111 ريض - حساب التكامل
الفصل الدراسي الأول 1444 هـ
حل الاختبار الفصلي
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السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_{-1}^{4} (2x+1)\, dx$.

الحل : $[a,b]=[-1,4]$ و $f(x)=2x+1$.

$$\Delta_x = \frac{b-a}{n} = \frac{4-(-1)}{n} = \frac{4+1}{n} = \frac{5}{n}$$

$$x_k = a + k\,\Delta_x = -1 + k\left(\frac{5}{n}\right) = -1 + \frac{5k}{n}$$

$$f(x_k) = 2\left(-1+\frac{5k}{n}\right) + 1 = -2 + \frac{10k}{n} + 1 = \frac{10k}{n} - 1$$

$$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{10k}{n}-1\right)\ \left(\frac{5}{n}\right) = \sum_{k=1}^{n}\left(\frac{50k}{n^2}-\frac{5}{n}\right) = \sum_{k=1}^{n}\frac{50k}{n^2} - \sum_{k=1}^{n}\frac{5}{n}$$

$$= \frac{50}{n^2}\sum_{k=1}^{n} k - \frac{5}{n}\sum_{k=1}^{n} 1 = \frac{50}{n^2}\ \frac{n(n+1)}{2} - \frac{5}{n}\,(n) = 25\left(\frac{n+1}{n}\right) - 5$$

$$\int_{-1}^{4}(2x+1)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[25\left(\frac{n+1}{n}\right)-5\right] = 25(1) - 5 = 20$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\sqrt{3x}}^{\ln 2x} \frac{1}{\sqrt{t^2-2t}}\, dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{\sqrt{3x}}^{\ln 2x} \frac{1}{\sqrt{t^2-2t}}\, dt$$

$$= \frac{1}{\sqrt{(\ln 2x)^2 - 2\ln 2x}}\ \left(\frac{2}{2x}\right) - \frac{1}{\sqrt{\left(\sqrt{3x}\right)^2 - 2\sqrt{3x}}}\ \left(\frac{1}{2\sqrt{3x}}\,(3)\right)$$

$$= \frac{1}{x\sqrt{(\ln 2x)^2 - 2\ln 2x}} - \frac{3}{2\sqrt{3x}\,\sqrt{3x-2\sqrt{3x}}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 7^{x^2}\ \cosh^{-1}\left(x^2\right) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(7^{x^2}\ (2x)\ln 7\right)\ \cosh^{-1}\left(x^2\right) + 7^{x^2}\left[\frac{1}{\sqrt{\left(x^2\right)^2 - 1}}\,(2x)\right]$$

$$= 2x\ 7^{x^2}\ \ln 7\ \cosh^{-1}\left(x^2\right) + \frac{2x\ 7^{x^2}}{\sqrt{x^4-1}}$$

$$y = \sin^{-1}(2x)\ \log_5|3-\tan 6x| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-(2x)^2}}\,(2)\right)\log_5|3-\tan 6x| + \sin^{-1}(2x)\left(\frac{-\sec^2(6x)\ (6)}{3-\tan 6x}\ \frac{1}{\ln 5}\right)$$

$$= \frac{2\log_5|3-\tan 6x|}{\sqrt{1-4x^2}} - \frac{6\sec^2(6x)\ \sin^{-1}(2x)}{(3-\tan 6x)\ \ln 5}$$

$$y = (\tan x)^{\sin x} \quad (5)$$

الحل :

$$y = (\tan x)^{\sin x} \implies \ln|y| = \ln\left|(\tan x)^{\sin x}\right| = \sin x\ \ln|\tan x|$$

بإشتقاق الطرفين

$$\frac{y'}{y} = (\cos x)\ \ln|\tan x| + \sin x\left(\frac{\sec^2 x}{\tan x}\right)$$

$$y' = y\left[\cos x\ \ln|\tan x| + \frac{\sin x\ \sec^2 x}{\tan x}\right]$$

$$y' = (\tan x)^{\sin x}\ \left[\cos x\ \ln|\tan x| + \sec x\right]$$

$$y = \coth^{-1}\left(\frac{1}{x}\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1-\left(\frac{1}{x}\right)^2}\ \left(\frac{-1}{x^2}\right) = \frac{1}{x^2\left[1-\frac{1}{x^2}\right]} = \frac{1}{x^2-1}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$\int \frac{\csc^2\left(e^{2x}\right)}{e^{-2x}}\,dx \quad (1)$$

الحل :

$$\int \frac{\csc^2\left(e^{2x}\right)}{e^{-2x}}\,dx = \int \csc^2\left(e^{2x}\right)\ e^{2x}\,dx = \frac{1}{2}\int \csc^2\left(e^{2x}\right)\ e^{2x}\,(2)\,dx$$

$$= \frac{1}{2}\left(-\cot\left(e^{2x}\right)\right) + c = -\frac{1}{2}\cot\left(e^{2x}\right) + c$$

باستخدام القانون

$$\int \csc^2\left(f(x)\right)\ f'(x)\,dx = -\cot\left(f(x)\right) + c$$

$$\int \left(x^3+4\right)^3\ x^2\,dx \quad (2)$$

الحل :

$$\int \left(x^3+4\right)^3\ x^2\,dx = \frac{1}{3}\int \left(x^3+4\right)^3\ (3x^2)\,dx = \frac{1}{3}\ \frac{(x^3+4)^4}{4} + c$$

باستخدام القانون $\int [f(x)]^n\ f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$\int_0^1 x\ 10^{2-x^2}\,dx \quad (3)$$

الحل :

$$\int_0^1 x\ 10^{2-x^2}\,dx = \frac{1}{-2}\int_0^1 10^{2-x^2}(-2x)\,dx = \frac{1}{-2}\left[\frac{10^{2-x^2}}{\ln 10}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{10^{2-(1)^2}}{\ln 10} - \frac{10^{2-(0)^2}}{\ln 10}\right] = \frac{1}{-2}\left[\frac{10^1}{\ln 10} - \frac{10^2}{\ln 10}\right] = \frac{1}{-2}\left(\frac{-90}{\ln 10}\right) = \frac{45}{\ln 10}$$

باستخدام القانون $\int a^{f(x)}\ f'(x)\,dx = \dfrac{a^{f(x)}}{\ln a} + c$.

$$\int \frac{x+2}{x+1}\,dx \quad (4)$$

الحل :

$$\int \frac{x+2}{x+1}\,dx = \int \frac{(x+1)+1}{x+1}\,dx = \int \left(\frac{x+1}{x+1} + \frac{1}{x+1}\right)\,dx$$

$$= \int \left(1 + \frac{1}{x+1}\right)\,dx = \int 1\,dx + \int \frac{1}{x+1}\,dx = x + \ln|x+1| + c$$

باستخدام القانون $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$.

(5) $\int \frac{e^{2x}}{\sqrt{4+e^{4x}}}\,dx$

الحل :

$$\int \frac{e^{2x}}{\sqrt{4+e^{4x}}}\,dx = \int \frac{e^{2x}}{\sqrt{(2)^2+(e^{2x})^2}}\,dx = \frac{1}{2}\int \frac{2\,e^{2x}}{\sqrt{(2)^2+(e^{2x})^2}}\,dx$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{e^{2x}}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$

(6) $\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\,dx$

الحل :

$$\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\,dx = \int \tanh\left(\sqrt{x}\right)\;\left(\frac{1}{\sqrt{x}}\right)\,dx = 2\int \tanh\left(\sqrt{x}\right)\;\left(\frac{1}{2\sqrt{x}}\right)\,dx$$

$$= 2\ln\left|\cosh\left(\sqrt{x}\right)\right| + c$$

باستخدام القانون

$$\int \tanh(f(x))\;f'(x)\,dx = \ln|\cosh(f(x))| + c$$

(7) $\int \frac{3}{9x^2+4}\,dx$

الحل :

$$\int \frac{3}{9x^2+4}\,dx = \int \frac{3}{(3x)^2+(2)^2}\,dx = \frac{1}{2}\;\tan^{-1}\left(\frac{3x}{2}\right) + c$$

باستخدام القانون

$$. \int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right)+c$$

$$\int (x+3)\,\sqrt{x+2}\,dx \quad (8)$$

الحل : بوضع $u = x+2 \implies u+1 = x+3$

$$dx = du$$

$$\int (x+3)\,\sqrt{x+2}\,dx = \int (u+1)\sqrt{u}\,du = \int (u+1)u^{\frac{1}{2}}\ du$$

$$= \int \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)\ du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}}+\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{2}{5}\ (x+2)^{\frac{5}{2}}+\frac{2}{3}\ (x+2)^{\frac{3}{2}}+c$$

$$\int e^{3x}\cosh x\,dx \quad (9)$$

الحل :

$$\int e^{3x}\cosh x\,dx = \int e^{3x}\left(\frac{e^{x}+e^{-x}}{2}\right)\ dx = \int \left(\frac{e^{4x}+e^{2x}}{2}\right)\ dx$$

$$= \int \left(\frac{e^{4x}}{2}+\frac{e^{2x}}{2}\right)\ dx = \frac{1}{2}\,\frac{1}{4}\int e^{4x}\,(4)\,dx + \frac{1}{2}\,\frac{1}{2}\int e^{2x}(2)\,dx$$

$$= \frac{1}{8}\,e^{4x}+\frac{1}{4}\,e^{2x}+c = \frac{e^{4x}}{8}+\frac{e^{2x}}{4}+c$$

باستخدام القانون $\int e^{f(x)}\ f'(x)\,dx = e^{f(x)}+c$