King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S1-1441H
M 380 - Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t+s \mid T > s) = pr(T > t) \quad \forall\, t,\ s \geq 0$

b) The lifetime T of a certain component has an exponential distribution with parameter $\lambda = 0.02$. Find $\mathrm{pr}(T \leq 120 \mid T > 100)$

Q2: [4+5]

a) The joint probability density function of the two random variables X and Y is $f(x,y) = 8xy$, $0 \leq x \leq y \leq 1$. Find $f_{Y|X}(y \mid \tfrac{1}{3})$

b) Given the joint probability mass functions of two random variables X and Y as in the following table:

X \ Y	1	2	3
0	1/8	0	0
1	0	1/4	1/8
2	0	1/4	1/8
3	1/8	0	0

i) Find $\rho(X,Y)$

ii) Determine whether X and Y are two independent random variables or not? Justify your answer.

Q3: [4+4]

a) Find the moment generating function $M_X(t)$ of X, where $X \sim \exp(\lambda)$

b) Given $V = (X_1, X_2, \ldots, X_n)$ is a multivariate random variable where $X_1, X_2, \ldots, X_n$ be independent random variables that are exponentially distributed with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Identify the distribution of V such that $\min V = \min\{X_1, X_2, \ldots, X_n\}$.

The Model Answer

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t+s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

Proof:

$$pr(T > t+s | T > s) = \frac{pr(T > t+s, T > s)}{pr(T > s)}$$

$$= \frac{pr(T > t+s)}{pr(T > s)}$$

$\because\ T \sim \exp(\lambda)$

$$\therefore\ pr(T > t+s | T > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = R(t)$$

$$= pr(T > t)$$

b)

$$pr(T \leq 120 | T > 100) = 1 - pr(T > 120 | T > 100)$$

$$= 1 - pr(T > 20 + 100 | T > 100)$$

$$= 1 - pr(T > 20)$$

$$= pr(T \leq 20)$$

$$= 1 - e^{-\lambda t}$$

$$= 1 - e^{-0.02(20)}$$

$$\therefore pr(T \leq 120 | T > 100) = 1 - e^{-0.4} \simeq 0.33$$

Q2: [4+5]

a) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$f_{X,Y}(x,y) = 8xy\ ,\ \ 0 \leq x \leq y \leq 1$$

$$\because \ f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_x^1 8xydy$$

$$= 8x\left[\frac{y^2}{2}\right]_x^1$$

$$\therefore \ f_X(x) = 4x(1-x^2), \quad 0 \le x \le 1$$

$$\therefore \ f_{Y|X}(y|x) = \frac{8xy}{4x(1-x^2)}$$

$$= \frac{2y}{1-x^2}$$

$$f_{Y|X}(y|\tfrac{1}{3}) = \tfrac{9}{4}y, \quad 0 \le y \le 1$$

b)

Y / X	1	2	3	$P_X(x)$
0	1/8	0	0	1/8
1	0	1/4	1/8	3/8
2	0	1/4	1/8	3/8
3	1/8	0	0	1/8
$P_Y(y)$	2/8	4/8	2/8	Sum=1

$E(X) = \tfrac{3}{2}$, $E(X^2) = 3$, $Var(X) = \tfrac{3}{4}$

$E(Y) = 2$, $E(Y^2) = \tfrac{9}{2}$, $Var(Y) = \tfrac{1}{2}$

$E(XY) = 3$

$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0$$

$\Rightarrow$ X and Y are not correlated

$\because$ for example, P(X=1,Y=1)=0, but P(X=1)P(Y=1) $= \frac{3}{8}(\frac{2}{8}) = \frac{3}{32}$
$\Rightarrow$ P(X=1,Y=1) $\neq$ P(X=1)P(Y=1)

$\therefore$ X and Y are not independent r.vs

Q3: [4+4]

a)

$$\because \ M_X(t) = E(e^{tX}), \quad t \in \mathbb{R}$$

$$\therefore \ M_X(t) = \int_{-\infty}^{\infty} e^{tX} f_X(x)dx$$

For $X \sim \exp(\lambda)$

$$M_X(t) = \int_0^{\infty} e^{tX} \lambda e^{-\lambda x} dx$$

$$M_X(t) \quad = \lambda \int_0^{\infty} e^{-(\lambda - t)X} dx, \quad t<\lambda$$

$$\because \int_0^{\infty} e^{-(\lambda - t)X} dx \quad = \lim_{p \to \infty} \int_0^{p} e^{-(\lambda - t)x} dx$$

$$= \lim_{p \to \infty} \left[\frac{e^{-(\lambda - t)p} - 1}{-(\lambda - t)} \right]$$

$$\therefore \int_0^{\infty} e^{-(\lambda - t)X} dx = \frac{1}{\lambda - t}, \quad t<\lambda$$

$$\therefore M_X(t) = \frac{\lambda}{\lambda - t}, \quad t<\lambda$$

b)

Given $X_1 \sim \exp(\lambda_1), X_2 \sim \exp(\lambda_2), \ldots, X_n \sim \exp(\lambda_n)$ are independent r,vs
To get distribution of V s.t min V=min $(X_1, X_2, X_3, \ldots, X_n)$
Let min V=v, $\quad v \in \mathbb{R}$
$\therefore$ pr(V>v)=pr(X_1>v)pr(X_2>v) ... pr(X_n>v)

$$=e^{-\lambda_1 v} e^{-\lambda_2 v} \ldots e^{-\lambda_n v}$$

$$\therefore \ pr(V>v) = e^{-(\sum_{i=1}^{n} \lambda_i)v}$$

$\therefore$ V is exponentially distributed with parameter $\sum_{i=1}^{n} \lambda_i$