King Saud University
College of Sciences
Department of Mathematics

First Mid Term Mu Exam,
S2 1442
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+5]

a) The joint probability density function of the two random variables X and Y is $f(x,y)=8xy$, $0 \le x \le y \le 1$. Find $f_{Y|X}(y|\frac{1}{3})$

b) Given the joint probability mass function of two random variables X and Y as in the following table:

X \ Y	1	2	3
0	1/8	0	0
1	0	1/4	1/8
2	0	1/4	1/8
3	1/8	0	0

i) Find $\rho(X,Y)$

ii) Determine whether X and Y are two independent random variables or not? Justify your answer.

Q2: [4+4]

a) Let $X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \ldots\ +\xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k)=\mu$, $E(N)=\upsilon$ and $Var(\xi_k)=\sigma^2$, $Var(N)=\tau^2$

Prove that $E(X)=\mu\upsilon$ and $Var(X)=\upsilon\sigma^2+\mu^2\tau^2$

b) The following experiment is performed: An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials.

i) Formulate Z as a random sum and thereby determine its mean and variance.

ii) What is the distribution of Z?

Q3: [3+2+3]

An oil drilling company drills at a large number of locations in search of oil. The probability of success at any location is 0.25 and the locations may be regarded as independent.

a) What is the probability that the driller will experience 1 success if 10 locations are drilled?

b) The driller feels that he will go bankrupt if he drills 10 times before experiencing his first success. What is the probability that he will go bankrupt?

c) What is the probability that he will get the first success on the 10^{th} trial?

The Model Answer

Q1: [4+5]

a) $f_{Y|X}(y|x)=\dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$f_{X,Y}(x,y)=8xy\ ,\ \ 0\le x\le y\le 1$$

$$\because\ f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_{x}^{1} 8xydy$$

$$=8x\left[\frac{y^2}{2}\right]_x^1$$

$$\therefore\ f_X(x)=4x(1-x^2),\qquad 0\le x\le 1$$

$$\therefore\ f_{Y|X}(y|x)=\frac{8xy}{4x(1-x^2)}$$

$$=\frac{2y}{1-x^2}$$

$$f_{Y|X}(y|\tfrac{1}{3})=\tfrac{9}{4}y,\qquad 0\le y\le 1$$

b)

X \ Y	1	2	3	$P_X(x)$
0	1/8	0	0	1/8
1	0	1/4	1/8	3/8
2	0	1/4	1/8	3/8
3	1/8	0	0	1/8
$P_Y(y)$	2/8	4/8	2/8	Sum=1

$E(X)=\frac{3}{2}$, $E(X^2)=3$, $Var(X)=\frac{3}{4}$

$E(Y)=2$, $E(Y^2)=\frac{9}{2}$, $Var(Y)=\frac{1}{2}$

$E(XY)=3$

$Cov(X,Y)=E(XY)-E(X)E(Y)=0$

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=0$$

$\Rightarrow$ X and Y are not correlated

$\because$ for example, $P(X=1,Y=1)=0$, but $P(X=1)P(Y=1)=\frac{3}{8}(\frac{2}{8})=\frac{3}{32}$

$\Rightarrow P(X=1,Y=1)\neq P(X=1)P(Y=1)$

$\therefore$ X and Y are not independent r.vs

Q2: [4+4]

a)

(i) To prove that $E(X)=\mu\upsilon$

$$\because\ E(X)=\sum_{n=0}^{\infty}E[X|N=n]P_N(n)$$ Def. of Total Expectation

$$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_N|N=n]P_N(n)$$ Def. of Random Sum

$$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_n|N=n]P_N(n)$$ Prop. of Conditional Expectation

$$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_n]P_N(n)$$ where N is independent of $\xi_1, \xi_2, ...$

$\because\ E(\xi_k)=\mu,\quad k=1,2,\ ...\ ,n$

$$\therefore\ E(X)=\sum_{n=1}^{\infty}n\mu P_N(n)$$

$$\therefore\ E(X)=\mu\sum_{n=1}^{\infty}nP_N(n)$$

$\therefore\ E(X) = \mu E(N) = \mu\upsilon$

(ii) To prove that $\text{Var(X)}=\upsilon\sigma^2 + \mu^2\tau^2$

$$\text{Var(X)}=\text{E}[(\text{X}-\mu\upsilon)^2]$$
$$=\text{E}[\text{X}-N\mu+N\mu-\upsilon\mu]^2$$
$$\text{Var(X)}=\text{E}[(\text{X}-N\mu)^2]+\text{E}[\mu^2(N-v)^2]+2\text{E}[\mu(\text{X}-N\mu)(N-v)] \qquad (1)$$

$$\because \text{E}[(\text{X}-N\mu)^2]=\sum_{n=0}^{\infty}\text{E}[(\text{X}-N\mu)^2 \,|N=n]P_N(n)$$
$$=\sum_{n=1}^{\infty}\text{E}[(\xi_1+\xi_2+\ \dots\ +\xi_n-n\mu)^2 \,|N=n]P_N(n)$$
$$\therefore \text{E}[(\text{X}-N\mu)^2]=\sum_{n=1}^{\infty}\text{E}(\xi_1+\xi_2+\ \dots\ +\xi_n-n\mu)^2]P_N(n)$$

$$\because \text{Var}(\xi_k)=\text{E}(\xi_k-\mu)^2=\sigma^2, \quad k=1,2,\dots,n$$

$$\therefore \text{E}[(\text{X}-N\mu)^2]=\sum_{n=1}^{\infty}n\sigma^2P_N(n)$$
$$=\sigma^2\sum_{n=1}^{\infty}nP_N(n)$$
$$\therefore \text{E}[(\text{X}-N\mu)^2]=\upsilon\sigma^2, \text{ where } \sum_{n=1}^{\infty}nP_N(n)=\upsilon \qquad (2)$$

$$\text{E}[\mu^2(N-v)^2]=\mu^2\text{E}[(N-v)^2]$$
$$\therefore \text{E}[\mu^2(N-v)^2]=\mu^2 Var(N)=\mu^2\tau^2 \qquad (3)$$

Also,

$$\text{E}[\mu(\text{X}-N\mu)(N-v)]=\mu\sum_{n=1}^{\infty}\text{E}[(\text{X}-n\mu)(n-\upsilon)\,|N=n]P_N(n)$$
$$=\mu\sum_{n=1}^{\infty}(n-\upsilon)\text{E}[(\text{X}-n\mu)\,|N=n]P_N(n)$$
$$=0 \qquad (4)$$

where $\text{E}[(\text{X}-n\mu)\,|N=n]=\text{E}(\text{X}-n\mu)$ independent prop.

$$=\text{E}(\xi_1+\xi_2+\ \dots\ +\xi_n-n\mu)$$
$$=n\mu-n\mu=0$$

Substitute (2), (3) and (4) in (1), we get

Var(X)= $\upsilon\sigma^2 + \mu^2\tau^2$

b)

i) $Z = \xi_1 + \xi_2 + ... + \xi_N, \; N > 0$

$E(\xi_k) = \mu = p,$ Var(ξ_k)=$\sigma^2 = p(1-p)$
$E(N) = v = \lambda,$ Var(N)=$\tau^2 = \lambda$
$\because E(Z) = \mu v$
$\therefore E(Z) = \lambda p$
$\because$ Var(Z)=$v\sigma^2 + \mu^2\tau^2$
$\therefore$ Var(Z)=$\lambda p(1-p) + p^2\lambda$
=λp

ii) $Z \sim$ Poisson(λp)

Q3: [3+2+3]

a) This implies that n=10, p=0.25 and X=1

$\therefore$ pr(x=1)=$\binom{10}{1} p^1 q^9$
=$10 \times 0.25 \times 0.75^9$
=0.1877

b) The probability that he will go bankrupt is given by

pr(x=0)=$\binom{10}{0} p^0 q^{10}$
=$0.25^0 \times 0.75^{10}$
=0.0563

c) What is the probability that he will get the first success on the 10^{th} trial?

pr(x=10)=p(1-p)^9
=0.25(0.75)^9
=0.0188