King Saud University: Mathematics Department Math-254
Second Semester 1445 H First Midterm Exam.
Solution
Maximum Marks = 25 Time: 90 mins.

Question 1: Show that the x-value of the intersection point (x, y) of the graphs $y = x^3+2x-1$ and $y = \sin x$ is lying in the interval $[0.5, 1]$. Then use Secant method to find its second approximation, when $x_0 = 0.5$ and $x_1 = 0.55$. Also, find the intersection point. [6 Marks]

Solution. For the intersection of the graphs, we mean that $x^3 + 2x - 1 = \sin x$ and it gives, $x^3 + 2x - 1 - \sin x = 0$. Thus, $f(x) = x^3 + 2x - \sin x - 1$. Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5) = -0.3544,\ f(1.0) = 1.1585$, which shows that $f(0.5)f(1.0) < 0$. Hence the x-value (or root of $f(x) = 0$) lies in the interval $[0.5, 1.0]$. Applying Secant iterative formula to find the approximation of this root of the equation, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^3 + 2x_n - \sin x_n - 1)}{(x_n^3 + 2x_n - \sin x_n - 1) - (x_{n-1}^3 + 2x_{n-1} - \sin x_{n-1} - 1)}, \qquad n \geq 1.$$

Finding the first approximation, taking $n = 1$, we get

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^3 + 2x_1 - \sin x_1 - 1)}{(x_1^3 + 2x_1 - \sin x_1 - 1) - (x_0^3 + 2x_0 - \sin x_0 - 1)},$$

and using the initial approximations $x_0 = 0.5$ and $x_1 = 0.55$, we get

$$x_2 = 0.55 - \frac{(0.55 - 0.5)((0.55)^3 - 2(0.55) - \sin(0.55) - 1)}{((0.55)^3 - 2(0.55) - \sin(0.55) - 1) - ((0.5)^3 - 2(0.5) - \sin(0.5) - 1)} = 0.6806.$$

Finding the second approximation, taking $n = 2$,

$$x_3 = x_2 - \frac{(x_2 - x_1)(x_2^3 + 2x_2 - \sin x_2 - 1)}{(x_2^3 + 2x_2 - \sin x_2 - 1) - (x_1^3 + 2x_1 - \sin x_1 - 1)},$$

and using the initial approximations $x_1 = 0.55$ and $x_2 = 0.6806$, we get

$$x_3 = 0.6806 - \frac{(0.6806 - 0.55)((0.6806)^3 - 2(0.6806) - \sin(0.6806) - 1)}{((0.6806)^3 - 2(0.6806) - \sin(0.6806) - 1) - ((0.55)^3 - 2(0.55) - \sin(0.55) - 1)},$$

$$x_3 = 0.6603.$$

Hence $x_3 = 0.6603$ is the second approximation of the x-value and $(0.6603, 0.61)$ is approximation of the intersection point. •

Question 2: Find a value of constant $\lambda(\neq -1)$ to ensure the rapid convergence to the root 1.4650 of the iterative scheme

$$x_{n+1} = \frac{\lambda x_n + x_n^{-2} + 1}{\lambda + 1}, \qquad n \geq 0.$$

Use Newton's method to find the absolute error $|\alpha - x_2|$, when $x_0 = 1.5$. [6 Marks]

Solution. Since $\alpha = 1.4650$ and we have

$$g(x) = \frac{\lambda x + x^{-2} + 1}{\lambda + 1}, \quad g'(x) = \frac{\lambda - 2x^{-3}}{\lambda + 1}.$$

Hence for rapid convergence, $g'(\alpha) = g'(1.4650) = \dfrac{\lambda - 2(1.4650)^{-3}}{\lambda + 1} = 0$, gives, $\lambda = 0.6361$.

Thus the iterative scheme becomes

$$x_{n+1} = \frac{(0.6361)x_n + x_n^{-2} + 1}{0.6361 + 1}, \qquad n \geq 0.$$

To use Newton's method we need $f(x)$, which can be obtained as

$$f(x) = g(x) - x = \frac{(0.6361)x + x^{-2} + 1}{1.6361} - x = \frac{x^{-2} + 1 - x}{1.6361} = 0,$$

or

$$f(x) = x^{-2} + 1 - x = 0, \quad \text{and} \quad f'(x) = -2x^{-3} - 1.$$

Using these functions values in the Newton's iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{-2} + 1 - x_n}{(-2x_n^{-3} - 1)}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 1.5$, we get

$$x_1 = x_0 - \frac{x_0^{-2} + 1 - x_0}{(-2x_0^{-3} - 1)} = 1.4651,$$

and the second approximation of the root using the first approximation $x_1 = 1.4656$, we get

$$x_2 = x_1 - \frac{x_1^{-2} + 1 - x_1}{(-2x_1^{-3} - 1)} = 1.4656.$$

Thus

$$AbsE = |\alpha - x_2| = |1.4650 - 1.4656| = 0.0006,$$

is the possible absolute error. •

Question 3: Show that the equation $(1-x)^2 e^{1-x} = 0$ has multiple root $\alpha = 1$ with order of multiplicity 2. Develop the Modified Newton's formula for computing the approximation of this root and use it to find the second approximation x_2 using $x_0 = 0.75$. Show that the developed formula converges only quadratically to the root. [7 Marks]

Solution. Since $\alpha = 1$ is a root of $f(x) = (1-x)^2 e^{1-x}$, so

$$\begin{array}{lllll} f(x) &=& (1-x)^2 e^{1-x}, & f(1) &=& 0, \\ f'(x) &=& -2(1-x)e^{1-x} - (1-x)^2 e^{1-x} = (x-3)(1-x)e^{1-x}, & f'(1) &=& 0, \\ f''(x) &=& (7 - 6x + x^2)e^{1-x}, & f''(1) &=& 2 \neq 0, \end{array}$$

the function $f(x)$ has multiple root $\alpha = 1$ with order of multiplicity 2. Using modified Newton's iterative formula, we get

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{(1-x_n)^2 e^{1-x_n}}{(x_n - 3)(1-x_n)e^{1-x_n}} = x_n - 2\frac{(1-x_n)}{(x_n - 3)}, \quad n \geq 0.$$

Now evaluating this at the give approximation $x_0 = 0.75$, gives

$$x_1 = x_0 - 2\frac{(1-x_0)}{(x_0 - 3)} = 0.9722 \quad \text{and} \quad x_2 = x_1 - 2\frac{(1-x_0)}{(x_1 - 3)} = 0.9996.$$

The fixed point form of the developed modified Newton's formula is

$$x_{n+1} = g(x_n) = x_n - 2\frac{(1-x_n)}{(x_n-3)},$$

where

$$g(x) = x - 2\frac{(1-x)}{(x-3)}.$$

By taking first derivative, we have

$$g'(x) = 1 - 2\left(\frac{2}{(x-3)^2}\right), \quad g'(x) = 1 - \left(\frac{4}{(x-3)^2}\right),$$

and

$$g'(1) = 1 - \frac{4}{(1-3)^2} = 1 - 1 = 0.$$

Now taking the second derivative, we have

$$g''(x) = 0 - \left(\frac{-8}{(x-3)^3}\right), \quad g''(1) = 0 - 1 = -1 \neq 0.$$

Thus the method converges only quadratically to the given root. •

Question 4: Consider the nonlinear system

$$\begin{array}{rcl} x^3 + \alpha y^2 & = & 21 \\ x^2 + 2y + 2 & = & 0 \end{array}$$

If the determinant of the Jacobian matrix J is 18 when the initial approximation is $(x_0, y_0)^T = (1, -1)^T$, then use Newton's method to find the first approximation $(x_1, y_1)^T$. [6 Marks]

Solution. Given

$$\begin{array}{rcl} f_1(x,y) & = & x^3 + \alpha y^2 - 21, \quad f_{1x} = 3x^2, \quad f_{1y} = 2\alpha y, \\ f_2(x,y) & = & x^2 + 2y + 2, \quad\quad f_{2x} = 2x, \quad\quad f_{2y} = 2. \end{array}$$

The Jacobian matrix J at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 & 2\alpha y \\ 2x & 2 \end{pmatrix},$$

and the Jacobian matrix J at the given initial approximation $(1, -1)^T$ can be calculated as

$$J = \begin{pmatrix} 3 & -2\alpha \\ 2 & 2 \end{pmatrix}.$$

Given the determinant of the Jacobian matrix is 18, so we have

$$|J| = \begin{vmatrix} 3 & -2\alpha \\ 2 & 2 \end{vmatrix} = 6 + 4\alpha = 18, \quad \text{gives} \quad \alpha = 3.$$

Thus using $\alpha = 3$, we have the function values

$$f_1(1,-1) = 1 + 3(-1)^2 - 21 = -17, \quad \text{and} \quad f_2(1,-1) = (1)^2 + 2(-1) + 2 = 1,$$

and the Jacobian matrix can be obtained as

$$J = \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix}.$$

Thus the inverse of the Jacobian matrix can be obtained as

$$J^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix}.$$

Now using Newton's formula for finding the first approximation we do as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}\begin{pmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{pmatrix},$$

or

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - J^{-1}\begin{pmatrix} f_1(1,-1) \\ f_2(1,-1) \end{pmatrix},$$

so using the given information, we get

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{18}\begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -17 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5556 \\ -3.0556 \end{pmatrix},$$

the required first approximation. •