

Name:

ID:

Section:

Mark:

King Saud University
College of Sciences, Department of Mathematics
1444/Semester-2/ MATH 380/ Quiz-2

Marks: 10

Max. Time: 35 Minutes

Answer the following questions.

Q1: [3]

The number of accidents occurring in a factory in a week is a Poisson random variable with mean 3. The number of individuals injured in different accidents is independently distributed, each with mean 2 and variance 4. Determine the mean and variance of the number of individuals injured in a week.

Q2: [4]

Let $\zeta_1, \zeta_2, \zeta_3, \dots$ be independent Bernoulli random variables with parameter p , $0 < p < 1$. Show that $X_0 = 1$ and $X_n = p^{-n} \zeta_1 \zeta_2 \dots \zeta_n$, $n = 1, 2, \dots$, defines a nonnegative martingale.

Q3: [3]

Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability β and is followed by a defective item with probability $1 - \beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth item ?

The Model Answer

Q1: [3]

Let $Z = \xi_1 + \xi_2 + \dots + \xi_N$, where

N is # of accidents in a week and

ξ_k is # of individuals injured for k th accident.

$N \sim \text{Poisson}(3)$, consequently $E(N) = 3$, $\text{Var}(N) = 3$.

and $\therefore E(\xi_k) = 2$, $\text{Var}(\xi_k) = 4$

$\therefore E(Z) = \mu\nu = 2(3) = 6$

and $\text{Var}(Z) = \nu\sigma^2 + \mu^2\tau^2$
 $= 3(4) + 4(3) = 24$

Q2: [4]

A stochastic process X_n is a martingale if for $n = 0, 1, 2, \dots$

(1) $E[|X_n|] < \infty$,

(2) $E[X_{n+1}|X_0, \dots, X_n] = X_n$.

\Rightarrow

(1) $E[|X_n|] = E[X_n] = E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n]$, and as $\zeta_{i's}$ are independent,
 $= p^{-n} E[\zeta_1] \dots E[\zeta_n]$
 $= p^{-n} p^n = 1$, as $E[\zeta_k] = p$,

$\therefore E[|X_n|] = E[X_n] = 1 < \infty$.

(2) $E[X_{n+1}|X_0, \dots, X_n] = E[p^{-(n+1)}\zeta_1\zeta_2 \dots \zeta_{n+1}|X_0, \dots, X_n]$
 $= E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n p^{-1}\zeta_{n+1}|X_0, \dots, X_n]$
 $= p^{-n}\zeta_1\zeta_2 \dots \zeta_n E[p^{-1}\zeta_{n+1}|X_0, \dots, X_n]$,
as $\zeta_1\zeta_2 \dots \zeta_n$ are determined by X_0, \dots, X_n
 $= p^{-n}\zeta_1\zeta_2 \dots \zeta_n p^{-1}E[\zeta_{n+1}|X_0, \dots, X_n]$

$E[X_{n+1}|X_0, \dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n p^{-1}E[\zeta_{n+1}]$, as ζ_{n+1} is independent of $X_{i's}$,
 $= p^{-n}\zeta_1\zeta_2 \dots \zeta_n p^{-1}p$

$\therefore E[X_{n+1}|X_0, \dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n = X_n$.

We have proved from (1) and (2) that X_n defines a nonnegative martingale.

Q3: [3]

$$\begin{aligned} & \Pr\{X_2 = G, X_3 = G, X_4 = G, X_5 = D | X_1 = G\} \\ &= \Pr\{X_5 = D, X_4 = G, X_3 = G, X_2 = G | X_1 = G\} \\ &= \Pr\{X_5 = D | X_4 = G\} \cdot \Pr\{X_4 = G | X_3 = G\} \cdot \Pr\{X_3 = G | X_2 = G\} \cdot \Pr\{X_2 = G | X_1 = G\} \\ &= p_{GD} p_{GG}^3 \\ &= (1 - \beta)\beta^3 \\ &= \beta^3(1 - \beta) \end{aligned}$$

Also, you can solve it as follows.

$$\begin{aligned} p_1 p_{12} p_{23} p_{34} p_{45}, p_1 &= \Pr(X_1 = G) = 1 \\ &= p_G p_{GG}^3 p_{GD}, p_G = 1 \\ &= \beta^3 (1 - \beta) \end{aligned}$$
