

5.1 Eigenvalues and Eigenvectors

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector \mathbf{x} in R^n is called an *eigenvector* of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ . The scalar λ is called an *eigenvalue* of A (or of T_A), and \mathbf{x} is said to be an *eigenvector corresponding to λ* .

THEOREM 5.1.1 If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0 \quad (1)$$

This is called the *characteristic equation* of A .

When the determinant $\det(\lambda I - A)$ in (1) is expanded, the characteristic equation of A takes the form

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0 \quad (3)$$

where the left side of this equation is a polynomial of degree n in which the coefficient of λ^n is 1 (Exercise 37). The polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n \quad (4)$$

is called the *characteristic polynomial* of A .

► **EXAMPLE 3 Eigenvalues of a 3 x 3 Matrix**

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \quad (5)$$

To solve this equation, we will begin by searching for integer solutions. This task can be simplified by exploiting the fact that all integer solutions (if there are any) of a polynomial equation with *integer coefficients*

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

must be divisors of the constant term, c_n . Thus, the only possible integer solutions of (5) are the divisors of -4 , that is, $\pm 1, \pm 2, \pm 4$. Successively substituting these values in (5) shows that $\lambda = 4$ is an integer solution and hence that $\lambda - 4$ is a factor of the left side of (5). Dividing $\lambda - 4$ into $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ shows that (5) can be rewritten as

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Thus, the remaining solutions of (5) satisfy the quadratic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

which can be solved by the quadratic formula. Thus, the eigenvalues of A are

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}$$

▶ EXAMPLE 4 Eigenvalues of an Upper Triangular Matrix

Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Solution Recalling that the determinant of a triangular matrix is the product of the entries on the main diagonal (Theorem 2.1.2), we obtain

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix} \\ &= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) \end{aligned}$$

Thus, the characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

and the eigenvalues are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \lambda = a_{33}, \quad \lambda = a_{44}$$

which are precisely the diagonal entries of A . ◀

THEOREM 5.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A .*

▶ **EXAMPLE 5 Eigenvalues of a Lower Triangular Matrix**

By inspection, the eigenvalues of the lower triangular matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$, and $\lambda = -\frac{1}{4}$. ◀

The following theorem gives some alternative ways of describing eigenvalues.

THEOREM 5.1.3 *If A is an $n \times n$ matrix, the following statements are equivalent.*

- (a) λ is an eigenvalue of A .
- (b) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
- (c) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
- (d) There is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$.

*Finding Eigenvectors and
Bases for Eigenspaces*

Definition

Let $A \in M_n(\mathbb{R})$ and λ an eigenvalue of the matrix A . We define

$$E_\lambda = \{X \in \mathbb{R}^n; AX = \lambda X\}$$

This space is called the eigenspace associated to the eigenvalue λ .

Thus, we can find the eigenvectors of A corresponding to λ by finding the nonzero vectors in the solution space of this linear system. This solution space, which is called the *eigenspace* of A corresponding to λ , can also be viewed as:

1. the null space of the matrix $\lambda I - A$
2. the kernel of the matrix operator $T_{\lambda I - A}: \mathbb{R}^n \rightarrow \mathbb{R}^n$
3. the set of vectors for which $Ax = \lambda x$

► **EXAMPLE 6 Bases for Eigenspaces**

Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Solution The characteristic equation of A is

$$\begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda + 1) - 6 = (\lambda - 2)(\lambda + 3) = 0$$

so the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$. Thus, there are two eigenspaces of A one for each eigenvalue.

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to an eigenvalue λ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$ that is,

$$\begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the case where $\lambda = 2$ this equation becomes

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose general solution is

$$x_1 = t, \quad x_2 = t$$

(verify). Since this can be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 2$. We leave it for you to follow the pattern of these computations and show that

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -3$. ◀

► **EXAMPLE 7 Eigenvectors and Bases for Eigenspaces**

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, or in factored form, $(\lambda - 1)(\lambda - 2)^2 = 0$ (verify). Thus, the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so there are two eigenspaces of A .

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if \mathbf{x} is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$, or in matrix form,

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

In the case where $\lambda = 2$, Formula (6) becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1 = -s, \quad x_2 = t, \quad x_3 = s$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent (why?), these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then (6) becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields (verify)

$$x_1 = -2s, \quad x_2 = s, \quad x_3 = s$$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$. ◀