

4.1 Real Vector Spaces

DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called *scalars*. By *addition* we mean a rule for associating with each pair of objects \mathbf{u} and \mathbf{v} in V an object $\mathbf{u} + \mathbf{v}$, called the *sum* of \mathbf{u} and \mathbf{v} ; by *scalar multiplication* we mean a rule for associating with each scalar k and each object \mathbf{u} in V an object $k\mathbf{u}$, called the *scalar multiple* of \mathbf{u} by k . If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m , then we call V a *vector space* and we call the objects in V *vectors*.

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V , called a *zero vector* for V , such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .
5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a *negative* of \mathbf{u} , such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and \mathbf{u} is any object in V , then $k\mathbf{u}$ is in V .
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

In this text scalars will be either real numbers or complex numbers. Vector spaces with real scalars will be called *real vector spaces* and those with complex scalars will be called *complex vector spaces*. There is a more general notion of a vector space in which scalars can come from a mathematical structure known as a “field,” but we will not be concerned with that level of generality. For now, we will focus exclusively on real vector spaces, which we will refer to simply as “vector spaces.”

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on R^n . The only requirement is that the ten vector space axioms be satisfied.

▶ **EXAMPLE 1 The Zero Vector Space**

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalars k . It is easy to check that all the vector space axioms are satisfied. We call this the *zero vector space*. ◀

▶ **EXAMPLE 2 R^n Is a Vector Space**

Let $V = R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n -tuples; that is,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \\ k\mathbf{u} &= (ku_1, ku_2, \dots, ku_n)\end{aligned}$$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce n -tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1. ◀

► **EXAMPLE 4 The Vector Space of 2×2 Matrices**

Let V be the set of 2×2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \\ k\mathbf{u} &= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \end{aligned} \quad (1)$$

The set V is closed under addition and scalar multiplication because the foregoing operations produce 2×2 matrices as the end result. Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold. Some of these are standard properties of matrix operations. For example, Axiom 2 follows from Theorem 1.4.1(a) since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

Similarly, Axioms 3, 7, 8, and 9 follow from parts (b), (h), (j), and (e), respectively, of that theorem (verify). This leaves Axioms 4, 5, and 10 that remain to be verified.

To confirm that Axiom 4 is satisfied, we must find a 2×2 matrix $\mathbf{0}$ in V for which $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ for all 2×2 matrices in V . We can do this by taking

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this definition,

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$. To verify that Axiom 5 holds we must show that each object \mathbf{u} in V has a negative $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ and $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. This can be done by defining the negative of \mathbf{u} to be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition,

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. Finally, Axiom 10 holds because

$$1\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

► **EXAMPLE 5 The Vector Space of $m \times n$ Matrices**

Example 4 is a special case of a more general class of vector spaces. You should have no trouble adapting the argument used in that example to show that the set V of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication is a vector space. We will denote this vector space by the symbol M_{mn} . Thus, for example, the vector space in Example 4 is denoted as M_{22} .

► **EXAMPLE 6 The Vector Space of Real-Valued Functions**

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \quad (2)$$

$$(k\mathbf{f})(x) = kf(x) \quad (3)$$

The set V with these operations is denoted by the symbol $F(-\infty, \infty)$. We can prove that this is a vector space.

► **EXAMPLE 7 A Set That Is Not a Vector Space**

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then

$$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$

$$k\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$$

The addition operation is the standard one from R^2 , but the scalar multiplication is not. In the exercises we will ask you to show that the first nine vector space axioms are satisfied. However, Axiom 10 fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, V is not a vector space with the stated operations. ◀

THEOREM 4.1.1 *Let V be a vector space, \mathbf{u} a vector in V , and k a scalar; then:*

- (a) $0\mathbf{u} = \mathbf{0}$
- (b) $k\mathbf{0} = \mathbf{0}$
- (c) $(-1)\mathbf{u} = -\mathbf{u}$
- (d) *If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.*

4.2 Subspaces

DEFINITION 1 A subset W of a vector space V is called a *subspace* of V if W is itself a vector space under the addition and scalar multiplication defined on V .

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V , then W is a subspace of V if and only if the following conditions are satisfied.*

- (a) *If \mathbf{u} and \mathbf{v} are vectors in W , then $\mathbf{u} + \mathbf{v}$ is in W .*
- (b) *If k is a scalar and \mathbf{u} is a vector in W , then $k\mathbf{u}$ is in W .*

Theorem 4.2.1 states that W is a subspace of V if and only if it is closed under addition and scalar multiplication.

► **EXAMPLE 1 The Zero Subspace**

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for any scalar k . We call W the *zero subspace* of V .

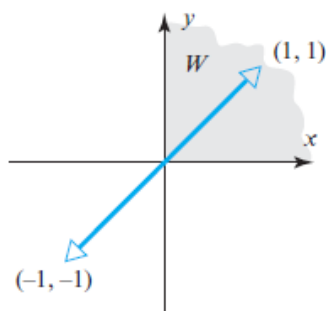
Note that every vector space has at least two subspaces, itself and its zero subspace.

Table 1

Subspaces of R^2	Subspaces of R^3
<ul style="list-style-type: none"> • $\{0\}$ • Lines through the origin • R^2 	<ul style="list-style-type: none"> • $\{0\}$ • Lines through the origin • Planes through the origin • R^3

► **EXAMPLE 4 A Subset of R^2 That Is Not a Subspace**

Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$ (the shaded region in Figure 4.2.4). This set is not a subspace of R^2 because it is not closed under scalar multiplication. For example, $v = (1, 1)$ is a vector in W , but $(-1)v = (-1, -1)$ is not.



▲ **Figure 4.2.4** W is not closed under scalar multiplication.

▶ **EXAMPLE 5 Subspaces of M_{nn}**

$$A^T = A$$

We know from Theorem 1.7.2 that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric. Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn} . Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn} .

▶ **EXAMPLE 6 A Subset of M_{nn} That Is Not a Subspace**

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn} , failing on two counts—it is not closed under addition and not closed under scalar multiplication. We will illustrate this with an example in M_{22} that you can readily adapt to M_{nn} . Consider the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

