4.5 Dimension

THEOREM 4.5.1 *All bases for a finite-dimensional vector space have the same number of vectors.*

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

▶ **EXAMPLE 1 Dimensions of Some Familiar Vector Spaces**

$\dim(R^n) = n$ [The standard basis has n vectors.]

$\dim(P_n) = n + 1$ [The standard basis has $n + 1$ vectors.]

$\dim(M_{mn}) = mn$ [The standard basis has mn vectors.]

▶ EXAMPLE 2 Dimension of Span(S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span(S) is expressible as a linear combination of the vectors in S. Thus, if the vectors in S are *linearly independent*, they automatically form a basis for span(S), from which we can conclude that

$$\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

▶ EXAMPLE 3 Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution In Example 6 of Section 1.2 we found the solution of this system to be

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space. We leave it for you to check that these vectors are linearly independent by showing that none of them is a linear combination of the other two (but see the remark that follows). Thus, the solution space has dimension 3. ◀

Remark It can be shown that for any homogeneous linear system, the method of the last example *always* produces a basis for the solution space of the system. We omit the formal proof.

THEOREM 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(*a*) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.*

(*b*) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,*

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

EXAMPLE 4 Applying the Plus/Minus Theorem

Show that $\mathbf{p}_1 = 1 - x^2$, $\mathbf{p}_2 = 2 - x^2$, and $\mathbf{p}_3 = x^3$ are linearly independent vectors.

Solution The set $S = \{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent since neither vector in S is a scalar multiple of the other. Since the vector $\mathbf{p}_3$ cannot be expressed as a linear combination of the vectors in S (why?), it can be adjoined to S to produce a linearly independent set $S \cup \{\mathbf{p}_3\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. ◀

In general, to show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, one must show that the vectors are linearly independent and span V. However, if we happen to know that V has dimension n (so that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ contains the right number of vectors for a basis), then it suffices to check *either* linear independence *or* spanning—the remaining condition will hold automatically. This is the content of the following theorem.

THEOREM 4.5.4 *Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.*

To put it yet another way, suppose we have a set of vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in a subspace V. Then if any two of the following statements is true, the third must also be true:

1. $\mathcal{B}$ is linearly independent,
2. $\mathcal{B}$ spans V, and
3. $\dim V = m$.

EXAMPLE 5 Bases by Inspection

(a) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.

(b) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3.

Solution (a) Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis by Theorem 4.5.4.

Solution (b) The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the xz-plane (why?). The vector $\mathbf{v}_3$ is outside of the xz-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Since R^3 is three-dimensional, Theorem 4.5.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space R^3. ◀

1. Every spanning set for a subspace is either a basis for that subspace or has a basis as a subset.
2. Every linearly independent set in a subspace is either a basis for that subspace or can be extended to a basis for it.

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

We conclude this section with a theorem that relates the dimension of a vector space to the dimensions of its subspaces.

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(*a*) *W is finite-dimensional.*

(*b*) $\dim(W) \leq \dim(V)$.

(*c*) $W = V$ *if and only if* $\dim(W) = \dim(V)$.

If $S = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is a set of vectors and F the vector sub-space generated by S. We have the following two algorithms to construct a basis of F.

First Algorithm

1. Construct the matrix A such that its rows are the vectors of S
2. The non zeros rows of any row echelon form of the matrix A are a basis of the vector space $F = \langle S \rangle$.

Second Algorithm

1. Construct the matrix A such that its columns are the vectors of S
2. Take any row echelon form C of the matrix A.
3. Let $C_{k_1}, \ldots C_{k_p}$ be the columns which contain a leading number and $k_1 < \ldots < k_p$. Then $\{v_{k_1}, \ldots, v_{k_p}\}$ is a basis of the vector space $F = \langle S \rangle$.

Ex. Find a basis for the subspace $W \subseteq \mathbb{R}^5$, generated by $(1,-1,2,0,-1)$, $(2,-1,-2,0,1)$, $(-1,0,4,0,-2)$, and $(0,-1,6,0,-3)$.

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 2 & -1 & -2 & 0 & 1 \\ -1 & 0 & 4 & 0 & -2 \\ 0 & -1 & 6 & 0 & -3 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -6 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the subspace W is

$B = \{(1,-1,2,0,-1), (0,1,-6,0,3)\}$

Q. What is the dimension of the subspace W?

Q3: Let V be the subspace of $\mathbb{R}^3$ **spanned** by the set S={ v_1=(1, 2,3), v_2=(2, 4,6), v_3=(4, 6, 6)}. Find a **subset** of S that forms a basis of V.

Answer:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 3 & 6 & 6 \end{bmatrix} \xrightarrow[-3R_{13}]{-2R_{12}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{6R_{23}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 and 3 have leading ones, then v_1 and v_3 forms a basis of V.

Homework

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 0, 1), \quad \mathbf{v}_4 = (0, 0, -1)$$

18. Find a basis for the subspace of R^4 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = (2, 2, 2, 0), \quad \mathbf{v}_3 = (0, 0, 0, 3),$$
$$\mathbf{v}_4 = (3, 3, 3, 4)$$