King Saud University
College of Sciences
Department of Mathematics

First Mid Term Mu Exam,
S1 1441-1442
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [5+3]

a) Given the following joint distribution. Calculate E(X), E(Y), Var(X), Var(Y), Cov(X,Y), ρ(X,Y), and verify E(X) using the law of total Expectation.

Y \ X	0	1
0	0.1	0.3
1	0.4	0.2

b) The lifetime, in years, of a certain class of light bulbs has an exponential distribution with parameter $\lambda = 2$. What is the probability that a bulb selected at random from this class will last more than 1.5 years? What is the probability that a bulb selected at random will last exactly 1.5 years?

Q2: [4+4]

a) Given independent exponentially distributed random variables S and T with common parameter λ, determine the probability density function of the sum R=S+T and identify its type by name.

b) Let X and Y two random variables have the joint normal (bivariate normal) distribution. What value of α that minimizes the variance of $Z=\alpha X+(1-\alpha)Y$? Simplify your result when X and Y are independent

Q3: [3+3+3]

a) Let X and Y are jointly distributed random variables having the density function $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$ for x,y >0 find $f_{X|Y}(x|y)$

b) If $T \sim \exp(\lambda)$ prove that: $pr(T > t + s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

c) Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10000 miles. If a person desires to take a 9000-mile trip, what is the probability that he will be able to complete his trip without having to replace the car battery?

The Model Answer

Q1: [5+3]

a)

X \ Y	0	1	$P_Y(y)$
0	0.1	0.3	0.4
1	0.4	0.2	0.6
$P_X(x)$	0.5	0.5	Sum=1

$E(X)=0.5,\ E(X^2)=0.5,\ Var(X)=0.25$

$E(Y)=0.6,\ E(Y^2)=0.6,\ Var(Y)=0.24$

$E(XY)=0.2,\ Cov(X,Y)=-0.10,\ \rho(X,Y)=-0.4$

$$P(X|Y=y)=\frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P(X=0|Y=0)=\frac{0.1}{0.4}=\frac{1}{4},\quad P(X=1|Y=0)=\frac{0.3}{0.4}=\frac{3}{4}$$

$$P(X=0|Y=1)=\frac{0.4}{0.6}=\frac{2}{3},\quad P(X=1|Y=1)=\frac{0.2}{0.6}=\frac{1}{3}$$

$X\|Y$	0	1	$E[X\|Y]$
y=0	1/4	3/4	3/4
y=1	2/3	1/3	1/3

$$E(X)=\sum_y E(X|Y=y)P_Y(y)$$

$$E(X)=\frac{3}{4}P_Y(0)+\frac{1}{3}P_Y(1)$$

$$E(X)=\frac{3}{4}(0.4)+\frac{1}{3}(0.6)=0.5$$

b) $X \sim \exp(2)$

i) Pr(T>1.5)=$e^{-3} = 0.0498$

ii) Pr(T=1.5)=0

Q2: [4+4]

a)

$\because$ S, T ~ exp(λ) , R=S+T

$\therefore$ R ~ Gamma(2,λ)

$\Rightarrow f_R(r) = \frac{\lambda^2}{\Gamma(2)} r^{2-1} e^{-\lambda r}$

$\therefore f_R(r) = \lambda^2 r e^{-\lambda r}, \quad r \geq 0$

Which is the Gamma probability density function.

b) $Z = \alpha X + (1-\alpha)Y$

Var(Z)=$\alpha^2\sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$

To get α^* that minimizes Var(Z) let $\frac{\partial V}{\partial \alpha} = 0$

$\Rightarrow$

$$\alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}$$

For independent random variables X and Y , ρ=0

Consequently, $\alpha^* = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$

Q3: [3+3+3]

a) $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$ for x, y >0

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty f(x,y)\ dx$$

$$= \int_0^\infty \frac{1}{y} e^{-(x/y)-y}\ dx$$

$$= \frac{e^{-y}}{y} \int_0^\infty e^{-x/y}\ dx$$

$$= \frac{e^{-y}}{y} \left[\frac{e^{-x/y}}{-1/y} \right]_0^\infty$$

$$= e^{-y} [0+1]$$

$$\therefore\ f_Y(y) = e^{-y},\ y>0$$

Note: $\lim_{x\to\infty} e^{-x} = 0$

$$\therefore\ f_{X|Y}(x|y) = \frac{1}{y} e^{-x/y} \quad \text{for x, y >0}$$

b)

$$pr(T > t+s | T > s) = \frac{pr(T > t+s, T > s)}{pr(T > s)}$$

$$= \frac{pr(T > t+s)}{pr(T > s)}$$

$$\because\ T \sim \exp(\lambda)$$

$$\therefore\ pr(T > t+s | T > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = R(t)$$

$$= pr(T > t)$$

c)

$$\because\ X \sim \exp(\tfrac{1}{10000})$$

$$\therefore\ Pr(X>9000) = e^{-0.9}$$

$$\approx 0.4066$$