1

Lecture (33)

Brownian Motion (Wiener process)

Textbook Ch 8 p 391

* The Brownian motion process is an example of a continuous-time, continuous state-space Markov process.

Let $B(t)$ be the y-component (as a function of time) of a particle in Brownian motion. Let x be the position of the particle at time t_0 i.e $B(t_0) = x$. Let $p(y,t|x)$ be the probability density function, in y, where $y = B(t_0 + t)$, given that $B(t_0) = x$.

$\therefore$ $p(y,t|x)$ is a pdf in y

$$\therefore p(y,t|x) \geq 0 \text{ and } \int_{-\infty}^{\infty} p(y,t|x)\,dy = 1 \qquad (1)$$

Note that:

(1) $\lim_{t \to 0} p(y,t|x) = 0$ for $y \neq x$ (2)

(2) $p(y,t|x)$ must satisfy the partial differential equation (p.d.e)

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} \qquad (3)$$

which is called the diffusion equation and σ^2 is the diffusion coefficient.

If $\sigma^2 = 1$, eqn (3) becomes

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \qquad (*)$$

and we can verify that

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(y-x)^2\right) \qquad (4)$$

is the solution of (*) which is a normal probability density function whose mean is x and whose variance is t.

i.e the mean position is the initial location x, and the variance is the time of observation t.

2/5

So, we can write ④ as

$$\left.\begin{aligned} P(y,t\mid x) &= \frac{1}{\sqrt{t}}\,\phi\left(\frac{y-x}{\sqrt{t}}\right) \\ &= \phi_t(y-x) \end{aligned}\right\} \quad (5)$$

and

$$pr\{B(t)\le y \mid B(0)=x\} = \Phi\left(\frac{y-x}{\sqrt{t}}\right) \quad (6)$$

in terms of probability density function and cumulative distribution function of standard Normal distⁿ $N(0,1)$.

- the probability density fn in ④ and ⑤ gives only the probability distⁿ of $B(t)-B(0)$.

the complete description of the Brownian motion process with diffusion coefficient σ^2 is given by the following definition.

- Defⁿ

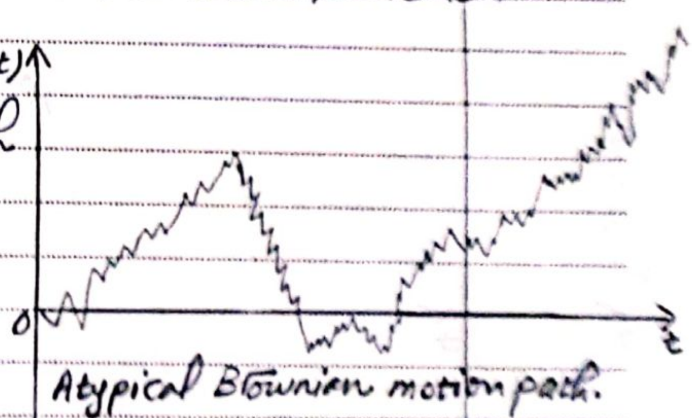
Brownian motion with diffusion coefficient σ^2 is a stochastic process $\{B(t): t\geq 0\}$ with the properties:

(a) Every increment $B(s+t)-B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$, $\sigma^2>0$

(b) $B(t_2)-B(t_1)$, $B(t_4)-B(t_3)$ ...
are independent random variables.

(c) $B(0)=0$, and $B(t)$ is continuous as a function of t.

A typical Brownian motion path is illustrated in Fig. 8.1 p. 395 Textbook (as in opp. Fig.)



A typical Brownian motion path.

* For a standard Brownian motion ($\sigma^2=1$) we have

$$\left.\begin{aligned} &pr\{B(s+t)\le y \mid B(s)=x\} \\ &= pr\{B(s+t)-B(s)\le y-x\} = \Phi\left(\frac{y-x}{\sqrt{t}}\right). \end{aligned}\right\} \quad (7)$$

★ Remark

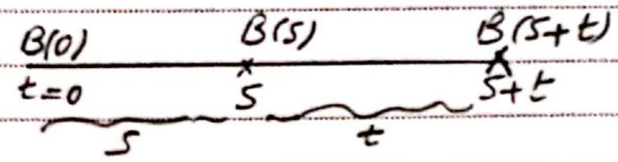
To show that the variance of the displacement in Brownian motion is a linear function of time.

We have

$$B(s+t)-B(0)=[B(s)-B(0)]+[B(s+t)-B(s)]$$

∵ the increment $B(s+t)-B(s)$ depends only on the duration t,

and also $B(s)-B(0)$ depends on the duration s.



∴ We can conclude that the variance of $B(s+t)$ is given by

$$Var[B(s+t)]=Var[B(s)]+Var[B(t)]$$

Thus, the only nonnegative solution to such an equation is to have the variance of the displacement a linear function of time t.

★ The Covariance function

Using the independent increments assumption in Brownian motion we determine the covariance as follows:

$$B(s+t)-B(s)\sim N(0,\sigma^2 t)$$

i.e. $E\{B(t)\}=0$, $Var\{B(t)\}=\sigma^2 t$

$$\therefore E\{[B(t)]^2\}=\sigma^2 t$$

Then, for $0\le s<t$

$$Cov[B(s),B(t)]=E[B(s)B(t)]-\underbrace{E[B(s)]E[B(t)]}_{0}$$

Note

$$Var\{B(t)\}=E\{[B(t)]^2\}-\underbrace{\cancel{[E\{B(t)\}]^2}}_{0}$$

$=\sigma^2 t$

$$\therefore Cov[B(s),B(t)]=E[B(s)B(t)]$$
$$=E[B(s)\{B(t)-B(s)+B(s)\}]$$
$$=E[[B(s)]^2]+E[B(s)\{B(t)-B(s)\}]$$
$$=\underbrace{E[[B(s)]^2]}_{\sigma^2 s}+\underbrace{E[B(s)]}_{0}E[B(t)-B(s)]$$

4/

$\because E[[B(s)]^2] = \sigma^2 s$, $E[B(s)] = 0$, We get

$Cov[B(s), B(t)] = \sigma^2 s$, where $0 \leq s < t$ (1)

Similarly, if $0 \leq t < s$, we obtain

$Cov[B(s), B(t)] = \sigma^2 t$. (2)

Thus, From (1) and (2), we can say that

$Cov[B(s), B(t)] = \sigma^2 \min\{s, t\}$ for $s, t \geq 0$.

Pb 8.1.1 p. 402 Textbook

let $\{B(t): t \geq 0\}$ be a standard Brownian motion

(a) Evaluate $pr\{B(4) \leq 3 \mid B(0) = 1\}$

(b) Find the number c for which $pr\{B(9) > c \mid B(0) = 1\} = 0.10$.

Ans:

(a) $pr\{\underset{t}{B(4)} \leq \underset{y}{3} \mid B(0) = \underset{x}{1}\} = \Phi\left(\frac{y - x}{\sqrt{t}}\right)$

, See Eq. (6)

$= \Phi\left(\frac{3-1}{\sqrt{4}}\right)$

$\therefore pr\{B(4) \leq 3 \mid B(0) = 1\} = \Phi(1) = 0.8413$

from SND Table

(b) $\because pr\{B(9) > c \mid B(0) = 1\} = 0.10$

$\therefore 1 - pr\{B(9) \leq c \mid B(0) = 1\} = 0.10$

$\therefore pr\{\underset{t}{B(9)} \leq \underset{y}{c} \mid B(0) = \underset{x}{1}\} = 0.90$

$\therefore \Phi\left(\frac{c-1}{3}\right) = 0.90$

$\therefore \frac{c-1}{3} = 1.282$ by using SND Table

$\therefore c = 3(1.282) + 1 = 4.846$

5

pb 8.1.2 p. 402 Textbook

Let $\{B(t); t \geq 0\}$ be a standard Brownian motion and $c > 0$ a constant. Show that the process defined by $W(t) = c\,B(t/c^2)$ is a standard Brownian motion.

Ans:

$\because \mathrm{Cov}\{B(s), B(t)\} = \sigma^2 s \quad$ for $0 \leq s < t$

For standard Brownian motion $(\sigma^2 = 1)$, we have

$$\mathrm{Cov}\{B(s), B(t)\} = s \quad \text{for } 0 \leq s < t$$

$$\Rightarrow \mathrm{Cov}\left\{B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right\} = \frac{s}{c^2}, \quad 0 \leq \frac{s}{c^2} < \frac{t}{c^2}$$

Multiply both sides by c^2 we get

$$\therefore c^2\,\mathrm{Cov}\left\{B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right\} = s$$

$$\therefore \mathrm{Cov}\left\{cB\left(\frac{s}{c^2}\right), cB\left(\frac{t}{c^2}\right)\right\} = s$$

$\because W(t) = c\,B(t/c^2)$

Note
$\mathrm{Cov}(\alpha X, \alpha Y) = \alpha^2\,\mathrm{Cov}(X, Y), \alpha > 0$

$$\therefore \mathrm{Cov}\{W(s), W(t)\} = s, \quad 0 \leq s < t \quad (1)$$

Similarly, if $0 \leq t < s$, we can obtain

$$\mathrm{Cov}\{W(s), W(t)\} = t \quad (2)$$

So, from (1) and (2), we conclude that

$$\mathrm{Cov}\{W(s), W(t)\} = \min\{s, t\} \quad \text{for } s, t \geq 0$$

$\therefore W(t) = c\,B(t/c^2)$ is a standard Brownian motion.

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