King Saud University
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Department of Mathematics

Final Exam, S1 1444
M 380 - Stochastic Processes
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [4+4]

(a) The number of accidents occurring in a factory in a week is a Poisson random variable with mean 3. The number of individuals injured in different accidents is independent exponentially distributed, each with mean 0.5. Determine the mean and variance of the number of individuals injured in a weak.

(b) Suppose $X_1, X_2, X_3, \ldots$ are identically independent distributed random variables where

$\Pr\{X_k = 1\} = \Pr\{X_k = -1\} = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$. Show that S_n is a martingale.

Q2: [5+4]

(a) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

b) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{cc} 0 & 1 \\ \hline 0.89 & 0.11 \\ 0.02 & 0.98 \end{array}$$

i) What is the probability that the fourth item is good given that the first item is defective?

ii) In the long run, what is the probability that an item produced by this system is good or it's defective?

Q3: [5+2]

(a) Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.5; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.7; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.2. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the limiting distribution and the long run fraction of days in which it is sunny.

(b) The probability of the thrower winning in the dice game is $\mathrm{p} = 0.5071$. Suppose player A is the thrower and begins the game with \$10, and player B, his opponent, begins with \$5. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

Q4: [5+4]

(a) If $X(t)$ represents a size of a population where $X(0) = 1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\ (p),\;\; p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0,\; \mu_1 = 2,\; \mu_2 = 3$ and $\mu_3 = 4.$ Determine $P_n(t)$ for $n = 0,1,2,3.$

Q5: [4+3]

(a) Using the independent increment assumption for Brownian motion $\{B(t);\; t \ge 0\}$, show that the covariance is given by $\mathrm{Cov}\left[B(s), B(t)\right] = \sigma^2 \min\{s,t\},$ for $s,t \ge 0.$

(b) Let $\{B(t);\; t \ge 0\}$ be a standard Brownian motion. Evaluate $\Pr\{B(4) \le 5 | B(0) = 1\}.$

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Model Answer

Q1: [4+4]

(a)

$N \sim$ Poisson (3)
N is the # of accidents in aweek
ξ_k is the # of individuals injured for kth accident
$E(\xi_k) = \mu = 0.5, \ \text{var}(\xi_k) = \sigma^2 = 0.25$
$E(N) = v = 3, \ \text{var}(N) = \tau^2 = 3$
$\therefore \ E(X) = \mu v = 0.5(3) = 1.5$
$\text{var}(X) = v\sigma^2 + \mu^2\tau^2$
$\therefore \ \text{var}(X)=3(0.25)+0.25(3)=1.5$

(b)

A stochastic process $\{X_n; n = 0,1,2,...\}$ is a martingale if

(i) $E\left[|X_n|\right] < \infty,$

(ii) $E\left[X_{n+1} | X_0, ..., X_n\right] = X_n.$

(1) To show that $E\left[|S_n|\right] < \infty,$

$$|S_n| = |X_1 + ... + X_n| \leq |X_1| + ... + |X_n|$$
$$\leq 1 + ... + 1 = n$$

$$E\left[|S_n|\right] \leq E[n] = n < \infty.$$

(2) To show that $E\left[S_{n+1} | X_1, ..., X_n\right] = S_n,$

$$\begin{aligned} E\left[S_{n+1} | X_1, ..., X_n\right] &= E\left[S_n + X_{n+1} | X_1, ..., X_n\right] \\ &= E\left[S_n | X_1, ..., X_n\right] + E\left[X_{n+1} | X_1, ..., X_n\right] \\ &= S_n + E\left[X_{n+1}\right], \end{aligned}$$

where S_n is determined by $X_1, ..., X_n$ and X_{n+1} is independent of $X_{i's}$, and $\because E\left[X_{n+1}\right]=(1).\Pr\left\{X_{n+1}=1\right\}+(-1).\Pr\left\{X_{n+1}=-1\right\}$

$$=(1)(1/2)+(-1)(1/2)=0$$

$\therefore E\left[S_{n+1}\middle|X_1,...,X_n\right]=S_n$

That is from (1) and (2), we have proved that S_n is a martingale.

Q2: [5+4]

(a)

$$\mathbf{P}=\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0.2 & 0.4 & 0.3 & 0.1 \\ 2 & 0.1 & 0.5 & 0.3 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$u_i=pr\left\{X_T=0\middle|X_0=i\right\}$ for i=1,2,

and v_i =E[$T\middle|X_0=i$] for i=1,2.

(i)

$$u_1=p_{10}+p_{11}u_1+p_{12}u_2$$
$$u_2=p_{20}+p_{21}u_1+p_{22}u_2$$

$\Rightarrow$

$$u_1=0.2+0.4u_1+0.3u_2$$
$$u_2=0.1+0.5u_1+0.3u_2$$

$\Rightarrow$

$$6u_1-3u_2=2 \qquad (1)$$
$$5u_1-7u_2=-1 \qquad (2)$$

Solving (1) and (2), we get

$$u_1=\tfrac{17}{27} \text{ and } u_2=\tfrac{16}{27}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2=u_{20}=\tfrac{16}{27}=0.5926$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.3v_2$$
$$v_2 = 1 + 0.5v_1 + 0.3v_2$$

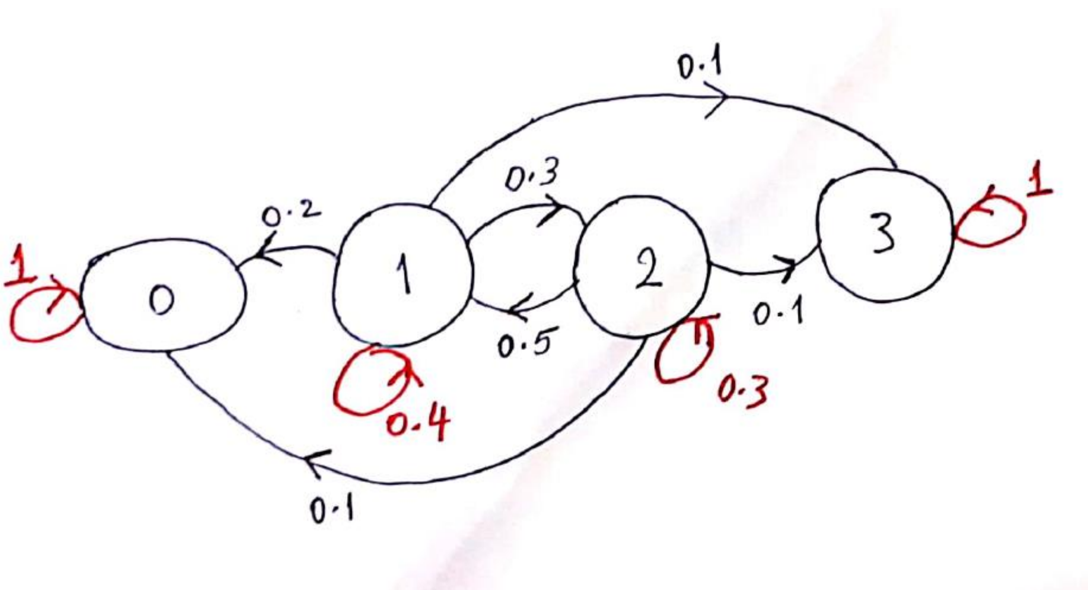
$\Rightarrow$

$$6v_1 - 3v_2 = 10 \qquad (1)$$
$$5v_1 - 7v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$$v_2 = v_{20} = \tfrac{110}{27}$$
$$\simeq 4.0741$$

(iii) It's an absorbing Markov Chain.



Markov Chain Diagram

(b)

i)

$$P^2 = \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix} \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.7943 & 0.2057 \\ 0.0374 & 0.9626 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.7110 & 0.2890 \\ 0.0525 & 0.9475 \end{bmatrix}$$

$$pr\left\{X_3 = 0 \middle| X_0 = 1\right\} = p_{10}^3 = 0.0525$$

$$= 5.25\% ,$$

which is the probability that the fourth item is good given that the first item is defective

ii) In the long run, the probability that an item produced by this system is good is given by:

$$b/(a+b) = \frac{0.02}{0.02+0.11}$$

$$= \frac{2}{13} = 15.38\ \%$$

In the long run, the probability that an item produced by this system is defective is given by:

$$a/(a+b) = \frac{0.11}{0.02+0.11}$$

$$= \frac{11}{13} = 84.62\ \% ,$$

where $\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

Clearly, $1 - \frac{2}{13} = \frac{11}{13}$=84.62 %

Q3: [5+2]

(a)

	(S,S)	(S,C)	(C,S)	(C,C)
(S,S)	0.7	0.3	0	0
(S,C)	0	0	0.4	0.6
(C,S)	0.5	0.5	0	0
(C,C)	0	0	0.2	0.8

In the long run, the limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$0.7\pi_0 + 0.5\pi_2 = \pi_0 \Rightarrow \pi_2 = \frac{3}{5}\pi_0 \qquad (1)$$

$$0.3\pi_0 + 0.5\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{3}{5}\pi_0 \qquad (2)$$

$$0.6\pi_1 + 0.8\pi_3 = \pi_3 \Rightarrow \pi_3 = \frac{9}{5}\pi_0 \qquad (3)$$

And $\because\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$

$$\therefore\ \pi_0 = 0.25$$
$$\Rightarrow \pi = (0.25, 0.15, 0.15, 0.45)$$

The long run fraction of days in which it is sunny is

$$\pi_0 + \pi_1 = 0.25 + 0.15$$
$$= 0.4$$

(b)

$i = \$10$ fortune for player A

$N = \$10 + \$5 = \$15$

$p = 0.5071 \Rightarrow$ q=0.4929

$$u_i = pr\{X_n \text{ reaches state 0 before state } N | X_0 = i\}$$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}, \quad p \neq q$$

$$u_i = \frac{\left[\left(\frac{0.4929}{0.5071}\right)^{10} - \left(\frac{0.4929}{0.5071}\right)^{15}\right]}{\left[1-\left(\frac{0.4929}{0.5071}\right)^{15}\right]}$$
$$= 0.2873$$

Q4: [5+4]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t),\ n=1,2,3,\ \ldots \qquad (2)$$

The initial condition is $X(0)=1 \quad \Rightarrow\ p_1(0)=1$

$$\Rightarrow\ p_n(0) = \begin{cases} 1 & ,\ n=1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

For $\lambda_0 = 0 \qquad (1) \Rightarrow \frac{dp_0(t)}{dt} = 0$

$\therefore\ p_0(t) = 0$, where $p_0(0) = 0$ (3)

$$(2) \Rightarrow\ \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$

$$\Rightarrow\ \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t),\ n=1,2,\ \ldots$$

$\because\ \lambda_n = n\lambda,\ \ \lambda_{n-1} = (n-1)\lambda$

$\therefore\ \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t),$ n=1,2, ...

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt} + n\lambda p_n(t)\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\therefore\ \frac{d}{dt}\left[p_n(t) e^{n\lambda t}\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x) e^{n\lambda x}\right] = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\therefore\ \left[p_n(x) e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\Rightarrow\ p_n(t) = e^{-n\lambda t}\left[p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx\right],\ n=1,2,\ \ldots \qquad (4)$$

which is a recurrence relation.

at $n=1$

$$p_1(t) = e^{-\lambda t}\left[p_1(0) + 0\right] = e^{-\lambda t} \qquad (5)$$

at $n=2$

$$p_2(t)=e^{-2\lambda t}\left[p_2(0)+\lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5)\Rightarrow\ p_1(x)=e^{-\lambda x}$

$\therefore\ p_2(t)=e^{-2\lambda t}\left[\lambda\int_0^t e^{-\lambda x}e^{2\lambda x}dx\right]$

$\therefore\ p_2(t)=\lambda e^{-2\lambda t}\int_0^t e^{\lambda x}dx$

$=e^{-\lambda t}(1-e^{-\lambda t})^1$ (6)

Similarly as (5) and (6), we deduce that

$p_n(t)=e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$

$=p(1-p)^{n-1},\ \ p=e^{-\lambda t},\ n=1,2,\ \ldots$

$\therefore X(t)\sim geom(p),\ p=e^{-\lambda t}$

$Mean\left[X(t)\right]=1/p=e^{\lambda t},$

$Variance\left[X(t)\right]=\dfrac{1-p}{p^2}=\dfrac{1-e^{-\lambda t}}{e^{-2\lambda t}}$

(b) The transition probabilities are given by

$p_N(t)=e^{-\mu_N t}$ (1)

and for $n<N$

$p_n(t)=pr\left\{X(t)=n\,\middle|\,X(0)=N\right\}$

$=\mu_{n+1}\mu_{n+2}\ \ldots\ \mu_N\left[A_{n,n}e^{-\mu_n t}+\ \ldots\ +A_{k,n}e^{-\mu_k t}+\ \ldots\ +A_{N,n}e^{-\mu_N t}\right]$ (2)

where $A_{k,n}=\prod_{i=N}^{n}\dfrac{1}{(\mu_i-\mu_k)}$, $i\neq k,\ n\leq k\leq N,\ i=N,N\text{-}1,\ \ldots,\ n$ (3)

For N=3 $(1)\Rightarrow \mathrm{p}_3(\mathrm{t})=\mathrm{e}^{-\mu_3 t}$

$\therefore\ \mathrm{p}_3(\mathrm{t})=\mathrm{e}^{-4t}$ (I)

For n=2 $(2)\Rightarrow \mathrm{p}_2(\mathrm{t})=\mu_3\left[A_{2,2}e^{-\mu_2 t}+A_{3,2}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow \; A_{2,2} = \prod_{i=3}^{2} \frac{1}{(\mu_i - \mu_2)} \;, \; i \neq 2$$

$$= \frac{1}{\mu_3 - \mu_2} = 1,$$

$$A_{3,2} = \prod_{i=3}^{2} \frac{1}{(\mu_i - \mu_3)}, \; i \neq 3$$

$$= \frac{1}{\mu_2 - \mu_3} = -1$$

$$\therefore \; p_2(t) = 4\left[e^{-3t} - e^{-4t}\right] \qquad \text{(II)}$$

For n=1 $\quad (2) \Rightarrow p_1(t) = \mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t} + A_{2,1}e^{-\mu_2 t} + A_{3,1}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow \; A_{1,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_1)} \;, \; i \neq 1$$

$$= \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} = \frac{1}{2},$$

$$A_{2,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_2)} \;, \; i \neq 2$$

$$= \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} = -1,$$

$$A_{3,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_3)} \;, \; i \neq 3$$

$$= \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} = \frac{1}{2}$$

$$p_1(t) = 12\left[\frac{1}{2}e^{-2t} - e^{-3t} + \frac{1}{2}e^{-4t}\right]$$

$$\therefore \; p_1(t) \; = 6\left[e^{-2t} - 2e^{-3t} + e^{-4t}\right] \qquad \text{(III)}$$

Using (I), (II) and (III) we can get $p_0(t)$ as follows

$$\therefore\ p_0(t) = 1 - [p_1(t) + p_2(t) + p_3(t)]$$
$$= 1 - \left[6e^{-2t} - 12e^{-3t} + 6e^{-4t} + 4e^{-3t} - 4e^{-4t} + e^{-4t}\right]$$
$$= 1 - 6e^{-2t} + 8e^{-3t} - 3e^{-4t} \qquad \text{(IV)}$$

Q5: [4+3]

(a)

$$\because B(s+t) - B(s) \sim N(0, \sigma^2 t)$$
$$\therefore E[B(t)] = 0, \quad \text{var}[B(t)] = \sigma^2 t$$
$$\therefore E\left[[B(t)]^2\right] = \sigma^2 t$$
$$\because \text{Cov}[B(s), B(t)] = E[B(s)B(t)] - E[B(s)]E[B(t)]$$
$$\therefore \text{Cov}[B(s), B(t)] = E[B(s)B(t)]$$

Then, for $0 \le s < t$

$$\because \text{Cov}[B(s), B(t)] = E[B(s)B(t)]$$
$$= E\left[B(s)[B(t) - B(s) + B(s)]\right]$$
$$= E\left[[B(s)]^2\right] + E[B(s)]E[B(t) - B(s)] \text{ (by independence property)}$$

So, we obtain $\text{Cov}[B(s), B(t)] = \sigma^2 s \qquad (1)$

where $E\left[[B(s)]^2\right] = \sigma^2 s$ and $E[B(s)] = 0$

Similarly, if $0 \le t < s$, we obtain $\text{Cov}[B(s), B(t)] = \sigma^2 \text{t} \qquad (2)$

Thus, from (1) and (2) we deduce that

$$\text{Cov}[B(s), B(t)] = \sigma^2 \min\{s, t\} \quad \text{for } s, t \ge 0$$

(b)

$$\because \Pr\{B(4) \le 5 | B(0) = 1\} = \Phi\left(\tfrac{y-x}{\sqrt{t}}\right)$$
$$= \Phi\left(\tfrac{5-1}{\sqrt{4}}\right)$$
$$\therefore \Pr\{B(4) \le 5 | B(0) = 1\} = \Phi(2) = 0.9772$$
