King Saud University
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Department of Mathematics

Final Exam, S3 1444
M 380 - Stochastic Processes
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 4)

Q1: 8[4+1+3]

(a) Suppose that X has a binomial distribution with parameters p and N, where N has a Poisson distribution with mean λ. What is the marginal distribution for X?

(b) Define a martingale.

(c) Suppose $X_1, X_2, X_3, \ldots$ are identically independent distributed random variables where

$\Pr\{X_k = 1\} = \Pr\{X_k = -1\} = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$. Show that S_n is a martingale.

Q2: 6[2+2+2]

A Markov chain $X_0, X_1, X_2, \ldots$ with states 0, 1, and 2, has the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

and initial distribution $p_0 = pr\{X_0 = 0\} = 0.4$ and $p_1 = pr\{X_0 = 1\} = 0.6$. Determine the following probabilities:

(i) $pr\{X_2 = 0\}$ (ii) $pr\{X_3 = 0\}$

(iii) If the process starts from state $X_0 = 2$, find $pr\{X_0 = 2, X_1 = 0, X_2 = 1\}$

Q3: [4]

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.5 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{array} \right\| \end{array}$$

Every period that the process spends in state 0 incurs a cost \$4. Every period that the process spends in state 1 incurs a cost of \$7. Every period that the process spends in state 2 incurs a cost of \$5. What is the long run mean cost per period associated with this Markov chain?

Q4: 6[4+2]

(a) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are middle class in the long run?

(b) The probability of the thrower winning in the dice game is $p = 0.5071$. Suppose player A is the thrower and begins the game with \$10, and player B, his opponent, begins with \$5. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

Q5: 9[5+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$
$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t),\ n = 1,2,3,... \quad (2)$$

where all birth parameters are the same constant λ with initial condition $X(0)=0$,

Show that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!},\ n = 0,1,2,...$

(b) Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

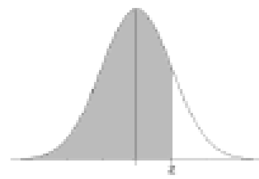
Q6: 7[2+5]

Let $\{B(t);\ t \ge 0\}$ be a standard Brownian motion.

(a) Evaluate $\Pr\{B(9) \le 7 | B(0) = 1\}$.

(b) Show that the process defined by $W(t) = cB(t/c^2),\ c > 0$ is a standard Brownian motion.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Model Answer

Q1: 8[4+1+3]

(a) $X \sim Bin(p, N), \quad N \sim Poisson(\lambda)$

$P_{X|N}(x|n) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2, \dots, n$

and $P_N(n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0,1,2, \dots.$

So, $pr(X = x) = \sum_{n=0}^{\infty} P_{X|N}(x|n) \; P_N(n)$

$$= \sum_{n=0}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \lambda^x p^x e^{-\lambda} \sum_{n=x}^{\infty} \frac{n!}{x!(n-x)!} \frac{(1-p)^{n-x} \lambda^{n-x}}{n!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!}$$

Let $n - x = r$,

$$pr(X = x) = \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^r}{r!}$$

$$= \frac{(\lambda p)^x e^{-\lambda} e^{\lambda(1-p)}}{x!}$$

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!}, \quad x = 0,1,2, \dots$$

So, $X \sim Poisson(\lambda p)$ with mean and variance λp.

(b)

A stochastic process $\{X_n; n = 0,1,2,\dots\}$ is a martingale if

(i) $E[|X_n|] < \infty$,

(ii) $E[X_{n+1} | X_0, \dots, X_n] = X_n$.

(c)

(1) To show that $E[|S_n|] < \infty$,

$$|S_n| = |X_1 + \ldots + X_n| \le |X_1| + \ldots + |X_n|$$
$$\le 1 + \ldots + 1 = n$$

$$E\left[|S_n|\right] \le E[n] = n < \infty.$$

(2) To show that $E\left[S_{n+1} | S_1, \ldots, S_n\right] = S_n$,

$$E\left[S_{n+1} | S_1, \ldots, S_n\right] = E\left[S_n + X_{n+1} | S_1, \ldots, S_{nn}\right]$$
$$= E\left[S_n | S_1, \ldots, S_n\right] + E\left[X_{n+1} | S_1, \ldots, S_n\right]$$
$$= S_n + E[X_{n+1}],$$

where S_n is determined by $S_1, \ldots, S_n$ and X_{n+1} is independent of $S_1, \ldots, S_n$

and $\because E[X_{n+1}] = (1).\Pr\{X_{n+1} = 1\} + (-1).\Pr\{X_{n+1} = -1\}$
$$= (1)(1/2) + (-1)(1/2) = 0$$

$$\therefore E\left[S_{n+1} | S_1, \ldots, S_n\right] = S_n$$

That is from (1) and (2), we have proved that S_n is a martingale.

Q2: 6 [2+2+2]

i) $pr\{X_2 = 0\} = pr\{X_2 = 0 | X_0 = 0\}\, pr\{X_0 = 0\} + pr\{X_2 = 0 | X_0 = 1\}\, pr\{X_0 = 1\}$
$$= p_{00}^2\ p_0 + p_{10}^2\ p_1$$
$$= (0.36)(0.4) + (0.32)(0.6) = 0.336,$$

where $p^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.24 & 0.4 \\ 0.32 & 0.28 & 0.4 \\ 0.32 & 0.24 & 0.44 \end{bmatrix}$

ii) $pr\{X_3 = 0\} = pr\{X_3 = 0 | X_0 = 0\}\, pr\{X_0 = 0\} + pr\{X_3 = 0 | X_0 = 1\}\, pr\{X_0 = 1\}$
$$= p_{00}^3\ p_0 + p_{10}^3\ p_1$$
$$= (0.328)(0.4) + (0.336)(0.6) = 0.3328,$$

where $p_{00}^3 = [0.2\ \ 0.3\ \ 0.5] \begin{bmatrix} 0.36 \\ 0.32 \\ 0.32 \end{bmatrix} = 0.328$

and $p_{10}^3 = [0.4 \;\; 0.1 \;\; 0.5]\begin{bmatrix}0.36\\0.32\\0.32\end{bmatrix} = 0.336$

iii) $pr\{X_0 = 2, X_1 = 0, X_2 = 1\} = p_2\, p_{20}\, p_{01}$
$= 1(0.4)(0.3) = 0.12,$
where $p_2 = pr\{X_0 = 2\} = 1.$

Q3: [4]

$$\pi_j = \sum_{k=0}^{2} \pi_k P_{kj}$$

at $j = 0$

$$\Rightarrow \pi_0 = 0.5\pi_0 + 0.5\pi_1 + 0.3\pi_2$$
$$\therefore 5\pi_0 - 5\pi_1 - 3\pi_2 = 0 \qquad (1)$$

at $j = 1$

$$\Rightarrow \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2$$
$$\therefore 2\pi_0 - 9\pi_1 + 2\pi_2 = 0 \qquad (2)$$

and $\because \; \pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$

$\therefore$ By solving equations (1), (2) and (3)

We get $\pi_0 = 0.4205, \; \pi_1 = 0.1818, \; \pi_2 = 0.3977$

The long run mean cost per unit period is

$$C = \sum_{j=0}^{2} \pi_j c_j$$
$$= \pi_0 c_0 + \pi_1 c_1 + \pi_2 c_2$$
$$= \$ \; 4.9431$$

Q4: 6 [4+2]

(a)

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$

$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$

$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \qquad (1)$$
$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 34,\ \Delta_0 = \begin{vmatrix} 0 & -2 & -1 \\ 0 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 12$$

$$\Delta_1 = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 14,\ \Delta_2 = \begin{vmatrix} 3 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 8$$

$\therefore\ \pi_0 = \frac{\Delta_0}{\Delta} = \frac{6}{17},\ \pi_1 = \frac{\Delta_1}{\Delta} = \frac{7}{17},\ \pi_2 = \frac{\Delta_2}{\Delta} = \frac{4}{17}$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (6/17, 7/17, 4/17)$

$\therefore$ In the long run, approximately 41.18% of families are middle class.

(b)

$i = \$10$ fortune for player A

$N = \$10 + \$5 = \$15$

$p = 0.5071 \Rightarrow$ q=0.4929

$\mathrm{u}_i = pr\left\{X_n \text{ reaches state 0 before state } N \middle| X_0 = i\right\}$

$$\mathrm{u}_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}, \quad p \neq q$$

$$\mathrm{u}_i = \frac{\left[\left(\frac{0.4929}{0.5071}\right)^{10} - \left(\frac{0.4929}{0.5071}\right)^{15}\right]}{\left[1-\left(\frac{0.4929}{0.5071}\right)^{15}\right]}$$

$= 0.2873$

Q5: 9 [5+4]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \; n = 1,2,3,... \qquad (2)$$

Let $X(t)$ represents the size of the population, and the initial condition is

$X(0) = 0 \quad \Rightarrow \; p_0(0) = 1$

$$\Rightarrow \; p_n(0) = \begin{cases} 1 & , \; n = 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$(1) \Rightarrow \frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

$$\therefore \quad \frac{dp_0(t)}{p_0(t)} = -\lambda dt$$

$$\int_0^t \frac{dp_0(u)}{p_0(u)} = -\lambda \int_0^t du$$

$$\left[\ln p_0(u)\right]_0^t = -\lambda t$$

$$\ln p_0(t) - \ln p_0(0) = -\lambda t$$

$\ln p_0(t) - \ln 1 = -\lambda t$, where $p_0(0) = 1$

$$\therefore \quad \ln p_0(t) = -\lambda t \;\Rightarrow p_0(t) = e^{-\lambda t} \qquad (3)$$

$$(2) \Rightarrow \frac{dp_n(t)}{dt} = p_{n-1}(t) - \lambda p_n(t), \; n = 1, 2, 3, \ldots$$

$$\therefore \quad \frac{dp_n(t)}{dt} + \lambda p_n(t) = \lambda p_{n-1}(t)$$

Multiply both sides by $e^{\lambda t}$ (integrating factor)

$$e^{\lambda t}\left[\frac{dp_n(t)}{dt} + \lambda p_n(t)\right] = \lambda p_{n-1}(t) e^{\lambda t}$$

$$\therefore \quad \frac{d}{dt}\left[e^{\lambda t} p_n(t)\right] = \lambda p_{n-1}(t) e^{\lambda t}$$

$\therefore$ By separation of variables and Integration from 0 to t, we get

$$\int_0^t d\left[e^{\lambda x} p_n(x)\right] = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$\left[e^{\lambda x} p_n(x)\right]_0^t = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$e^{\lambda t} p_n(t) - p_n(0) = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx, \; n = 1, 2, 3, \ldots$$

$$p_n(t) = \lambda e^{-\lambda t} \int_0^t p_{n-1}(x) e^{\lambda x} dx, \; n = 1, 2, 3, \ldots \qquad (4)$$

which is a recurrence relation

$at\ n=1$

$$(4) \Rightarrow p_1(t) = \lambda e^{-\lambda t}\int_0^t p_0(x) e^{\lambda x} dx$$

$\because p_0(x) = e^{-\lambda x}$ from eq. (3)

$$\therefore p_1(t) = \lambda e^{-\lambda t}\int_0^t e^{-\lambda x} e^{\lambda x} dx$$

$$= \lambda e^{-\lambda t}\int_0^t dx$$

$$\therefore p_1(t) = \lambda t e^{-\lambda t} \qquad (5)$$

$at\ n=2$

$$(4) \Rightarrow p_2(t) = \lambda e^{-\lambda t}\int_0^t p_1(x) e^{\lambda x} dx$$

$\because p_1(x) = \lambda x e^{-\lambda x}$ from eq. (5)

$$\therefore p_2(t) = \lambda e^{-\lambda t}\int_0^t \lambda x e^{-\lambda x} e^{\lambda x} dx$$

$$= \lambda^2 e^{-\lambda t}\int_0^t x dx$$

$$\therefore p_2(t) = \lambda^2 e^{-\lambda t}\left[\frac{x^2}{2}\right]_0^t$$

$$\therefore p_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \qquad (6)$$

From (3), (5) and (6), we can deduce that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!},\ n = 0,1,2,\ldots$

(b)

$\because X \sim \text{Poisson}(\alpha),\ \ Y \sim \text{Poisson}(\beta)$

$\therefore X + Y \sim \text{Poisson}(\alpha + \beta)$

$$\Rightarrow \Pr\{X=k|N=n\} = \Pr\{X=k|X+Y=n\}$$

$$= \frac{\Pr\{X=k\}\cap\Pr\{X+Y=n\}}{\Pr\{X+Y=n\}}$$

$$= \frac{\Pr\{X=k\}\cap\Pr\{Y=n-k\}}{\Pr\{X+Y=n\}}$$

$$= \frac{e^{-\alpha}\alpha^{k}/k!\;.\;e^{-\beta}\beta^{n-k}/(n-k)!}{e^{-(\alpha+\beta)}(\alpha+\beta)^{n}/n!}$$

$$= \alpha^{k}\beta^{n-k}\left(\frac{1}{\alpha+\beta}\right)^{n}\frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^{k}\left(\frac{\beta}{\alpha+\beta}\right)^{n-k}$$

$$\therefore \Pr\{X=k|N=n\} = \binom{n}{k}p^{k}(1-p)^{n-k},\quad p=\frac{\alpha}{\alpha+\beta}$$

which is the p.m.f of binomial distribution.

Q6: 7 [2+5]

(a)

$$\because \Pr\{B(9)\le 7|B(0)=1\} = \Phi(\tfrac{y-x}{\sqrt{t}})$$

$$= \Phi(\tfrac{7-1}{\sqrt{9}})$$

$$\therefore \Pr\{B(9)\le 7|B(0)=1\} = \Phi(2) = 0.9772$$

(b)

$$\because \text{Cov}[B(s),B(t)] = \sigma^2 \text{s} \quad \text{for } 0\le s<t$$

For standard Brownian motion $(\sigma^2=1)$, we have

$$\text{Cov}[B(s),B(t)] = \text{s} \quad \text{for } 0\le s<t$$

$$\Rightarrow \text{Cov}\left[B\left(\frac{s}{c^2}\right),B\left(\frac{t}{c^2}\right)\right] = \frac{s}{c^2} \quad \text{for } 0\le\frac{s}{c^2}<\frac{t}{c^2}$$

Multiply both sides by c^2, we get

$$c^2 \operatorname{Cov}\left[B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right] = s$$

$$\therefore \operatorname{Cov}\left[cB\left(\frac{s}{c^2}\right), cB\left(\frac{t}{c^2}\right)\right] = s$$

$$\because W(t) = cB(t / c^2)$$

$$\therefore \operatorname{Cov}\left[W(s), W(t)\right] = s, \ \ 0 \le s < t \qquad (1)$$

Similarly, we can obtain

$$\operatorname{Cov}\left[W(s), W(t)\right] = t, \ \ 0 \le t < s \qquad (2)$$

We can deduce from (1) and (2) that $\operatorname{Cov}\left[W(s), W(t)\right] = \min\{s, t\} \ \text{ for } s, t \ge 0$

$\therefore W(t) = cB(t / c^2), \ \ c > 0$ is a standard Brownian motion.
