King Saud University: Mathematics Department Math-254
Second Semester 1443 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:—————————— **I.D. No.** ——————————

Name of the Teacher:—————————— **Section No.** ——————————

Note: Check the total number of pages are Seven (7).

(**16 Multiple choice questions and Three (3) Full questions**)

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d																

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 16		24
Q. 17		6
Q. 18		5
Q. 19		5
Total		40

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATH1)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	b	a	b	b	a	c	b	c	a	c	b	a	c	a	c	a

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATH2)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	a	c	c	a	d	b	a	a	c	b	a	b	b	c	a	b

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATH3)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	c	b	a	c	b	a	c	b	b	a	c	c	a	b	b	c

Question 17: Use LU decomposition by Dollittle's method to find the value(s) of nonzero α for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix},$$

is inconsistent and consistent. Solve the consistent system.

Solution. Since we know that

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = 2\alpha/\alpha = 2 = l_{21}$, $m_{31} = 1/\alpha = l_{31}$, and $m_{32} = \dfrac{3\alpha - 4}{(-9\alpha)} = l_{32}(\alpha \neq 0)$, gives

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & (3\alpha-4)/\alpha & (\alpha^2-1)/\alpha \end{pmatrix} \equiv \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & (\alpha^2-1)/\alpha \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/\alpha & (3\alpha-4)/(-9\alpha) & 1 \end{pmatrix} \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & (\alpha^2-1)/\alpha \end{pmatrix},$$

which is the required decomposition of A. The given linear system has no solution or infinitely many solution if

$$det(A) = det(U) = -9\alpha(\alpha^2 - 1)/\alpha = (\alpha^2 - 1) = 0,$$

gives, $\alpha = -1$ or $\alpha = 1$.
To find the solution of the given system when $\alpha = -1$ and it gives

$$\begin{pmatrix} -1 & 4 & 1 \\ -2 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -7/9 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -7/9 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 4]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 4 \end{pmatrix}.$$

Last row gives, $0x_1 + 0x_2 + 0x_3 = 4$, which is not possible, and so no solution. To find the solution of the given system when $\alpha = 1$ and it gives

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/9 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/9 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 0]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 0 \end{pmatrix}.$$

Last row gives, $0x_1 + 0x_2 + 0x_3 = 0$, which means we have many solutions. Performing backward substitution yields

$$\begin{array}{ccccccc} x_1 & + & 4x_2 & + & x_3 & = & 6 \\ & & -9x_2 & & & = & -9 \end{array}$$

and it gives, $[x_1, x_2, x_3]^T = [2 - t, 1, t]^T$, for any nonzero t.

Question 18: Use the following table to find the best approximation of $f(0.6)$ by using quadratic Lagrange interpolating polynomial for equally spaced data points

x	0.15	0.2	0.3	0.5	0.55	0.8	1
$f(x)$	-0.0427	-0.0644	-0.1286	-0.1733	-0.1808	-0.1428	0

The above table is for $f(x) = x^2 \ln x$. Compute the absolute error and an error bound for the approximation.

Solution. Since given $x = 0.6$, therefore, the best points for the equally spaced data points Lagrange quadratic polynomial are, $x_0 = 0.3, x_1 = 0.55$, and $x_2 = 0.8$ with $h = 0.25$. Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{1}$$

$$f(0.6) \approx p_2(0.6) = L_0(0.6)(-0.1286) + L_1(0.6)(-0.1808) + L_2(0.6)(-0.1428). \tag{2}$$

The Lagrange coefficients can be calculate as follows:

$$L_0(0.6) = \frac{(0.6-0.55)(0.6-0.8)}{(0.3-0.55)(0.3-0.8)} = -2/25 = -0.08,$$

$$L_1(0.6) = \frac{(0.6-0.3)(0.6-0.8)}{(0.55-0.3)(0.55-0.8)} = 24/25 = 0.96,$$

$$L_2(0.6) = \frac{(0.6-0.3)(0.6-0.55)}{(0.8-0.3)(0.8-0.55)} = 3/25 = 0.12.$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-1/9))(-0.0644) + (8/9)(-0.1733) + (2/9)(-0.1428) = -0.1821,$$

which is the required approximation of the given exact solution $0.36 + \ln 0.6 \approx -0.1839$.
The desired absolute error is

$$|f(0.6) - p_2(0.6)| = |0.36 + \ln 0.6 - (-0.17854| = |-0.1839 + 0.1821| = 0.0018.$$

To compute an error bound for the approximation of the given function in the interval $[0.3, 0.8]$, we use the following quadratic error formula

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}}.$$

As

$$M = \max_{0.3 \leq x \leq 0.8} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x \ln x + x, \qquad f''(x) = 2 \ln x + 3, \qquad f^{(3)}(x) = \frac{2}{x},$$

$$M = \max_{0.3 \leq x \leq 0.8} \left| \frac{2}{x} \right| = 20/3 = 6.6667.$$

Hence

$$|f(0.6) - p_2(0.6)| \leq \frac{(6.6667)(0.25)^3}{9\sqrt{(3)}} = 0.0067,$$

which is desired error bound.

Question 19: Approximate the integral $\int_1^{1.8} \frac{(e^x + e^{-x})}{2}\,dx$ by using best composite rule for $h = 0.2$. Compute the absolute error and an error bound for the approximation.

Solution. Given $f(x) = \frac{(e^x + e^{-x})}{2}$, and $h = 0.2$, gives $n = 4$. we need the equally spaced data points, so we have to take $x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6$ and $x_4 = 1.8$.
By using the best composite formula for $n = 4$, we have:

Simpson's Rule
By using the composite formula for $n = 4$, we have

$$I(f) = \int_1^{1.8} \frac{(e^x + e^{-x})}{2}\,dx \approx \frac{0.2}{3}\Big[f(1) + 4[f(1.2) + f(1.6)] + 2f(1.4) + f(1.8)\Big].$$

Now using the given values, we obtain

$$I(f) = \int_1^{1.8} \frac{(e^x + e^{-x})}{2}\,dx \approx \frac{0.2}{3}[1.5431 + 4(1.8107 + 2.5775) + 2(2.1509) + 3.1075] = 1.7670,$$

the required approximation. The absolute error is

$$|I(f) - S_4(f)| = |1.7670 - 1.7670| = 0.0000.$$

To find error bound we first compute the derivatives of the given function as

$$f'(x) = \frac{(e^x - e^{-x})}{2} \quad \text{and} \quad f''(x) = \frac{(e^x + e^{-x})}{2}.$$

$$f'''(x) = \frac{(e^x - e^{-x})}{2} \quad \text{and} \quad f^{iv}(x) = \frac{(e^x + e^{-x})}{2}.$$

Since $\eta(x)$ is unknown point in $(1, 1.8)$, therefore, the bound $|f^{iv}|$ on $[1, 1.8]$ is

$$M = \max_{1 \le x \le 1.8} |f^{iv}(x)| = \left|\frac{(e^x + e^{-x})}{2}\right| = 3.1075.$$

Thus the error bound formula becomes

$$|E_{S_4}(f)| \le \frac{(0.2)^4(1.8 - 1)}{180}(3.1075) = 2.2098 \times 10^{-5},$$

which is the possible maximum error in our approximation using Simpson's rule.