2.3 Properties of Determinants;

Basic Properties of Determinants

Suppose that A and B are $n \times n$ matrices and k is any scalar. We begin by considering possible relationships among $\det(A)$, $\det(B)$, and

$$\det(kA), \quad \det(A+B), \quad \text{and} \quad \det(AB)$$

$$\det(kA) = k^n \det(A) \tag{1}$$

For example,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Unfortunately, no simple relationship exists among $\det(A)$, $\det(B)$, and $\det(A+B)$. In particular, $\det(A+B)$ will usually *not* be equal to $\det(A) + \det(B)$. The following example illustrates this fact.

EXAMPLE 1 det($A + B$) ≠ det(A) + det(B)

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A+B) = 23$; thus

$$\det(A+B) \neq \det(A) + \det(B)$$

THEOREM 2.3.1 *Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then*

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

Thus

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det\begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

▶ EXAMPLE 2 Sums of Determinants

We leave it to you to confirm the following equality by evaluating the determinants.

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$ ◀

Determinant Test for Invertibility

Our next theorem provides an important criterion for determining whether a matrix is invertible.

THEOREM 2.3.3 *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

▶ **EXAMPLE 3 Determinant Test for Invertibility**

Since the first and third rows of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

are proportional, $\det(A) = 0$. Thus A is not invertible. ◀

THEOREM 2.3.4 *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B)$$

EXAMPLE 4 Verifying that $\det(AB) = \det(A)\det(B)$

Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

We leave it for you to verify that

$$\det(A) = 1, \quad \det(B) = -23, \quad \text{and} \quad \det(AB) = -23$$

Thus $\det(AB) = \det(A)\det(B)$, as guaranteed by Theorem 2.3.4.

THEOREM 2.3.5 *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof Since $A^{-1}A = I$, it follows that $\det(A^{-1}A) = \det(I)$. Therefore, we must have $\det(A^{-1})\det(A) = 1$. Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$. ◀

Adjoint of a Matrix

In a cofactor expansion we compute $\det(A)$ by multiplying the entries in a row or column by their cofactors and adding the resulting products. It turns out that if one multiplies the entries in any row by the corresponding cofactors from a *different* row, the sum of these products is always zero. (This result also holds for columns.) Although we omit the general proof, the next example illustrates this fact.

EXAMPLE 5 Entries and Cofactors from Different Rows

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

We leave it for you to verify that the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so, for example, the cofactor expansion of $\det(A)$ along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

Suppose, however, we multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products. The result is

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Or suppose we multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products. The result is again zero since

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$ ◀

DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from*** A. The transpose of this matrix is called the ***adjoint of*** A and is denoted by $\text{adj}(A)$.

EXAMPLE 6 Adjoint of a 3 x 3 Matrix

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Remark

For every $n \times n$ matrix A, we have

$$A \cdot \mathrm{adj}(A) = (\det(A))I_n$$

EXAMPLE

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

We leave it for you to verify that the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

and the adjoint of A is

$$\mathrm{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Therefore,

$$A\,\mathrm{adj}(A) = \det(A)I$$

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \tag{6}$$

EXAMPLE 7 Using the Adjoint to Find an Inverse Matrix

Use Formula (6) to find the inverse of the matrix A in Example 6.

Solution We showed in Example 5 that $\det(A) = 64$. Thus,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

THEOREM 2.3.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A can be expressed as a product of elementary matrices.*

(*g*) $\det(A) \neq 0$.