King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S1-1443H
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

(Note that SND Table is attached in page 3)

Q1: [4+5]

a) A company has three suppliers, designated A, B and C. The relative amounts of a certain product purchased from each of the suppliers are 50%, 35% and 15%, respectively. If the proportion defective produced by each supplier are 1%, 2% and 3%, respectively. Find the overall proportion defective of that product.

b) Suppose that the price X of a particular stock at closing has a log-normal distribution with mean $30 and variance 5. What is the probability that the price exceeds $35 ?

Q2: [4+5]

a) Let X be an exponentially distributed random variable with parameter λ. Determine the mean and median of X, then compare between them.

b) Given the following joint distribution

X \ Y	-2	0	5
1	0.15	0.25	0.20
3	0.20	0.05	0.15

Find ρ(X,Y).

Q3: [3+4]

a) The state space of the discrete random variable Z is $\{0, 1, 2, 3\}$, and the probability mass function is given by

$$p(0) = \frac{1}{4},\ p(1) = \frac{1}{8},\ p(2) = \frac{1}{2}, \text{ and } p(3) = \frac{1}{8}$$

(i) Plot the corresponding distribution function.

(ii) Determine the mean $E(Z)$ and the variance $Var(Z)$

b) Suppose X has an exponential distribution with parameter Λ, where Λ has a gamma distribution with parameters α and θ. Determine the marginal density function of X.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+5]

a)

let $P(D)$ represents the overall proportion defective of that product, then by using the law of total probability, we have

$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$$
$$\therefore P(D) = 0.01\times 0.5 + 0.02\times 0.35 + 0.03\times 0.15 = 0.0165$$

b)

$$E(X) = e^{\mu+\sigma^2/2} = 30 \qquad (1)$$
$$Var(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = 5 \qquad (2)$$

$$(1) \Rightarrow \mu + \sigma^2/2 = \ln(30) \qquad (3)$$

Substitute (3) in (2)

$$e^{2\ln 30}(e^{\sigma^2} - 1) = 5$$
$$\Rightarrow$$
$$e^{\ln 900}(e^{\sigma^2} - 1) = 5$$
$$(e^{\sigma^2} - 1) = \frac{5}{900}$$
$$e^{\sigma^2} = 1 + \frac{5}{900}$$
$$\therefore \ \sigma^2 = \ln(1 + 5/900) \approx 0.0055$$

$$(3) \Rightarrow$$
$$\mu = \ln(30) - \tfrac{1}{2}\ln(1 + 5/900)$$
$$\approx 3.3984$$

$\therefore\ \Pr(X > 35) = 1 - \Pr(X \le 35)$

$= 1 - \Pr(\ln X \le \ln 35)$

$= 1 - \Phi\left(\dfrac{\ln 35 - 3.3984}{\sqrt{0.0055}}\right)$

$= 1 - \Phi(2.12)$

$\therefore\ \Pr(X > 35) = 1 - 0.9830 \approx 0.02$

Q2: [4+5]

a)

$\because X \sim \exp(\lambda)$

$\therefore$ The mean is $\mu = E(X)$

$$= \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

where the pdf is $f(x) = \lambda e^{-\lambda x}$

Let $u = \lambda x$ then,

$$\mu = \frac{1}{\lambda}\int_0^\infty u e^{-u}\,du$$

We know that

$$\int_0^\infty u e^{-u}\,du = [-ue^{-u}]_0^\infty + \int_0^\infty e^{-u}\,du \quad \text{(by using integration by parts)}$$

$$\therefore \int_0^\infty u e^{-u}\,du = 0 + [-e^{-u}]_0^\infty = 1$$

$\therefore$ The mean is $\mu = \dfrac{1}{\lambda}$ (1)

For exp. distribution, the CDF is

$F(x) = 1 - e^{-\lambda x},\quad x \ge 0$

To get the median, solve the equation

$F(m) = 1 - e^{-\lambda m} = 0.5$

so, $e^{-\lambda m} = 0.5$

$\Rightarrow -\lambda m = \ln 0.5$

$\therefore m = \dfrac{-\ln 0.5}{\lambda}$

$\therefore$ The median is $m = \dfrac{\ln 2}{\lambda}$ (2)

Clearly, we deduce from (1) and (2) that mean$>$median, where $\ln 2 \approx 0.7$

b)

Y \ X	-2	0	5	$P_X(x)$
1	0.15	0.25	0.20	0.6
3	0.20	0.05	0.15	0.4
$P_Y(y)$	0.35	0.30	0.35	Sum=1

$$E(X) = 1.8,\ E(X^2) = 4.2,\ \ Var(X) = 0.96$$
$$E(Y) = 1.05,\ E(Y^2) = 10.15,\ \ Var(Y) = 9.0475$$
$$E(XY) = 1.75$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= -0.14 \end{aligned}$$

$$\begin{aligned} \rho(X,Y) &= \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\ &= \frac{-0.14}{\sqrt{0.96}\sqrt{9.0475}} \\ &\approx -0.05 \end{aligned}$$

Q3: [3+4]

a)

(i) graph

(ii) $E(Z) = 1.5,\ E(Z^2) = 3.25,\ \ Var(Z) = 1$

b)

$$\because X \mid \Lambda \sim \exp(\lambda), \quad \Lambda \sim \text{gamma}(\alpha, \theta)$$

$$\Rightarrow f_X(x) = \int_0^\infty f_{X|\Lambda}(x \mid \lambda) f(\lambda) d\lambda$$

$$f_{X|\Lambda}(x \mid \lambda) = \lambda e^{-\lambda x}, \; f(\lambda) = \frac{\theta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\theta\lambda}$$

$$\therefore f_X(x) = \int_0^\infty \lambda e^{-\lambda x} \frac{\theta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x} \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda$$

$$\therefore f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)} d\lambda$$

Let $u = \lambda(x+\theta) \Rightarrow \lambda = \frac{u}{(x+\theta)}, \; \lambda^\alpha = \frac{u^\alpha}{(x+\theta)^\alpha}, \; d\lambda = \frac{du}{(x+\theta)}$

$$\therefore f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} du$$

$$\therefore f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} \Gamma(\alpha+1), \; \int_0^\infty u^\alpha e^{-u} du = \Gamma(\alpha+1) \text{ is the gamma function, } \alpha > 0$$

$$\therefore f_X(x) = \frac{\theta^\alpha \alpha \Gamma(\alpha)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}}$$

$$\therefore f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$$

which is known as a Pareto density function.