King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1444
M 380 - Stochastic Processes
Time: 3 hours - Marks: 40

Answer the following questions.

Q1: [3+4]

(a)The number of accidents occurring in a factory in a week is a Poisson random variable with mean 2. The number of individuals injured in different accidents is independently distributed, each with mean 3 and variance 4. Determine the mean and variance of the number of individuals injured in a week.

(b) Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \zeta_1 + \zeta_2 + \ldots + \zeta_n$ be the sum of n independent random variables, each exponentially distributed with mean $E[\zeta_k] = 1$. Show that: $X_n = 2^n e^{-Sn}$, $n \geq 0$, defines a martingale.

Q2: [2+3]

(a) A Markov chain $X_0,\ X_1,\ X_2,\ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$.

(b) Let $\{X(t);\ t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the numerical values to two decimal places for the following probabilities:

(i) $\Pr\{X(1) \leq 2\}$.

(ii) $\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$.

Q3: [3+4]

(a) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1-\alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(ii) Determine the probability that a correct signal is received at stage 2.

(b) Let X_n denote the condition of a machine at the end of period n for $n = 1, 2, \ldots$. Let X_0 be the condition of the machine at the start. Consider the condition of the machine at any time can be observed and classified as being in one of the following three states:

State 1: Good operating order, State 2: Deteriorated operating order and State 3: In repair.

Assume that $\{X_n\}$ is a Markov chain with transition probability matrix

$$
\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}
$$

and starts in state $X_0 = 1$.

(i) Find $\Pr\{X_4 = 1\}$.

(ii) Calculate the limiting distribution.

(iii) What is the long run rate of repairs per unit time?

Q4: [3.5+3.5]

(a) A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1,\ \lambda_1 = 2,\ \lambda_2 = 3$ and $\lambda_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

(b) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0,\ \mu_1 = 2,\ \mu_2 = 3$ and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

Q5: [5+2]

(a) If $X(t)$ represents a size of a population where $X(0)=1$. Using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\,(p),\;\; p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1). Show that $pr\{X(U)=k\} = p^k/(\beta k)$ for $k=1,2,\ldots$, with $p = 1-e^{-\beta}$.

Q6: [4+3]

(a) Prove that the covariance for a Brownian motion, for any times $0 \le s,t$, is given by:

$$Cov[B(s),B(t)] = \sigma^2 \, min\{s,t\}.$$

(b) Determine the covariance function $Cov[W(s),\, W(t)]$ for the stochastic process:

$W(t) = t\, B\left(\frac{1}{t}\right)$, with $W(0)=0$ and $B(t)$ is a standard Brownian motion.

Model Answer

Q1: [3: (1.5+1.5)+4: (2+2)]

(a) Let $N=$ number of accidents in a week, and ζ_k = number of individuals injured for k[th] accident. Let $X=\zeta_1+\zeta_2+\ldots+\zeta_N$, $N>0, N\sim Poisson(2)$.

Then:

$$E(N)=2, \qquad Var(N)=2,$$
$$E(\zeta_k)=3, \qquad Var(\zeta_k)=4,$$

So,

$$E(X)=\mu\ \nu=3(2)=6,$$
$$Var(X)=\nu\ \sigma^2+\mu^2\ \tau^2\ =2(4)+9(2)=26.$$

(b) (1) $E[|X_n|]=E[X_n]=E[2^n\ e^{-S_n}]$

$$= 2^n\ E[e^{-\zeta_1}\ldots.e^{-\zeta_n}]$$
$= 2^n\ E[e^{-\zeta_1}]\ldots.E[e^{-\zeta_n}]$, as $\zeta_{i's}$ are independent
$= 2^n\ \frac{1}{2}\ldots\ldots\frac{1}{2}\ = \frac{2^n}{2^n} = 1$, as

$$E\left[e^{-\zeta_n}\right]=\int_0^\infty e^{-x}\ e^{-x}\ dx$$
$$=\int_0^\infty e^{-2x}\ \ dx\ =\frac{1}{2}$$

So, $E[|X_n|]=1<\infty$.

(2) $E[X_{n+1}|X_0,\ldots..,X_n]=E[2^{n+1}\ e^{-S_{n+1}}|X_0,\ldots..,X_n],\ S_{n+1}=S_n+\zeta_{n+1}$

$$=E\left[2^n\ e^{-S_n}\ \ 2\ e^{-\zeta_{n+1}}\middle|X_0,\ldots..,X_n\right]$$
$$=2^n\ e^{-S_n}\ E\left[2\ e^{-\zeta_{n+1}}\middle|X_0,\ldots..,X_n\right]$$
$$=2^n\ e^{-S_n}\ 2\ E[e^{-\zeta_{n+1}}],$$

as ζ_{n+1} is independent of $X_{i's}$,

$$E[X_{n+1}|X_0,\ldots..,X_n]=2^n\ e^{-S_n}\ 2\,.\frac{1}{2}$$
$$=2^n\ e^{-S_n}$$
$$=X_n.$$

We have proved from (1) and (2) that X_n is a martingale.

Q2: [2+3: (1.5+1.5)]

(a)

$$\because pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = p_1 P_{10} P_{02}\ , \quad p_1 = pr\{X_0 = 1\} = 1$$
$$= 1(0.3)(0.1)$$
$$\therefore pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = 0.03$$

(b)

For Poisson Process

$$\Pr\{X(s+t) - X(s) = k\}$$
$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!},\ k = 0, 1, 2, \ldots$$

(i)

$$\Pr\{X(1) \le 2\} = \Pr\{X(1) = 0\} + \Pr\{X(1) = 1\} + \Pr\{X(1) = 2\}$$
$$= \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} = 5e^{-2} \simeq 0.68$$

(ii)

$$\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$$
$$= \Pr\{X(1) = 1\} \Pr\{X(2) = 3\}$$
$$= \Pr\{X(1) - X(0) = 1\} \Pr\{X(2) - X(1) = 2\}$$
$$= \frac{2^1 e^{-2}}{1!} . \frac{2^2 e^{-2}}{2!} = 4e^{-4} \simeq 0.07,$$

where the increments are independents random variables in Poisson process.

Q3: [3: (1.5+1.5)+4: (2+1.5+0.5)]

(a)

The transition probability matrix can be written as

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{array} \right\| \end{array}$$

(i) The probability that no error occurs up to stage $n = 2$ is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00} P_{00}, \quad \mathrm{p}_0 = pr\{X_0 = 0\} = 1$$
$$= 1 \times (1\text{-}\alpha) \times (1\text{-}\alpha)$$
$$= (1\text{-}\alpha)^2$$

where $p_0 = pr(X_0 = 0) = 1$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$$
$$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$$
$$= (1-\alpha)^2 + \alpha^2$$
$$= 1 - 2\alpha + 2\alpha^2$$

(b)

(i)

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}$$

$$\because \ \Pr\{X_4 = 1\} = \Pr\{X_4 = 1 | X_0 = 1\} \Pr\{X_0 = 1\}$$
$$= P_{11}^4, \quad \Pr\{X_0 = 1\} = 1$$

$$\mathrm{P}^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.81 & 0.18 & 0.01 \\ 0.1 & 0.81 & 0.09 \\ 0.9 & 0.1 & 0 \end{array} \right\| \end{array}$$

$$\text{and} \quad P^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.6831 & 0.2926 & 0.0243 \\ 0.2430 & 0.6831 & 0.0739 \\ 0.7390 & 0.2430 & 0.0180 \end{array} \right\| \end{array}$$

$$\therefore \Pr\{X_4 = 1\} = \Pr\{X_4 = 1 | X_0 = 1\} \Pr\{X_0 = 1\}$$
$$= P_{11}^4 p_1 = P_{11}^4 = 0.6831$$

(ii) To get the limiting distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\pi_G, \pi_D, \pi_R)$

Solving the following equations

$\pi_1 = 0.9\pi_1 + \pi_3$ (1)

$\pi_2 = 0.1\pi_1 + 0.9\pi_2$ (2)

$\pi_3 = 0.1\pi_2$ (3)

$\pi_1 + \pi_2 + \pi_3 = 1$ (4)

$(1) \Rightarrow \pi_3 = 0.1\pi_1$

$(2) \Rightarrow \pi_2 = \pi_1$

also, (3) $\pi_3 = 0.1\pi_2$

$(4) \Rightarrow \pi_1 + \pi_1 + 0.1\pi_1 = 1$

$\therefore \ \pi_1 = \frac{10}{21}$

$\Rightarrow \therefore \ \pi_2 = \frac{10}{21}$ and π_3

$\therefore \ \pi = (\frac{10}{21}, \frac{10}{21}, \frac{1}{21})$

(iii) $\pi_R = \pi_3 = \frac{1}{21} = 0.0476$

Q4: [3.5+3.5]

(a)

For pure birth process, the transition probabilities are given by

$$p_0(t) = e^{-\lambda_0 t}, \quad (1)$$

$$p_1(t) = \lambda_0 \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right], \quad (2)$$

and $p_n(t) = pr\{X(t) = n | X(0) = 0\}$

$$= \lambda_0 \lambda_1 \ldots \lambda_{n-1} \left[B_{0,n} e^{-\lambda_0 t} + \ldots + B_{k,n} e^{-\lambda_k t} + \ldots + B_{n,n} e^{-\lambda_n t} \right], \ n > 1, \quad (3)$$

where

$$B_{k,n} = \prod_{i=0}^{n}\left(\frac{1}{\lambda_i - \lambda_k}\right) \; i \neq k, \; 0 < k < n,$$

$$B_{0,n} = \prod_{i=1}^{n}\left(\frac{1}{\lambda_i - \lambda_0}\right)$$

and

$$B_{n,n} = \prod_{i=0}^{n-1}\left(\frac{1}{\lambda_i - \lambda_n}\right)$$

at $n = 0$ $(1) \Rightarrow p_0(t) = e^{-\lambda_0 t}, \quad \lambda_0 = 1$

$\therefore \; p_0(t) = e^{-t}$

at $n = 1$ $(2) \Rightarrow p_1(t) = \left[e^{-t} - e^{-2t}\right]$

at $n = 2$ $(3) \Rightarrow p_2(t) = \lambda_0\lambda_1\left[B_{0,2}e^{-\lambda_0 t} + B_{1,2}e^{-\lambda_1 t} + B_{2,2}e^{-\lambda_2 t}\right],$

where, $B_{0,2} = \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}$

$= \frac{1}{2},$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$= -1$

and

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$= \frac{1}{2}$

$\therefore \; p_2(t) = 2\left[\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right]$

at $n = 2$ $(3) \Rightarrow p_3(t) = \lambda_0\lambda_1\lambda_2\left[B_{0,3}e^{-\lambda_0 t} + B_{1,3}e^{-\lambda_1 t} + +B_{2,3}e^{-\lambda_2 t} + B_{3,3}e^{-\lambda_3 t}\right]$

where $B_{0,3} = \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)(\lambda_3 - \lambda_0)} = \frac{1}{8}$,

$B_{1,3} = \dfrac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} = -\frac{1}{3}$,

$B_{2,3} = \dfrac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} = \frac{1}{4}$,

and

$B_{3,3} = \dfrac{1}{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} = -\frac{1}{24}$.

$$\therefore\ p_3(t) = 6\left[\tfrac{1}{8}e^{-t} - \tfrac{1}{3}e^{-2t} + \tfrac{1}{4}e^{-3t} - \tfrac{1}{24}e^{-5t}\right]$$

(b)

For pure death process, the transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \qquad (1)$$

and for $n < N$

$$p_n(t) = pr\left\{X(t) = n \middle| X(0) = N\right\}$$
$$= \mu_{n+1}\mu_{n+2}\ \dots\ \mu_N\left[A_{n,n}e^{-\mu_n t} + \ \dots + A_{k,n}e^{-\mu_k t} + \ \dots + A_{N,n}e^{-\mu_N t}\right] \quad (2)$$

Where $A_{k,n} = \prod_{i=N}^{n} \dfrac{1}{(\mu_i - \mu_k)}$, $\quad i \neq k,\ n \leq k \leq N,\ i = N, N-1,\ \dots,\ n \quad (3)$

For N=3 $\quad (1) \Rightarrow \mathrm{p}_3(\mathrm{t}) = \mathrm{e}^{-\mu_3 t}$

$\therefore\ \mathrm{p}_3(\mathrm{t}) = \mathrm{e}^{-5t} \qquad$ (I)

For $n = 2 \quad (2) \Rightarrow \mathrm{p}_2(\mathrm{t}) = \mu_3\left[A_{2,2}e^{-\mu_2 t} + A_{3,2}e^{-\mu_3 t}\right]$

$(3) \Rightarrow\ A_{2,2} = \prod_{i=3}^{2} \dfrac{1}{(\mu_i - \mu_2)}$, $\ i \neq 2$

$$= \frac{1}{\mu_3 - \mu_2} = \frac{1}{2}$$

$$, A_{3,2}=\prod_{i=3}^{2}\frac{1}{(\mu_i-\mu_3)},\ i\neq 3$$

$$=\frac{1}{\mu_2-\mu_3}=-\frac{1}{2}$$

$$\therefore\ \text{p}_2(\text{t})=5\left[\frac{1}{2}e^{-3t}-\frac{1}{2}e^{-5t}\right]\qquad \text{(II)}$$

For $n=1$ $\quad (2)\Rightarrow \text{p}_1(\text{t})=\mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t}+A_{2,1}e^{-\mu_2 t}+A_{3,1}e^{-\mu_3 t}\right]$

$$(3)\Rightarrow\ A_{1,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_1)},\ i\neq 1$$

$$=\frac{1}{(\mu_3-\mu_1)(\mu_2-\mu_1)}=\frac{1}{3}$$

$$A_{2,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_2)},\ i\neq 2$$

$$=\frac{1}{(\mu_3-\mu_2)(\mu_1-\mu_2)}=-\frac{1}{2}$$

$$, A_{3,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_3)},\ i\neq 3$$

$$=\frac{1}{(\mu_2-\mu_3)(\mu_1-\mu_3)}=\frac{1}{6}$$

$$\therefore\ \text{p}_1(\text{t})=15\left[\frac{1}{3}e^{-2t}-\frac{1}{2}e^{-3t}+\frac{1}{6}e^{-5t}\right]\qquad \text{(III)}$$

Using (I), (II) and (III) we can get $\text{p}_0(\text{t})$ as follows

$$\therefore\ \text{p}_0(\text{t})=1-\left[\text{p}_1(\text{t})+\text{p}_2(\text{t})+\text{p}_3(\text{t})\right]$$

$$=1-\left[5e^{-2t}-\frac{15}{2}e^{-3t}+\frac{5}{2}e^{-3t}+\frac{5}{2}e^{-5t}-\frac{5}{2}e^{-5t}+e^{-5t}\right]$$

$$=1-5e^{-2t}+5e^{-3t}-e^{-5t}\qquad \text{(IV)}$$

Q5: [5+2]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \dots \quad (2)$$

The initial condition is $X(0) = 1 \quad \Rightarrow \; p_1(0) = 1$

$$\Rightarrow \; p_n(0) = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda_0 = 0 \qquad (1) \Rightarrow \frac{dp_0(t)}{dt} = 0$$

$$\Rightarrow \; p_0(t) = 0 \qquad (3)$$

$$(2) \Rightarrow \; \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$

$$\Rightarrow \; \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t), \; n = 1, 2, \dots$$

$$\because \; \lambda_n = n\lambda, \;\; \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \; \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t), \; \text{n=1,2, ...}$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt} + n\lambda p_n(t)\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\therefore \; \frac{d}{dt}\left[p_n(t) e^{n\lambda t}\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x) e^{n\lambda x}\right] = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\therefore \; \left[p_n(x) e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\Rightarrow \; p_n(t) = e^{-n\lambda t}\left[p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx\right], \; n = 1, 2, \dots \quad (4)$$

which is a recurrence relation.

at $n = 1$

$$p_1(t) = e^{-\lambda t}\left[p_1(0) + 0\right] = e^{-\lambda t} \qquad (5)$$

at $n = 2$

$$p_2(t) = e^{-2\lambda t}\left[p_2(0) + \lambda \int_0^t p_1(x) e^{2\lambda x} dx \right]$$

$$(5) \Rightarrow \; p_1(x) = e^{-\lambda x}$$

$$\therefore \; p_2(t) = e^{-2\lambda t}\left[\lambda \int_0^t e^{-\lambda x} e^{2\lambda x} dx \right]$$

$$\therefore \; p_2(t) = \lambda e^{-2\lambda t} \int_0^t e^{\lambda x} dx$$

$$= e^{-\lambda t}(1 - e^{-\lambda t})^1 \qquad (6)$$

Similarly as (5) and (6), we deduce that

$$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$$

$$= p(1 - p)^{n-1}, \;\; p = e^{-\lambda t}, \;\; n = 1, 2, \; ...$$

$$\therefore X(t) \sim geom(p), \;\; p = e^{-\lambda t}$$

$$Mean[X(t)] = 1/p = e^{\lambda t},$$

$$Variance[X(t)] = \frac{1-p}{p^2} = \frac{1 - e^{-\lambda t}}{e^{-2\lambda t}}$$

(b) For Yule process,

$$p_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \geq 1$$

$$\Rightarrow$$

$$\because pr\{X(U) = k\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1} du$$

$$= \frac{1}{\beta} \int_0^1 (1 - e^{-\beta u})^{k-1}. \; \beta e^{-\beta u} du$$

$$= \frac{1}{\beta}\left[\frac{(1 - e^{-\beta u})^k}{k} \right]_0^1$$

$$= \frac{1}{\beta k}\left[(1 - e^{-\beta})^k \right]$$

$$\therefore pr\{X(U) = k\} = \frac{p^k}{\beta k}, \; k = 1, 2, \; ... \text{ where } p = 1 - e^{-\beta}$$

Q6: [4+3]

(a)

Using the independent increments property for Brownian motion, we can deduce that $E[B(t)] = 0$ and $E[B(t)^2] = \sigma^2 t$.

Then for $0 \le s < t$,

$$Cov[B(s),B(t)] = E[B(s)B(t)]$$
$$= E[B(s)\{B(t) - B(s) + B(s)\}]$$
$$= E\left[B(s)^2\right] + E[B(s)\{B(t) - B(s)\}]$$
$$= \sigma^2 s \; + \; E[B(s)]E[B(t) - B(s)]$$
$$= \sigma^2 s \quad (E[B(s)] = 0).$$

Similarly, if $0 \le t < s$, we obtain $Cov[B(s),B(t)] = \sigma^2 t$.

Therefore, we have $Cov[B(s),B(t)] = \sigma^2 \, min\{s,t\}, \quad for \; s,t \ge 0.$

(b)

$$Cov[W(s),W(t)] = Cov\left[sB\left(\frac{1}{s}\right), tB\left(\frac{1}{t}\right)\right]$$
$$= s \; t \; Cov\left[B\left(\frac{1}{s}\right), B\left(\frac{1}{t}\right)\right]$$
$$= s \; t \; min\left\{\frac{1}{s}, \frac{1}{t}\right\}$$
$$= s \; t \left(\frac{1}{s}\right), \quad \left(if \left(\frac{1}{s}\right) < \left(\frac{1}{t}\right)\right)$$
$$= t \quad (if \; 0 \le t < s)$$

and

$$= \; s \;\; (if \; 0 \le s < t),$$

So

$$= min\{s,t\},$$

$for \; s,t \ge 0.$