired values.

After r_s is determined, it must be examined for significance. Small samples allow one to reference a table of critical values, such as (Table B.7).

However, if the sample size n exceeds those available from the table, then a large sample approximation may be performed.

Example

Spearman Rank-Order Correlation (Small Data Samples without Ties)

Eight men were involved in a study to examine the resting heart rate regarding frequency of visits to the gym. The assumption is that the person who visits the gym more frequently for a workout will have a slower heart rate.

Table 2 shows the number of visits each participant made to the gym during the month the study was conducted. It also provides the mean heart rate measured at the end of the week during the final 3 weeks of the month.

TABLE 2

Participant	Number of visits	Mean heart rate
1	5	100
2	12	89
3	7	78
4	14	66
5	2	77
6	8	103
7	15	67
8	17	63

The values in this study do not possess characteristics of a strong interval scale.

For instance, the number of visits to the gym does not necessarily communicate duration and intensity of physical activity. In addition, heart rate has several factors that can result in differences from one person to another. Ordinal measures offer a clearer relationship to compare these values from one individual to the next.

Therefore, we will convert these values to ranks and use a Spearman rank-order correlation.

1. State the Null and Research Hypothesis

The null hypothesis states that there is no correlation between number of visits to the gym in a month and mean resting heart rate. The research hypothesis states that there is a correlation between the number of visits to the gym and the mean resting heart rate.

The null hypothesis is

$$H_O\colon \rho_s = 0$$

The research hypothesis is

$$H_A\colon \rho_s \neq 0$$

2. Set the Level of Risk (or the Level of Significance) Associated with the Null Hypothesis

The level of risk, also called an alpha (α), is frequently set at 0.05. We will use α = 0.05 in our example. In other words, there is a 95% chance that any observed statistical difference will be real and not due to chance.

3. Choose the Appropriate Test Statistic

As stated earlier, we decided to analyze the variables using an ordinal, or rank, procedure. Therefore, we will convert the values in each variable to ordinal data. In addition, we will be comparing the two variables, the number of visits to the gym in a month and the mean resting heart rate. Since we are comparing two variables in which one or both are measured on an ordinal scale, we will use the Spearman rank-order correlation.

4. Compute the Test Statistic

First, rank the scores for each variable separately as shown in Table 3. Rank them from the lowest score to the highest score to form an ordinal distribution for each variable.

TABLE 3

	Original scores		Ranked scores	
Participant	Number of visits	Mean heart rate	Number of visits	Mean heart rate
1	5	100	2	7
2	12	89	5	6
3	7	78	3	5
4	14	66	6	2
5	2	77	1	4
6	8	103	4	8
7	15	67	7	3
8	17	63	8	1

To calculate the Spearman rank-order correlation coefficient, we need to calculate the differences between rank pairs and their subsequent squares where D = rank (mean heart rate) − rank (number of visits).

It is helpful to organize the data to manage the summation in the formula (Table 4).

TABLE 4

Ranked scores		Rank differences	
Number of visits	Mean heart rate	D	D^2
2	7	5	25
5	6	1	1
3	5	2	4
6	2	−4	16
1	4	3	9
4	8	4	16
7	3	−4	16
8	1	−7	49
			$\sum D_i^2 = 136$

Next, compute the Spearman rank-order correlation coefficient:

$$r_s = 1 - \frac{6\sum D_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6(136)}{8(8^2-1)} = 1 - \frac{816}{8(64-1)}$$

$$= 1 - \frac{816}{8(63)} = 1 - \frac{816}{504}$$

$$= 1 - 1.619$$

$$r_s = -0.619$$

5. Determine the Value Needed for Rejection of the Null Hypothesis Using the Appropriate Table of Critical Values for the Particular Statistic

Table B.7 lists critical values for the Spearman rank-order correlation coefficient.

In this study, the critical value is found for $n = 8$ and $df = 6$.

Since we are conducting a two-tailed test and $\alpha = 0.05$, the critical value is 0.738. If the obtained value exceeds or is equal to the critical value, 0.738, we will reject the null hypothesis.

If the critical value exceeds the absolute value of the obtained value, we will not reject the null hypothesis.

6. Compare the Obtained Value with the Critical Value

The critical value for rejecting the null hypothesis is 0.738 and the obtained value is $|r_s| = 0.619$.

If the critical value is less than or equal to the obtained value, we must reject the null hypothesis. If instead, the critical value is greater than the obtained value, we must not reject the null hypothesis. Since the critical value exceeds the absolute value of the obtained value, we do not reject the null hypothesis.

7. Interpret the Results

We did not reject the null hypothesis, suggesting that there is no significant correlation between the number of visits the males made to the gym in a month and their mean resting heart rates.

8. Reporting the Results

The reporting of results for the Spearman rank order correlation should include such information as the number of participants (n), two variables that are being correlated, correlation coefficient (r_s), degrees of freedom (df), and p-value's relation to .

For this example, eight men ($n = 8$) were observed for 1 month. Their number of visits to the gym was documented (variable 1) and their mean resting heart rate was recorded during the last 3 weeks of the month (variable 2). These data were put in ordinal form for purposes of the analysis. The Spearman rank-order correlation coefficient was not significant ($r_{s(6)} = -0.619$, $p > 0.05$). Based on this data, we can state that there is no clear relationship between adult male resting heart rate and the frequency of visits to the gym.

Example:

Sample Spearman Rank-Order Correlation
(Small Data Samples with Ties)

The researcher repeated the experiment in the previous example using females. Table 5 shows the number of visits each participant made to the gym during the month of the study and their subsequent mean heart rates.

TABLE 5

Participant	Number of visits	Mean heart rate
1	5	96
2	12	63
3	7	78
4	14	66
5	3	79
6	8	95
7	15	67
8	12	64
9	2	99
10	16	62
11	12	65
12	7	76
13	17	61

As with the previous example, the values in this study do not possess characteristics of a strong interval scale, so we will use ordinal measures.

We will convert these values to ranks and use a Spearman rank-order correlation.

Steps 1–3 are the same as the previous example. Therefore, we will begin with step 4.

4. Compute the Test Statistic

First, rank the scores for each variable as shown in Table 6. Rank the scores from the lowest score to the highest score to form an ordinal distribution for each variable.

TABLE 6

	Original scores		Rank scores	
Participant	Number of visits	Mean heart rate	Number of visits	Mean heart rate
1	5	96	3	12
2	12	63	8	3
3	7	78	4.5	9
4	14	66	10	6
5	3	79	2	10
6	8	95	6	11
7	15	67	11	7
8	12	64	8	4
9	2	99	1	13
10	16	62	12	2
11	12	65	8	5
12	7	76	4.5	8
13	17	61	13	1

To calculate the Spearman rank-order correlation coefficient, we need to calculate the differences between rank pairs and their subsequent squares where D = rank (mean heart rate) − rank (number of visits). It is helpful to organize the data to manage the summation in the formula (Table 7).

TABLE 7

	Rank scores		Rank differences	
Participant	Number of visits	Mean heart rate	D	D^2
1	3	12	9	81
2	8	3	−5	25
3	4.5	9	4.5	20.25
4	10	6	−4	16
5	2	10	8	64
6	6	11	5	25
7	11	7	−4	16
8	8	4	−4	16
9	1	13	12	144
10	12	2	−10	100
11	8	5	−3	9
12	4.5	8	3.5	12.25
13	13	1	−12	144
				$\sum D_i^2 = 672.5$

Next, compute the Spearman rank-order correlation coefficient.

Since there are ties present in the ranks, we will use formulas that account for the ties. First, use Formula 3 and Formula 4. For the number of visits, there are two groups of ties.
The first group has two tied values (rank = 4.5 and t = 2) and the second group has
three tied values (rank = 8 and t = 3):

$$T_x = \sum_{i=1}^{g} (t_i^3 - t_i)$$
$$= (2^3 - 2) + (3^3 - 3) = (8 - 2) + (27 - 3)$$
$$= 6 + 24$$
$$T_x = 30$$

For the mean resting heart rate, there are no ties. Therefore, Ty = 0.

Now, calculate the Spearman rank-order correlation coefficient using Formula 2:

$$r_s = \frac{(n^3 - n) - 6\sum D_i^2 - (T_x + T_y)/2}{\sqrt{(n^3 - n)^2 - (T_x + T_y)(n^3 - n) + T_x T_y}}$$

$$= \frac{(13^3 - 13) - 6(672.5) - (30 + 0)/2}{\sqrt{(13^3 - 13)^2 - (30 + 0)(13^3 - 13) + (30)(0)}}$$

$$= \frac{2184 - 4035 - 15}{\sqrt{(2184)^2 - (30)(2184) + 0}}$$

$$= \frac{-1866}{\sqrt{4,704,336}} = \frac{-1866}{2169}$$

$$r_s = -0.860$$

5. Determine the Value Needed for Rejection of the Null Hypothesis Using the Appropriate Table of Critical Values for the Particular Statistic

Table B.7 lists critical values for the Spearman rank-order correlation coefficient.

To be significant, the absolute value of the obtained value, $|r_s|$, must be greater than or equal to the critical value on the table. In this study, the critical value is found for n = 13 and df = 11. Since we are conducting a two-tailed test and $\alpha = 0.05$, the critical value is 0.560.

6. Compare the Obtained Value with the Critical Value

The critical value for rejecting the null hypothesis is 0.560 and the obtained value is $|r_s| = 0.860$.

If the critical value is less than or equal to the obtained value, we must reject the null hypothesis. If instead, the critical value is greater than the obtained value, we must not reject the null hypothesis. Since the critical value is less than the absolute value of the obtained value, we reject the null hypothesis.

7. Interpret the Results

We rejected the null hypothesis, suggesting that there is a significant correlation between the number of visits the females made to the gym in a month and their mean resting heart rates.

8. Reporting the Results

The reporting of results for the Spearman rank order correlation should include such information as the number of participants (n), two variables that are being correlated, correlation coefficient (r_s), degrees of freedom (*df*), and p-value's relation to α .

For this example, 13 women (n = 13) were observed for 1 month. Their number of visits to the gym was documented (variable 1) and their mean resting heart rate was recorded during the last 3 weeks of the month (variable 2). These data were put in ordinal form for purposes of the analysis. The Spearman rank-order correlation coefficient was significant ($rs(11) = -0.860$, $p < 0.05$).

Based on this data, we can state that there is a very strong inverse relationship between adult female resting heart rate and the frequency of visits to the gym.

COMPUTING THE POINT-BISERIAL AND BISERIAL CORRELATION COEFFICIENTS

The point-biserial and biserial correlations are statistical procedures for use with dichotomous variables. A dichotomous variable is simply a measure of two conditions.

A dichotomous variable is either discrete or continuous. A discrete dichotomous variable has no particular order and might include such examples as gender (male vs. female) or a coin toss (heads vs. tails). A continuous dichotomous variable has some type of order to the two conditions and might include measurements such as pass/fail or young/old. Finally, since the point-biserial and biserial correlations each involves an interval scale analysis, they are special cases of the Pearson product-moment correlation.

Correlation of a Dichotomous Variable and an Interval Scale Variable

The point-biserial correlation is a statistical procedure to measure the relationship between a discrete dichotomous variable and an interval scale variable. Use Formula 8 to determine the point-biserial correlation coefficient r_{pb}:

$$r_{pb} = \frac{\overline{x}_p - \overline{x}_q}{s}\sqrt{P_p P_q} \quad (8)$$

where $\overline{x}_p$ is the mean of the interval variable's values associated with the dichotomous variable's first category, $\overline{x}_q$ is the mean of the interval variable's values associated with the dichotomous variable's second category, s is the standard deviation of the variable on the interval scale, P_p is the proportion of the interval variable values associated with the dichotomous variable's first category, and P_q is the proportion of the interval variable values associated with the dichotomous variable's second category.

The biserial correlation is a statistical procedure to measure the relationship between a continuous dichotomous variable and an interval scale variable. Use Formula 11 to determine the biserial correlation coefficient r_b:

$$r_b = \left[\frac{\overline{x}_p - \overline{x}_q}{s_x}\right]\frac{P_p P_q}{y} \qquad (11)$$

where $\overline{x}_p$ is the mean of the interval variable's values associated with the dichotomous variable's first category, $\overline{x}_q$ is the mean of the interval variable's values associated with the dichotomous variable's second category, s_x is the standard deviation of the variable on the interval scale, P_p is the proportion of the interval variable values associated with the dichotomous variable's first category, P_q is the proportion of the interval variable values associated with the dichotomous variable's second category, and y is the height of the unit normal curve ordinate at the point dividing P_p and P_q (see Fig. 5).

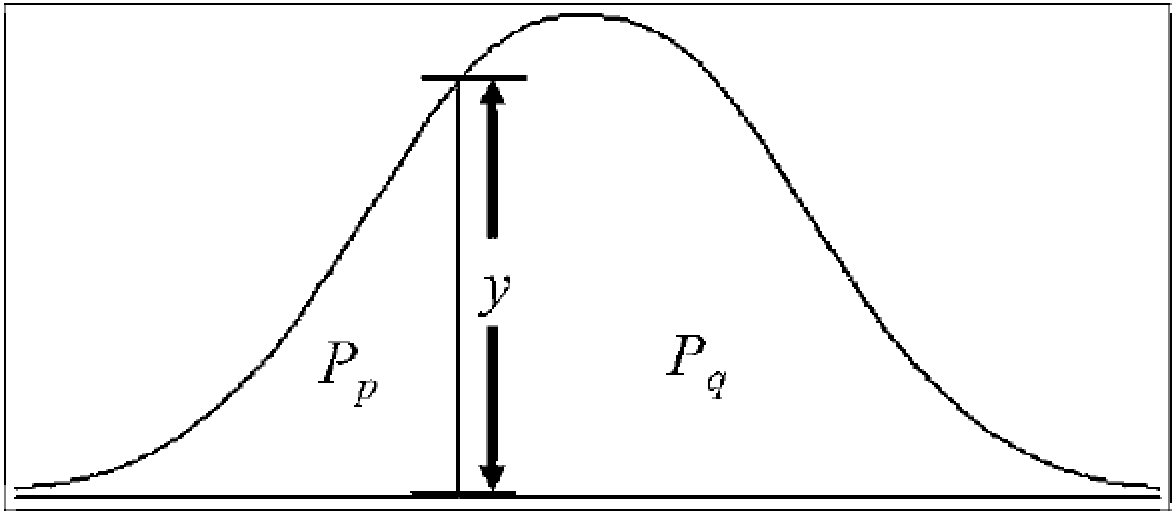

FIGURE 5

You may use Table B.1 or Formula 2 to find the height of the unit normal curve ordinate, y:

$$y = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad (12)$$

where e is the natural log base and approximately equal to 2.718282 and z is the z-score at the point dividing P_p and P_q.

Formula 13 is the relationship between the point-biserial and the biserial correlation coefficients. This formula is necessary to find the biserial correlation coefficient because SPSS only determines the point-biserial correlation coefficient:

$$r_b = r_{bp} \frac{\sqrt{P_p P_q}}{y} \qquad (13)$$

After the correlation coefficient is determined, it must be examined for significance.

Small samples allow one to reference a table of critical values, such as Table B.8.

However, if the sample size n exceeds those available from the table, then a large sample approximation may be performed.

Correlation of a Dichotomous Variable and a Rank-Order Variable

As explained earlier, the point-biserial and biserial correlation procedures earlier involve a dichotomous variable and an interval scale variable. If the correlation was a dichotomous variable and a rank-order variable, a slightly different approach is needed.

To find the point-biserial correlation coefficient for a discrete dichotomous variable and a rank-order variable, simply use the Spearman rank-order described earlier and assign arbitrary values to the dichotomous variable such as 0 and 1.
To find the biserial correlation coefficient for a continuous dichotomous variable and a rank-order variable, use the same procedure and then apply Formula 13 given earlier.

Example

Point-Biserial Correlation (Small Data Samples)

A researcher in a psychological lab investigated gender differences. She wished to compare male and female ability to recognize and remember visual details. She used 17 participants (8 males and 9 females) who were initially unaware of the actual experiment.

First, she placed each one of them alone in a room with various objects and asked them to wait. After 10 min, she asked each of the participants to complete a 30 question posttest relating to several details in the room.

Table 8 shows the participants' genders and posttest scores.

TABLE 8

Participant	Gender	Posttest score
1	M	7
2	M	19
3	M	8
4	M	10
5	M	7
6	M	15
7	M	6
8	M	13
9	F	14
10	F	11
11	F	18
12	F	23
13	F	17
14	F	20
15	F	14
16	F	24
17	F	22

The researcher wishes to determine if a relationship exists between the two variables and the relative strength of the relationship.

Gender is a discrete dichotomous variable and visual detail recognition is an interval scale variable. Therefore, we will use a point-biserial correlation.

1. State the Null and Research Hypothesis

The null hypothesis states that there is no correlation between gender and visual detail recognition. The research hypothesis states that there is a correlation between gender and visual detail recognition.

The null hypothesis is

$$H_O\colon \rho_{pb} = 0$$

The research hypothesis is

$$H_A\colon \rho_{pb} \neq 0$$

2. Set the Level of Risk (or the Level of Significance) Associated with the Null Hypothesis

The level of risk, also called an alpha (α), is frequently set at 0.05. We will use α = 0.05 in our example. In other words, there is a 95% chance that any observed statistical difference will be real and not due to chance.

3. Choose the Appropriate Test Statistic

As stated earlier, we decided to analyze the relationship between the two variables. A correlation will provide the relative strength of the relationship between the two variables. Gender is a discrete dichotomous variable and visual detail recognition is an interval scale variable.

Therefore, we will use a point-biserial correlation.

4. Compute the Test Statistic

First, compute the standard deviation of all values from the interval scale data. It is helpful to organize the data as shown in Table 9.

TABLE 9

Participant	Gender	Posttest score	$x_i - x$	$(x_i - \overline{x})^2$
1	M	7	–7.59	57.58
2	M	19	4.41	19.46
3	M	8	–6.59	43.40
4	M	10	–4.59	21.05
5	M	7	–7.59	57.58
6	M	15	0.41	0.17
7	M	6	–8.59	73.76
8	M	13	–1.59	2.52
9	F	14	–0.59	0.35
10	F	11	–3.59	12.88
11	F	18	3.41	11.64
12	F	23	8.41	70.76
13	F	17	2.41	5.82
14	F	20	5.41	29.29
15	F	14	–0.59	0.35
16	F	24	9.41	88.58
17	F	22	7.41	54.93
		$\sum x_i = 248$		$\sum (x_i - \overline{x})^2 = 550.12$

Using the summations from Table 9, calculate the mean and the standard deviation for the interval data:

$$\overline{x} = \sum x_i \div n$$

$$\overline{x} = 248 \div 17$$

$$\overline{x} = 14.59$$

$$s_x = \sqrt{\frac{\sum (x_i - \overline{x})^2}{n-1}} = \sqrt{\frac{550.12}{17-1}} = \sqrt{34.38}$$

$$s_x = 5.86$$

Next, compute the means and proportions of the values associated with each item from the dichotomous variable. The mean males' posttest score was

$$\overline{x}_p = \sum x_p \div n_M$$

$$= (7+19+8+10+7+15+6+13) \div 8$$

$$\overline{x}_p = 10.63$$

The mean females' posttest score was

$$\bar{x}_q = \sum x_q \div n_F$$
$$= (14 + 11 + 18 + 23 + 17 + 20 + 14 + 24 + 22) \div 9$$
$$\bar{x}_q = 18.11$$

The males' proportion was

$$P_p = n_M \div n$$
$$= 8 \div 17$$
$$P_p = 0.47$$

The females' proportion was

$$P_q = n_F \div n$$
$$= 9 \div 17$$
$$P_q = 0.53$$

Now, compute the point-biserial correlation coefficient using the values computed earlier:

$$
\begin{aligned}
r_{pb} &= \frac{\overline{x}_p - \overline{x}_q}{s_x}\sqrt{P_p P_q} \\
&= \frac{10.63 - 18.11}{5.86}\sqrt{(0.47)(0.53)} \\
&= \frac{-7.49}{5.86}\sqrt{0.25} = (-1.28)(0.50) \\
r_{pb} &= -0.637
\end{aligned}
$$

The sign on the correlation coefficient is dependent on the order we managed our dichotomous variable. Since that was arbitrary, the sign is irrelevant. Therefore, we use the absolute value of the point-biserial correlation coefficient:

$$r_{pb} = 0.637$$

5. Determine the Value Needed for Rejection of the Null Hypothesis

Using the Appropriate Table of Critical Values for the Particular Statistic Table B.8 lists critical values for the Pearson product-moment correlation coefficient.

Using the critical values, table requires that the degrees of freedom be known. Since df = n − 2 and n = 17, then df = 17 − 2.

Therefore, df = 15. Since we are conducting a two-tailed test and α = 0.05, the critical value is 0.482.

6. Compare the Obtained Value with the Critical Value

The critical value for rejecting the null hypothesis is 0.482 and the obtained value is $|r_{pb}| = 0.637$.

If the critical value is less than or equal to the obtained value, we must reject the null hypothesis. If instead, the critical value is greater than the obtained value, we must not reject the null hypothesis. Since the critical value is less than the absolute value of the obtained value, we reject the null hypothesis.

7. Interpret the Results

We rejected the null hypothesis, suggesting that there is a significant and moderately strong correlation between gender and visual detail recognition.

8. Reporting the Results

The reporting of results for the point-biserial correlation should include such information as the number of participants (n), two variables that are being correlated, correlation coefficient (r_{pb}), degrees of freedom (df), p-value's relation to α, and the mean values of each dichotomous variable.

For this example, a researcher compared male and female ability to recognize and remember visual details. Eight males ($n_M = 8$) and nine females ($n_F = 9$) participated in the experiment. The researcher measured participants' visual detail recognition with a 30 question test requiring participants to recall details in a room they had occupied. A point-biserial correlation produced significant results ($r_{pb(15)} = 0.637$, $p < 0.05$). These data suggest that there is a strong relationship between gender and visual detail recognition. Moreover, the mean scores on the detail recognition test indicate that males ($\overline{x}_M = 10.63$) recalled fewer details, while females ($\overline{x}_F = 18.11$) recalled more details.

SUMMARY

The relationship between two variables can be compared with a correlation analysis.

If any of the variables are ordinal or dichotomous, a nonparametric correlation is useful. The Spearman rank-order correlation, also called the Spearman's , is used to compare the relationship involving ordinal, or rank-ordered, variables. The point biserial and biserial correlations are used to compare the relationship between two variables if one of the variables is dichotomous. The parametric equivalent to these correlations is the Pearson product-moment correlation.

In this lecture, we described how to perform and interpret a Spearman rank order, point-biserial, and biserial correlations.