→ Probability space and σ-field

Def.n: A probability space is a triple (Ω, F, P), where:

(a) the set Ω is called the sample space consisting of all possible outcomes of the random experiment.

(b) F is a collection of events (subsets of Ω), called a σ-field, which satisfies the following properties: (1) $\Omega \in F$

(2) if $A \in F$, then $A^c \in F$,

and (3) if $A_1, A_2, \ldots, A_n, \ldots$ is a sequence of events belonging to F, then $\bigcup_{n=1}^{\infty} A_n \in F$.

(c) A probability measure P, which assigns to each event A a number $P(A)$ (the probability of A) in such a way that:

(i) $0 \leq P(A) \leq 1$ for any event A.

(ii) $P(\Omega) = 1$

(iii) If $A_1, A_2, \ldots, A_n, \ldots$ is a sequence of disjoint (mutually exclusive) events, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Examples

1- $\boxed{\text{Toss a coin two times}}$ in this case,

$$\Omega = \{\underset{\text{head}}{HH}, \underset{\text{tail}}{HT}, TT, TH\}$$

$\mathcal{F}$ is the collection of all possible subsets of Ω. $P(A)$ is the probability of A. If $A = \{HT, TH\}$, then $P(A) = \frac{1}{2}$.

2- $\boxed{\text{Choose a number randomly from the interval } [0,1]}$

In this case, $\Omega = [0,1]$. The basic events are intervals $[a,b]$, $0 \leq a \leq b \leq 1$.

Choose $\mathcal{F}$ so that it contains intervals.

For the probability P, we set:

$$P([a,b]) = \text{length of } [a,b] = b - a$$

This means that the chance that we select a number lying between a and b is proportional to the length of the interval.

3- $\boxed{\sigma\text{-field}}$

(i) $\mathcal{F} = \{\Omega, \phi\}$ is the smallest σ-field.

(ii) $\mathcal{F} = \{\Omega, A, A^c, \phi\}$ is a σ-field for any $A \subset \Omega$.

(iii) The collection of all subsets of Ω is the biggest σ-field.

Def.n For any collection D of subsets of Ω, the smallest σ-field G that contains D is called the σ-field generated by D. We write $G = \sigma(D)$.

For example, if $D = \{A\}$, then

$$G = \sigma(D) \quad \text{(the } \sigma\text{-field generated by } D)$$
$$= \{\phi, \Omega, A, A^c\}.$$

Remark

1- In discrete case, for a family of random variables $\{X_1, \ldots, X_n\}$, we denote by $F_n = \sigma(X_1, \ldots, X_n)$, the σ-field generated by $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are F_n-measurable and $F_n \subset F_{n+1}$ $\forall n \geq 0$.

2- In Continouse case, for random variable $(X_t)_{t \geq 0}$ we denote by $F_t = \sigma(X_u, u \leq t)$, the σ-field generated by $(X_u, u \leq t)$.

We say that $(X_u, 0 \leq u \leq t)$ are F_t-measurable and we have that $F_s \subset F_t$ if $s \leq t$.

3- In particular, we say that

X_n is F_n-measurable (in discreate Case), $n \geq 0$,

and

X_t is F_t-measurable (in Continouse case), $t \geq 0$.