College of Science.
Department of Mathematics

كلية العلوم
قسم الرياضيات

Final Exam
Academic Year 1445-1446 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	Complex Analysis التحليل المركب		اسم المقرر
Course Code	487 ريض		رمز المقرر
Exam Date	2023-12-13	1445-05-29	تاريخ الامتحان
Exam Time	01: 00 PM		وقت الامتحان
Exam Duration	3 hours	ثلاث ساعات	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 6 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
- Calculators are not allowed.

تعليمات عامة:

- عدد صفحات الامتحان 6 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
- يمنع استخدام الألات الحاسبة.

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	C.L.O 1.1	1		
2	C.L.O 1.2	2		
3	C.L.O 1.3	3		
4	C.L.O 2.1	4		
5	C.L.O 2.2	5&6		
6	C.L.O 2.3	7&8		
7				
8				

EXAM COVER PAGE

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total

Question 1

Prove in details that $\log z_1 z_2 = \log z_1 + \log z_2$. Is it true that $\log z^n = n \log z$, where $n \geq 2$, justify your answer.

See Book

Question 2

Let $f(z) = |z|^2$. Prove that f is differentiable only at one point. Is f analytic? Give reasons.

f is differentiable at 0 is simply to observe that

$$\lim_{z \to 0} \frac{|z|^2}{z} = \lim_{z \to 0} \bar{z} = 0.$$

Besides, if $z_0 \neq 0$, then

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{z \to z_0} \frac{|z| - |z_0|}{z - z_0} \left(|z| + |z_0|\right).$$

Now, if z approaches z_0 along the circle centered at 0 passing through z_0, then the previous limit is 0. And if z approaches z_0 along the ray $\left\{\lambda z_0 \mid \lambda \in (1, +\infty)\right\}$, then the previous limit is $2\overline{z_0} \neq 0$. Therefore the limit does not exist.

So, using the definition of differentiability I found out that the $f(z)$ is differentiable at zero. But since it is not differentiable in a neighborhood of zero therefore it cannot be analytic at zero and hence is nowhere analytic.

Or, using Cauchy Riemann equations,

$f(z) = |z|^2 = x^2 + y^2$

$\implies u(x, y) = x^2 + y^2$ and $v(x, y) = 0$

Then I started off by checking whether the Cauchy-Riemann equations were satisfied, and got,

$u_x = 2x \; , u_y = 2y$

$v_x = 0 \, , v_y = 0$

$f(z)$ can only be analytic at the origin

Or,

f(z) is function of z^* since $|z|^2 = zz^*$ so it is not analytic

Question 3

State Liouville's Theorem. Use this Theorem to prove that $\cos z$, is not bounded. Explain why we can't use this Theorem on $\tan z$.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant througout the plane.*

Note that $\cos z$ is not constant, but an entire function, hence it is necessary unbounded. For $f(z) = \tan z = \frac{\cos z}{\sin z}$, we can't apply Liouville's Theorem since $\tan z$ is not analytic for any $z = n\pi, n \in \mathbb{Z}$.

Question 4

State and prove the Cauchy's Residue Theorem.

See book.

Question 5

Find the integral

$$\int_\gamma \frac{e^z dz}{z^2(z^2+4)}$$

where γ is the positively oriented circle $|z| = 3.$

$f(z) = \frac{e^z}{z^2(z^2+4)}$ has 3 isolated singularities, $z = 0$ is a pole of degree 2 $and\ z = \pm 2i$ are simple poles

$Res_{\,z=0}\big(f(z)\big) = \frac{1}{4}$, $Res_{\,z=\pm c}\big(f(z)\big) = \lim_{z\to\pm c}\frac{e^z}{2z^3}$, where $c = 2i$, thus $Res_{\,z=2i} f(z) = \frac{-e^{2i}}{16\,i}$, and $Res_{\,z=-2i} f(z) = \frac{e^{-2i}}{16\,i}$, hence $\int_\gamma \frac{e^z dz}{z^2(z^2+4)} = 2\pi i\left(\frac{1}{4} - \frac{1}{8}\left(\frac{e^{2i}-e^{-2i}}{2i}\right)\right) = \frac{\pi i}{2}\left(1 - \frac{\sin 2}{2}\right).$

Question 6

Find Laurant series expansion of the function $f(z) = \frac{1}{z^2-5z+4}$ in the annulus $1 < |z| < 4$, then use this expansion to find the integral $\int_\gamma z^{14} f(z)dz,$ where γ is the positively oriented circle $|z| = 2.$

$$f(z) = \frac{1}{z^2-5z+4} = \frac{1}{(z-4)(z-1)} = \frac{-1}{3(z-4)} + \frac{1}{3(z-1)}$$

$$\frac{-1}{3(z-4)} = \frac{1}{3}\sum_{i=0}^{\infty}\frac{z^i}{4^{i+1}} \quad for\ all\ \left|\frac{z}{4}\right| < 1,$$

$$\frac{1}{3(z-1)} = \frac{1}{3}\sum_{i=0}^{\infty}\frac{1}{z^{i+1}} \quad for\ all\ \left|\frac{1}{z}\right| < 1\,,$$

$So, f(z) = \frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{z^i}{4^{i+1}} + \frac{1}{z^{i+1}}\right)$ *in the annulus* $1 < |z| < 4$

$$z^{14}f(z) = \frac{1}{3}\sum_{i=0}^{\infty}\frac{z^{i+14}}{4^{i+1}} + \frac{1}{3}\left(z^{13} + z^{12} + \cdots + \frac{1}{z}\right) + \frac{1}{3}\sum_{i=2}^{\infty}\frac{1}{z^i}$$

Thus, $Res_{\,z=0}\left(z^{14}f(z)\right) = \frac{1}{3}$, hence by Residue Theorem, $\int_\gamma z^{14}f(z)dz = \frac{2\pi i}{3}$

<u>Question 7</u>

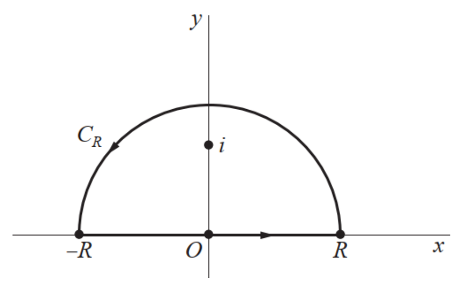
Use the Residue Theorem to find

$$\int_{-\infty}^{\infty}\frac{x\sin x\ dx}{x^4 + 3x^2 + 2}$$

All roots of $x^4 + 3x^2 + 2$ are complex, namely $\pm\sqrt{2}i, \pm i$.

Let $f(z) = \frac{ze^{iz}}{z^4+3z^2+2}$, Take $R > 2$ and consider the two positively oriented contours C_R as shown in the graph

Thus, $c_0 = i, c_1 = 2i$ are simple poles of $f(z)$ inside C_R with residues $\frac{1}{2e}, \frac{-e^{\sqrt{2}}}{2}$, respectively.



$$\int_{-R}^{R}\frac{ze^{iz}\,dz}{z^4+3z^2+2} = 2\pi i\left(\frac{1}{2e} - \frac{e^{\sqrt{2}}}{2}\right) - \int_{C_R} f(z)dz$$

Note that $|f(z)| = \left|\frac{ze^{iz}}{z^4+3z^2+2}\right| \le \frac{R}{(R^2-2)(R^2-1)}$, thus

$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi R^2}{R^4-3R^2+2} \underset{R\to\infty}{\longrightarrow} 0$, hence $\int_{-\infty}^{\infty}\frac{x\sin x\,dx}{x^4+3x^2+2} = Im\left[\lim_{R\to\infty}\int_{-R}^{R}\frac{ze^{iz}\,dz}{z^4+3z^2+2}\right] = \pi(\frac{1}{e} - e^{\sqrt{2}})$

<u>Question 8</u>

Use Residue Theorem to find

$$\int_0^{2\pi}\frac{d\theta}{1+\sin\theta\cos\theta}$$

Using substitutions, $z = e^{i\theta},\ 0 \le \theta \le 2\pi,\ \sin\theta = \frac{z-\frac{1}{z}}{2i}, \cos\theta = \frac{z-\frac{1}{z}}{2i}$ *and* $d\theta = \frac{dz}{iz}$ we obtain

$\int_0^{2\pi}\frac{d\theta}{1+\sin\theta\cos\theta} = \int_{|z|=1}\frac{4z\,dz}{z^4+4iz^2-1} = \int_{|z|=1}\frac{p(z)dz}{q(z)}$, and $q(z)$ has 4 different zeroes, that is 4 simple poles of $f(z) = \frac{p(z)}{q(z)}$. The only poles inside the unit ball are $c_{0,1} = \pm\left(\left(-2+\sqrt{3}\right)i\right)^{\frac{1}{2}}$, since $\left|c^2{}_{0,1}\right| < 1$, with residues;

$Res_{\,z=c_i}\left(f(z)\right) = \frac{p(c_i)}{q'(c_i)} = \frac{1}{(c_i)^2+2i} = \frac{-i}{\sqrt{3}}$, hence $\int_0^{2\pi}\frac{d\theta}{1+\sin\theta\cos\theta} = 2\pi i\left(\frac{-2i}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}}$