King Saud University: Mathematics Department Math-254
First Semester 1445 H Solution of Makeup First Midterm Exam.
Maximum Marks = 25 Time: 90 mins.

Questions: (6 + 6 + 7 + 6) Marks

Question 1:

Convert the equation $x^2 - 5 = 0$ to the fixed-point problem $x = x + k(x^2 - 5)$ with k a nonzero constant. Find a value of k to ensure rapid convergence of the scheme $x_{n+1} = x_n + k(x_n^2 - 5)$, for $n \geq 0$ to $\alpha = \sqrt{5}$. If $x_0 = 2$, compute absolute error $|\sqrt{5} - x_2|$.

Solution. Given $x^2 - 5 = 0$, and it can be written as for $k \neq 0$

$$k(x^2 - 5) = 0 \quad \text{or} \quad -x + x + k(x^2 - 5) = 0.$$

From this we have

$$x = x + k(x^2 - 5) = g(x),$$

and it gives the iterative scheme

$$x_{n+1} = x_n + k(x_n^2 - 5) = g(x_n), \qquad n \geq 0.$$

For guaranteed convergence of this scheme, we mean that

$$|g'(x)| < 1 \qquad \text{or} \qquad |1 + 2kx| < 1 \qquad \text{or} \qquad -1 < 1 + 2kx < 1.$$

Moreover, the convergence will be rapid if

$$g'(\alpha) = 1 + 2\alpha k = 0.$$

Since $\alpha = \sqrt{5}$, therefore, $1 + 2\sqrt{5}k = 0$. Thus, we have $k = -\dfrac{1}{2\sqrt{5}} = -0.2236$. Using $x_0 = 2$, we get

$$x_1 = x_0 + k(x_0^2 - 5) = 2 - 0.2236(2^2 - 5) = 2.2236,$$

and

$$x_2 = x_1 + k(x_1^2 - 5) = 2.2236 - 0.2236(2.2236^2 - 5) = 2.2360,$$

so $|\sqrt{5} - x_2| = |2.2361 - 2.2360| = 0.0001$, is the absolute error. •

Question 2:

Successive approximations x_n to the desired root are generated by the scheme

$$x_{n+1} = \frac{1 + 3x_n^2}{4 + x_n^3}, \qquad n \geq 0.$$

Find $f(x_n)$ and $f'(x_n)$ and then use the Newton's method to find the approximation of the root accurate to 10^{-2}, starting with $x_0 = 0.5$.

Solution. Given

$$x = \frac{1+3x^2}{4+x^3} = g(x),$$

and

$$x - g(x) = x - \frac{1+3x^2}{4+x^3} = \frac{x^4 - 3x^2 + 4x - 1}{4+x^3}.$$

Since, $f(x) = x - g(x) = 0$, therefore, we have, $f(x_n) = x_n^4 - 3x_n^2 + 4x_n - 1$ and $f'(x_n) = 4x_n^3 - 6x_n + 4$. Using these functions values in the Newton's iterative formula (**??**), we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 3x_n^2 + 4x_n - 1}{4x_n^3 - 6x_n + 4}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 0.5$, we get

$$x_1 = x_0 - \frac{x_0^4 - 3x_0^2 + 4x_0 - 1}{4x_0^3 - 6x_0 + 4} = 0.5 - \frac{0.3125}{1.5} = 0.2917.$$

Similarly, the other approximations can be obtained as

$$x_2 = 0.2917 - \frac{(-0.0813)}{2.3491} = 0.3263; \quad \text{and} \quad x_3 = 0.3263 - \frac{(-0.0029)}{2.1812} = 0.3276.$$

Notice that $|x_3 - x_2| = |0.3276 - 0.3263| = 0.0013 < 10^{-2}$. •

Question 3:

Show that the rate of convergence of Newton's method at the root $\alpha = 1$ of the equation $(x-1)^2 \sin x = 0$ is linear. Use quadratic convergence method to find x_2 using $x_0 = 1.5$. Compute the relative error.

Solution. Since

$$f(x) = (x-1)^2 \sin x \quad \text{and} \quad f'(x) = 2(x-1)\sin x + (x-1)^2 \cos x,$$

and $f'(1) = 0$, gives that $\alpha = 1$ is the multiple root. Using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{((x_n-1)^2 \sin x_n)}{(2(x_n-1)\sin x_n + (x_n-1)^2 \cos x_n)} = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)},$$

for $n \geq 0$. The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}.$$

Then

$$g(x) = x - \frac{((x-1)\sin x)}{(2\sin x + (x-1)\cos x)},$$

and

$$g'(x) = 1 - \frac{(2\sin x + (x-1)\cos x)(\sin x + (x-1)\cos x) - ((x-1)\sin x)(3\cos x - (x-1)\sin x)}{(2\sin x + (x-1)\cos x)^2}.$$

Thus

$$g'(1) = 1 - \frac{2(\sin 1)^2}{4(\sin 1)^2} = \frac{1}{2} \neq 0,$$

and so the Newton's method converges linearly. The quadratic convergent method for multiple root is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function. To find m, we check that

$$f''(x) = 2\sin x + 4(x-1)\cos x - (x-1)^2 \sin x, \quad \text{and} \quad f''(1) = 2\sin 1 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 1.5$, we have the following two approximations

$$x_1 = x_0 - 2\frac{((x_0-1)\sin x_0)}{(2\sin x_0 + (x_0-1)\cos x_0)} = 1.0087, \quad x_2 = x_1 - 2\frac{((x_1-1)\sin x_1)}{(2\sin x_1 + (x_1-1)\cos x_1)} = 1.0000,$$

with the relative, $|\alpha - x_2)|/|\alpha| = |1 - 1.0000|/1 = 0.0000$. •

Question 4:

For what values of α the following linear system has (i) Unique solution, (ii) No solution, (iii) Infinitely many solutions, by using the simple Gaussian elimination method. Use smallest positive integer value of α to get the unique solution of the system.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & \alpha x_3 & = & 4 \\ 2x_1 & - & x_2 & + & 2\alpha x_3 & = & 1 \\ \alpha x_1 & + & 5x_2 & + & x_3 & = & 6 \end{array}$$

Solution. Using the multiples $m_{21} = 2$, $m_{31} = \alpha$, and $m_{32} = \dfrac{5-3\alpha}{-7}$, gives matrix form

$$[A|\mathbf{b}] = \begin{pmatrix} 1 & 3 & \alpha & 4 \\ 2 & -1 & 2\alpha & 1 \\ \alpha & 5 & 1 & 6 \end{pmatrix} \equiv \begin{pmatrix} 1 & 3 & \alpha & 4 \\ 0 & -7 & 0 & -7 \\ 0 & 5-3\alpha & 1-\alpha^2 & 6-4\alpha \end{pmatrix} \equiv \begin{pmatrix} 1 & 3 & \alpha & 4 \\ 0 & -7 & 0 & -7 \\ 0 & 0 & 1-\alpha^2 & 1-\alpha \end{pmatrix} = [U|c].$$

So if $1-\alpha^2 \neq 0$, then we have the unique solution of the given system while for $\alpha = \pm 1$, we have no unique solution. If $\alpha = 1$, then we have infinitely many solution because third row of above matrix gives

$$0x_1 + 0x_2 + 0x_3 = 0,$$

and when $\alpha = -1$, we have

$$0x_1 + 0x_2 + 0x_3 = 2,$$

which is not possible, so no solution.
Since we can not take $\alpha = 1$ for the unique solution, so can take next positive integer $\alpha = 2$, which gives us upper-triangular system of the form

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & = & 4 \\ & - & 7x_2 & & & = & -7 \\ & & & - & 3x_3 & = & -1 \end{array}$$

Solving this system using backward substitution, we get, $x_1 = 1/3$, $x_2 = 1$, $x_3 = 1/3$, the required unique solution of the given system using smallest positive integer value of α. •