SECOND MID TERM EXAMINATION, SEMESTER I, 2023-24
DEPT. OF MATHEMATICS, COLLEGE OF SCIENCE, KSU
MATH: 107 FULL MARK: 25 TIME: 90 MINUTES

Q1. Marks. 2+2+3=7

Let $\mathbf{u} = \langle 3,1,0\rangle$, $\mathbf{v} = \langle 1,2,3\rangle$, and $\mathbf{w} = \langle -1,2,1\rangle$. Then

(a) Find all values of λ such that the vectors $\mathbf{u}$ and $(\lambda\mathbf{v} + \mathbf{w})$ are orthogonal.

(b) Find the $comp_{\mathbf{u}}\mathbf{w}$ and $comp_{\mathbf{u}}(-5\mathbf{w})$.

(c) Find the volume of the parallelepiped (box) with adjacent sides $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, and $\overrightarrow{AD} = \mathbf{w}$.

Q2. Marks. 3+3=6

Let P_1 and P_2 be two planes defined by their equations:

$$P_1: \; x - 2y + 2z = 3$$
$$P_2: \; 2x + 2y - z = 1$$

(a) Find the parametric equations of the line of intersection of the planes P_1 and P_2.

(b) Identify the surface $36x^2 + 9y^2 - 4z^2 - 36 = 0$, and sketch it.

Q3. Marks. 4+4+4=12

(a) Find the velocity, acceleration and speed of a moving point P at $t = \frac{\pi}{2}$ along $\mathbf{r}(t) = t(\cos 2t\mathbf{i} - \cos t\mathbf{j} + t\mathbf{k})$.

(b) Find the curvature, radius of curvature, and center of curvature for the curve $y = x^3$ at $P(2,8)$.

(c) If $\mathbf{r}(t) = 3t\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$, find the tangential and normal components of acceleration at $P(6,4,4)$.

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Model Answer for SMT Exam S1/1445

Q1 (7 Marks): 2+2+3

$\vec{u} = \langle 3,1,0\rangle$, $\vec{v} = \langle 1,2,3\rangle$ and $\vec{w} = \langle -1,2,1\rangle$

(a) $\vec{u}\cdot(\lambda\vec{v}+\vec{w}) = \langle 3,1,0\rangle\cdot\langle \lambda-1, 2\lambda+2, 3\lambda+1\rangle$
$= 3(\lambda-1)+2\lambda+2$

$\vec{u}$ and $(\lambda\vec{v}+\vec{w})$ are orthogonal iff
$\vec{u}\cdot(\lambda\vec{v}+\vec{w}) = 0$

i.e $\Leftrightarrow 3\lambda-3+2\lambda+2=0$
$\Leftrightarrow 5\lambda=1 \Leftrightarrow \boxed{\lambda=\frac{1}{5}}$ (2)

(b) $\text{Comp}_{\vec{u}}\vec{w} = \dfrac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|} = \dfrac{-3+2}{\sqrt{9+1}} = \dfrac{-1}{\sqrt{10}}$ (1)

, $\text{Comp}_{\vec{u}}(-5\vec{w}) = \dfrac{\langle 3,1,0\rangle\cdot\langle 5,-10,-5\rangle}{\sqrt{10}}$ (2)

$= \dfrac{15-10}{\sqrt{10}} = \dfrac{5}{\sqrt{10}}$ (1)

(c)

$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 3(2-6)-1(1+3)$

$= -12-4 = -16$

Volume $= |-16| = 16$ unit^3 (3)

Q2 (6 Marks): 3+3

(a) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 2 & 2 & -1 & 1 \end{array}\right] \xrightarrow{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 6 & -5 & -5 \end{array}\right]$

$\xrightarrow{\frac{1}{6}r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & \frac{-5}{6} & \frac{-5}{6} \end{array}\right]$

$\Rightarrow x-2y+2z=3$
$y-\frac{5}{6}z = -\frac{5}{6}$

let $z=t$, $t\in\mathbb{R}$

$\Rightarrow y = -\frac{5}{6}+\frac{5}{6}t$, $x = 3-\frac{5}{3}+\frac{5}{3}t-2t$ (3)

$\therefore x = \frac{4}{3}-\frac{1}{3}t$, $y = -\frac{5}{6}+\frac{5}{6}t$, $z=t$,
where $t\in\mathbb{R}$
are the parametric Eqns of the line of intersection of the planes P_1 and P_2.

* حل آخر الفقرة (a) في السؤال الثالث Q2 (a)

* The two planes are

$$P_1: x-2y+2z=3 \Rightarrow n_1=\langle 1,-2,2\rangle$$

$$P_2: 2x+2y-z=1 \Rightarrow n_2=\langle 2,2,-1\rangle$$

So, $\vec{a}=\vec{n_1}\times\vec{n_2}$

$$\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 2\\ 2 & 2 & -1\end{vmatrix}=-2\vec{i}+5\vec{j}+6\vec{k}$$

* Which is the direction vector for the intersection line of the two planes.

To get a point on that line, let $z=0$

$$\Rightarrow \begin{aligned} x-2y&=3 \quad (1)\\ 2x+2y&=1 \quad (2)\end{aligned}$$

By adding (1) & (2), we obtain

$$3x=4 \Rightarrow x=\frac{4}{3}$$

$$\therefore y=\left(\frac{4}{3}-3\right)\times\frac{1}{2}=-\frac{5}{6}$$

* $\therefore P\left(\frac{4}{3},\frac{-5}{6},0\right)$ is a point lies on the intersection line

So, The parametric Eqns of this line are given by

$$x=\frac{4}{3}-2t,\quad y=\frac{-5}{6}+5t \text{ and } z=6t,\quad t\in\mathbb{R}$$

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(b) $36x^2+9y^2-4z^2-36=0$

$\Rightarrow \frac{x^2}{1}+\frac{y^2}{4}-\frac{z^2}{9}=1$ 1 3

Hyperboloid of one sheet with z-axis 1

+ graph 1

Q3 (12 Marks): 4+4+4

(a) $\vec{r}(t)=t(\cos 2t\,\vec{i}-\cos t\,\vec{j}+t\,\vec{k})$

velocity $=t\cos 2t\,\vec{i}-t\cos t\,\vec{j}+t^2\vec{k}$

$\Rightarrow \vec{r}'(t)=(-2t\sin 2t+\cos 2t)\,\vec{i}$

Acceleration $+(t\sin t-\cos t)\vec{j}+2t\,\vec{k}$ 1

$\Rightarrow \vec{r}''(t)=(-4t\cos 2t-2\sin 2t-2\sin 2t)\vec{i}$

$+(t\cos t+\sin t+\sin t)\vec{j}+2\vec{k}$

$\vec{r}''(t)=(-4t\cos 2t-4\sin 2t)\vec{i}+(t\cos t+2\sin t)\vec{j}+2\vec{k}$ 1

At $t=\pi/2$, 4

$\vec{v}=\vec{r}'(\pi/2)=\langle -1,\pi/2,\pi\rangle$ 1

$\vec{a}=\vec{r}''(\pi/2)=\langle 2\pi,2,2\rangle$

Speed $=\|\vec{v}\|=\sqrt{1+\frac{\pi^2}{4}+\pi^2}=\sqrt{1+\frac{5\pi^2}{4}}$ 1

$\simeq 3.65$

(b) $y=x^3,\ y'=3x^2,\ y''=6x$ 1 4

$K=\frac{|y''|}{[1+(y')^2]^{3/2}}=\frac{|6x|}{(1+9x^4)^{3/2}}$ 1

$K\big|_{(2,8)}=\frac{12}{(145)^{3/2}}$, $\rho=\frac{(145)^{3/2}}{12}$ 1

Curvature radius of curvature

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$$h = x - \frac{y'[1+(y')^2]}{y''}$$

$$k = y + \frac{1+(y')^2}{y''}$$

$$\Rightarrow h = 2 - \frac{\cancel{12}(145)}{\cancel{12}} = -143,$$

$$k = 8 + \frac{145}{12} = \frac{241}{12} \simeq 20.08$$ (1)

The center of curvature is $\left(-143, \frac{241}{12}\right)$ #

(c) $\vec{r}(t) = \langle 3t, t^2, t^2 \rangle$

$$\Rightarrow \vec{r}'(t) = \langle 3, 2t, 2t \rangle, \quad \vec{r}''(t) = \langle 0, 2, 2 \rangle$$ (1)

$$a_T = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{\|\vec{r}'(t)\|} = \frac{\langle 3, 2t, 2t\rangle \cdot \langle 0, 2, 2\rangle}{\sqrt{9+4t^2+4t^2}}$$

$$\Rightarrow a_T = \frac{0+4t+4t}{\sqrt{9+8t^2}} = \frac{8t}{\sqrt{9+8t^2}}$$

Note that at $P(6,4,4)$ $\Rightarrow t=2$

$$a_T\Big|_{t=2} = \frac{16}{\sqrt{9+32}} = \frac{16}{\sqrt{41}}$$ (1)

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$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2t & 2t \\ 0 & 2 & 2 \end{vmatrix}$$

$$= (4t-4t)\vec{i} - (6-0)\vec{j} + (6-0)\vec{k}$$

$$= 0\vec{i} - 6\vec{j} + 6\vec{k}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle 0, -6, 6 \rangle$$ (1)

$$a_N\Big|_{t=2} = \frac{\|\vec{r}'(2) \times \vec{r}''(2)\|}{\|\vec{r}'(2)\|}$$

$$= \frac{\sqrt{2\times 36}}{\sqrt{9+16+16}} = \frac{6\sqrt{2}}{\sqrt{41}}$$ (1)

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