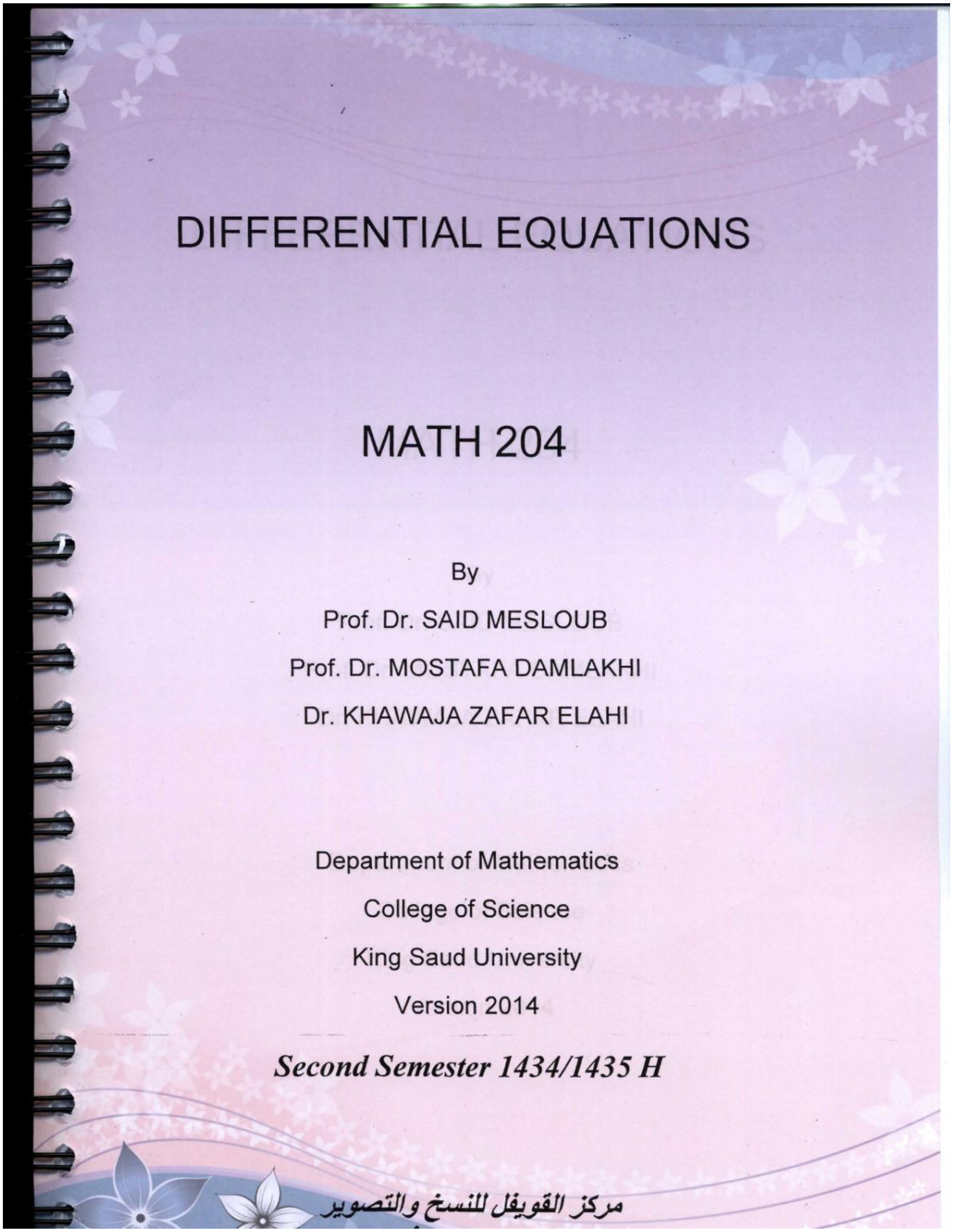
Exercises in:



Were E is the Young' modulus, I is the moment of Inertia and $f(x)$ is the load per unit length.

6) Growth and Decay

$$\frac{dP}{dt} = KP \quad ; \quad P(t_0) = P \ . \tag{17}$$

Where p is a given quantity and K is the constant of proportionality.

7) Newton' s of Heating and Cooling

$$\frac{dT}{dt} = k(T - T_s) \ . \tag{18}$$

Where $\frac{dT}{dt}$ is the rate of the body, $T - T_s$ is temperature difference between the body T and its surrounding T_s and k 'is the constant of proportionality.

Review exercises

1) Verify that $\sin(kx)$ is a solution of the equation

$$\frac{d^2x}{dt^2} + k^2x = 0 \ .$$

2) Verify that e^{-2x} and $3e^{-2x} + 4e^{x}$ are solution of the equation

$$y''' - 3y' + 2y = 0 \ .$$

3) Verify that

$$y = 2\ln x + 4.$$

is a solution of the differential equation

$$x^2y'' - xy' + y = 2\ln x \quad ; \quad x > 0.$$

4) Verify that $e^x \ln x$ is a solution of the differential equation

$$y'' - 2y' + y = \frac{1}{x}(y - y') \quad ; \quad x > 0 \ .$$

5) Show that the function

$$y(x) = c_1e^{-x} + c_2xe^{-x} + 1.$$

is a solution of the differential equation

$$y'' + 2y' + y = 1 \ .$$

for any values of the constants c_1 and c_2.

6) Show that the functions

$$y_1(x) \equiv 0,$$

and

$$y_2(x) = \frac{x^2}{4}, x \geqq 0,$$

are solutions of the differential equation

$$y' = \sqrt{y}\ .$$

In problems 7-10 verify that the indicated function $y = \phi(x)$ is an explicit solution of given first differential equation.

7) $(y - x)y' = y - x + 8 \quad ; \quad y = x + 4\sqrt{x+2}$.

8) $y' = 25 + y^2 \ ; \quad y = 5\tan 5x$.

9) $y' = 2xy^2 \quad ; \ y = \frac{-1}{4+x^2}$.

10) $2y' = y^3 \cos x \ ; \ y = (1 - \sin x)^{-\frac{1}{2}}$.

In problems 11-12 verify that the indicted expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case.

11) $\frac{dy}{dx} = (y - 1)(1 - 2y) \quad ; \ \ln(\frac{2y-1}{y-1}) = x$.

12) $2xydx + (x^2 - y)dy = 0 \ ; \ y^2 - 2x^2y = 1$.

13) *Bessel's* function of index zero is defined by the power series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}\ .$$

Verify that $J_0(x)$ is a solution of the differential equation

$$xy'' + y' + xy = 0 \ ; \quad x > 0\ .$$

14) Show that for $x > 0$, $y(x) = (\frac{2}{\sqrt{3}})x^{\frac{3}{2}}$ is a solution of the differential equation $yy'' = x$.

In problems 15- 24 in the following find the differential equation of the family of curves by elimination the arbitrary constants.

15) $y \sin x - xy^2 = c$.

16) $cy^2 = x^2 + y$.

17) $y = cx + c^2 + 1$.

18) $y = c_1x^2 + c_2x + c_3$.
19) $y = c_1e^x + c_2xe^x$.
20) $y = x^2 + c_1e^{2x} + c_2e^{3x}$.
21) $y = c_1e^{2x}\cos 3x + c_2e^{2x}\sin 3x$.
23) $y = c_1x^2 + c_2e^{-x}$.
24) $y = x^2 + c_1x + c_2e^{-x}$.

In problem 25- 35 obtain the differential equation of the family of plane curves described.

25) Straight lines through the origin.
26) Straight lines with slope and y-intercept equal.
27) Straight lines with slope and x-intercept equal.
28) Straight lines at a fixed distance p from the origin.
29) Circles with center at origin.
30) Circles with center on the x-axis
31) Circles with fixed radius r and tangent to the x-axis.
32) Parabolas with vertex on the x- axis , with axis parallel to the y-axis , and with distance from focus to vertex fixed as a.
33) Parabolas with vertex on the y - axis , with axis parallel to the x-*axis* , and with distance from focus to vertex fixed as a.
34) Parabolas with vertex and focus on the x-axis.
35) Parabolas with axis parallel to the x-axis.

Then the largest region for which the given IVP has a unique solution is R_1.

Example(6) Find the largest region of the xy- plane for which the initial value problem

$$\begin{cases} \sqrt{\frac{x}{y}}y' = \cos(x+y) \quad ; \quad y \neq 0 \\ y(1) = 1 \end{cases} .$$

Solution We have

$$y' = \cos(x+y)(\frac{x}{y})^{\frac{-1}{2}} = f(x,y).$$

Then

$$\frac{\partial f}{\partial y} = -\sin(x+y)(\frac{x}{y})^{\frac{-1}{2}} - \frac{1}{2}\cos(x+y)(\frac{x}{y})^{\frac{-3}{2}}(\frac{-x}{y^2}) .$$

So f and $\frac{\partial f}{\partial y}$ are continuous

$$R = \left\{(x,y) \quad ; \quad \frac{x}{y} > 0\right\}.$$

or

$$R = \{(x,y) \;\; ; \;\; x<0 \;\; \& \; y<0\} \cup \{(x,y) \;\; ; \;\; x>0 \;\; \& \;\; y>0\} .$$

But

$$(1,1) \in R_1 = \{(x,y) \;\; ; \;\; x>0 \;\; \& \;\; y>0\}.$$

Then the largest region for which the given IVP has a unique solution is R_1.

Exercises (2.1)

1) Determine and sketch the largest region of the $xy-$plane for which the following initial value problems has a unique solution

$$\begin{cases} \frac{dy}{dx} = \frac{y+2x}{y-2x} \\ y(1) = 0 \end{cases} .$$

In problems 2- 10 determine a region of the xyplane for which the given differential equation would have a unique solution whose graph passes through a point $(x_0 \, , y_0)$ in the region .

2) $\frac{dy}{dx} = y^{\frac{2}{3}}$.
3) $\frac{dy}{dx} = \sqrt{xy}$.
4) $x\,\frac{dy}{dx} = y^{\frac{1}{3}}$.
5) $\frac{dy}{dx} - \ln y = \sqrt{x}$.
6) $(4 - y^2)y' = x^2 y$.
7) $\ln(x-1)y' = \sin^{-1}(y)$.
8) $(x^2 + y^2)y' = \sqrt{y}\,x$.
9) $(y - x)y' = y + x^2$.
10) $(1 + y^3)y' = \tan^{-1}(x)$.

In problems 11-14 determine whether Theorem (2.1) guarantees that the differential equation

$$y' = \sqrt{y^2 - 9}.$$

possesses a unique solution through the given point .

11) (1,4) . 12) (5,3) .
13) (2,−3) . 14) (−1,1) .

In problems 15- 18 , $y = \frac{1}{x^2+c}$ is one -parameter family of solution of the first-order DE

$$y' + 3xy^2 = 0 \quad .$$

Find a solution of first -order IVP consisting of this differential equations and the given initial condition. Give the largest interval I over which the solution is defined.

15) $y(3) = \frac{1}{5}$. 16) $y(-2) = \frac{1}{2}$.
17) $y(0) = 2$. 18) $y(\frac{1}{3}) = -4$.

18) (a) Verify that $y = \tan(x + c)$ is one -parameter family of solution of the differential equation

$$y' = 1 + y^2 \ .$$

(b) Since

$$f(x, y) = 1 + y^2.$$

and $\frac{\partial f}{\partial y} = 2y$ are continuous everywhere, the region R in the Theorem (2.1) can be taken to be the entire $xy-$plane. Use family of solutions in part (a) to find an explicit solution of the first -order initial -value problem

$$\begin{cases} y' = 1 + y^2 \\ y(0) = 0 \end{cases} .$$

Even $x_0 = 0$ in the interval $-2 < x < 2$ explain why the solution is not defined on this interval ? .

19 (a) Verify that

$$3x^2 - y^2 = c.$$

is one -parameter family of solution of the differential equation

$$y\frac{dy}{dx} = 3x \ .$$

(b) Sketch the graph of the implicit solution

$$3x^2 - y^2 = 3.$$

Find all explicit solution $y = \phi(x)$ of the differential equation in part (a) defined by this relation .Give the interval I of definition of explicit solution.

(c) The point $(-2, 3)$ is not on the graph

$$3x^2 - y^2 = 3$$

but which the explicit solution in part (b) satisfy $y(-2) = 3$?.

2.2 Separable equations

We begin our study of methods for solving first -order differential equation by studying an equation of the form

$$M(x, y)dx + N(x, y)dy = 0.$$

where M and N are two functions of x and y . Some equations of this type are so simple that they can be written in the form

$$F(x)dx + G(y)dy = 0. \qquad (1)$$

that is , the variables can be separated. The a solution can be written immediately. For, it is only a matter of finding a function H such that

$$dH(x, y) = F(x)dx + G(y)dy = 0.$$

the the solution of (1) is $H(x, y) = c$ where c is an arbitrary constant.

Example (1) Find the solution of differential equation

$$2x(y^2 + y)dx + (x^2 - 1)ydy = 0 \quad ; \quad y \neq 0 \ . \qquad (2)$$

Exercises (2.2)

In exercises 1 through 14 find solution of the following differential equations .

1) $\sin x \sin y\ dx + \cos x \cos y\ dy = 0$.

2) $ye^{2x}dx = (4+e^{2x})dy$.

3) $x\cos^2(y)dx + \tan(y)dy = 0$.

4) $xy^3dx + (y+1)e^{-x}\ dy = 0$

5) $y\ln x\ \ln y\ dx + dy = 0$.

6) $y' = x - xy - y + 1$.

7) $\cos x \sin y\ dy$ -($\cos y \cos x$ +$\cos x$)$dx = 0$; $0 < x < \frac{\pi}{2}$.

8) $yy' = y^2x^3 + y^2x$; $y \neq 0$.

9) $(1+y^2)\cos x\ dx = (1+\sin^2 x)2ydy$.

10) $\frac{dy}{dx} = e^{4x-3y}$.

11) $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$.

12) $y\ln x\ \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$.

13) $(e^y+1)^2e^{-y}dx + (e^x+1)^3e^{-x}dy = 0$.

14) $\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$.

In problems 15 through 18, find an implicit and an explicit solution of the given initial-value problem.

15) $\begin{cases} \frac{dy}{dx} = 4(y^2+1) \\ y(\frac{\pi}{4}) = 1 \end{cases}$.

16) $\begin{cases} \frac{dy}{dx} = \frac{y^2-1}{x^2-1} \\ y(2) = 2 \end{cases}$.

17) $\begin{cases} x^2\frac{dy}{dx} = y - xy \\ y(-1) = -1 \end{cases}$.

18) $\begin{cases} (1+x^4)dy + x(1+4y^2)dx = 0 \\ y(1) = 0 \end{cases}$.

19) Find a solution of $x\frac{dy}{dx} = y^2 - y$ that passes through the indicated points.

(a) $(0,1)$. (b) $(0,0)$. (c) $(\frac{1}{2},\frac{1}{2})$ (d) $(2,\frac{1}{4})$.

20) Find the singular solution of the DE : $\frac{dy}{dx} = x\ \sqrt{1-y^2}$.

2.3 Equations with Homogeneous Coefficients

Definition (2.3.1) Let f be a function of x and y with domain D. The function f is called homogeneous of degree $k \in \mathbb{R}$ if

$$f(tx,ty) = t^k f(x,y) \quad \text{for all } t > 0 \quad ; \quad t \in \mathbb{R} .$$

hence

$$x\frac{dy}{dx} = \frac{y(1+xy)}{(1-xy)},$$

$$\frac{du}{dx} - y = \frac{y(1+xy)}{(1-xy)},$$

$$\frac{du}{dx} - \frac{u}{x} = \frac{u}{x}(\frac{1+u}{1-u}),$$

$$\frac{du}{dx} = \frac{2u}{x(1-u)},$$

by separating the variables we have

$$\frac{1}{2}\int(\frac{1}{u}-1)du = \int\frac{dx}{x},$$

$$\ln u - u - \ln x^2 = c \implies \frac{u}{x^2} = e^u c_1 \;;\quad c_1 = e^c ,$$

then the solutions of the differential equation (11) is given by

$$\frac{y}{x} = e^{xy}c_1, \text{ where } c_1 \neq 0 \text{ is an arbitrary constant.}$$

Exercises (2.3)

In exercises 1 through 13, obtain a family of solutions

1) $3(3x^2+y^2)dx - 2xydy = 0$.

2) $(x-y)dx + (2x+y)dy = 0$.

3) $x^2y' = 4x^2 + 7xy + 2y^2$.

4) $(x-y)(4x+y)dx + x(5x-y)dy = 0$.

5) $x(x^2+y^2)(ydx - xdy) + y^6dy = 0$.

6) $\left[x\csc\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$.

7) $xdx + \sin^2(\frac{y}{x})\left[ydx - xdy\right] = 0$.

8) $(x - y\ln y + y\ln x)dx + x(\ln y - \ln x)dy = 0$.

9) $\frac{dy}{dx} = \frac{x+3y}{3x+y}$.

10) $-ydx + (x+\sqrt{xy})dy = 0$.

11) $x\frac{dy}{dx} = y + \sqrt{x^2-y^2}$; $x > 0$.

12) $y^2dy = x(xdy - ydx)e^{\frac{x}{y}}$.

13) $\left[x - y\tan^{-1}(\frac{y}{x})\right]dx + x\tan^{-1}(\frac{y}{x})dy = 0$.

In exercises 14 through 20, find the solution of the initial value problem (IVP)

14) $\begin{cases} (x-y)dx + (3x+y)dy = 0 \\ y(3) = -2. \end{cases}$.

15) $\begin{cases} (y - \sqrt{x^2+y^2})dx - xdy = 0 \\ y(0) = 1 \end{cases}$.

16) $\begin{cases} \left[x\cos^2(\frac{y}{x}) - y\right] dx + xdy = 0 \\ y(1) = \frac{\pi}{4} \end{cases}$.

17) $\begin{cases} y^2dx + (x^2+3xy+4y^2)dy = 0 \\ y(2) = 1 \end{cases}$.

18) $\begin{cases} y(x^2+y^2)dx + x(3x^2-5y^2)dy = 0 \\ y(2) = 1. \end{cases}$.

19) $\begin{cases} (x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0 \\ y(1) = 0 \end{cases}$.

20) $\begin{cases} (x^2+2y^2)\frac{dx}{dy} = xy \\ y(-1) = 1 \end{cases}$.

21) Prove that with the aid of the substitution $y = ux$, you can solve any equation of the form

$$y^n f(x)dx + H(x,y)(ydx - xdy) = 0 \ .$$

where $H(x,y)$ is homogeneous in x and y.

22) If F is homogeneous of degree k in x andy, F can be written in the form

$$F = x^k\varphi(\frac{y}{x}) \ , \ x > 0 \ .$$

In exercises 23 through 3, solve the given differential equation by using an appropriate substitution .

23) $\frac{dy}{dx} = (x+y+1)^2$.

24) $\frac{dy}{dx} = \tan^2(x+y)$.

25) $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$.

26) $\frac{dy}{dx} = 1 + e^{y-x+5}$.

27) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$.

28) $(x+2y-4)dx - (2x+y-5)dy = 0$.

29) $(2x+3y-1)dx + (2x+3y+2)dy = 0$.

30) $x\,\frac{dy}{dx} = y\ln(xy)$.

31) $\frac{dy}{dx} = \frac{2y}{x} + \cos^2(\frac{y}{x^2})$, $x \neq 0$. (Hint put $u = \frac{y}{x^2}$) .

hence

$$\phi'(y) = y \text{ or } \phi(y) = \frac{1}{2}y^2 + C \ ,$$

So the solution of the differential equation (15) is defined implicitly by

$$F(x,y) = -\frac{1}{2}x^2y^2 + \frac{1}{2}\sin^2(x) + \frac{1}{2}y^2 + C = 0 \ .$$

Now from the initial condition $y(0) = 2$ we deduce that $C = -2$, hence the solution of the IVP is given by the curve

$$-\frac{1}{2}x^2y^2 + \frac{1}{2}\sin^2(x) + \frac{1}{2}y^2 - 2 = 0 \ .$$

Exercises (2.4)

Test each of the following equations for exactness and solve the equations. If some equations are not exact may be solved by methods discussed in the preceding sections.

1) $(6x + y^2)dx + y(2x - 3y)dy = 0$.
2) $(2xy - 3x^2)dx + (x^2 + y)dy = 0$.
3) $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$.
4) $(x - 2y)dx + 2(y - x)dy = 0$.
5) $(2xy + y)dx + (x^2 - x)dy = 0$.
6) $(1 + y^2)dx + (x^2y + y)dy = 0$.
7) $(1 + y^2 + xy^2)dx + (x^2y + y + 2xy)dy = 0.$
8) $(2xy - \tan y)dx + (x^2 - x\sec^2 y)dy = 0$.
9) $x(3xy - 4y^3 + 6)dx + (x^3 - 6x^2y^2 - 1)dy = 0.$
10) $(xy^2 + y - x)dx + x(xy + 1)dy = 0$.

Solve the following initial value problems :

11) $\begin{cases} (x - y)dx + (-x + y + 2)dy = 0 \\ y(1) = 1 \end{cases}$.

12) $\begin{cases} (x + y)dx + (x - y)dy = 0 \, , \\ y(0) = 2 \end{cases}$.

13) $\begin{cases} (x^2 + y^2)dx + 2xydy = 0 \\ y(1) = -1 \end{cases}$.

14) $\begin{cases} y' = \frac{y-x+1}{-x+y+3} \ , \ y(1) = 2 \\ y(1) = 2 \end{cases}$.

15) $\begin{cases} (e^x + y)dx + (2 + x + ye^y)dy = 0 \\ y(0) = 1 \end{cases}$.

hence

$$(n+k+5)y^n x^{k+3} + (k-n-1)x^k y^{n+1} = 0 \ ,$$

which implies that

$$\left\{ \begin{array}{l} n+k+5=0 \\ k-n-1=0 \end{array} \right| \Longrightarrow n=-3 \ , \quad k=-2 \ .$$

So the differential equation

$$(\frac{x}{y^2} - \frac{1}{yx^2})dx + (-\frac{x^2}{y^3} - \frac{1}{xy^2})dy = 0 \ , \tag{17}$$

is exact and it is easy to see that the solution of (17) is given by

$$F(x,y) = \frac{x^2}{2y^2} + \frac{1}{xy} + C = 0 \ .$$

2.5 Exercises

Solve each of the following equations.

1) $(x^2+y^2+1)dx + x(x-2y)dy = 0$.

2) $y(2x-y+1)dx + x(3x-4y+3)dy = 0$.

3) $(xy+1)dx + x(x+4y-2)dy = 0$.

4) $(2y^2+3xy-2y+6x)dx + x(x+2y-1)dy = 0$.

5) $y^2dx + (3xy+y^2-1)dy = 0$.

6) $2(2y^2+5xy-2y+4)dx + x(2x+2y-1)dy = 0$.

7) $y(2x^2-xy+10dx + (x-y)dy = 0$.

In problems 8- 12 solve the given differential equation by finding an appropriate integrating factor.

8) $(2y^2+3x)dx + 2xydy = 0$.

9) $\cos x \ dx + (1+\frac{2}{y})\sin x \ dy = 0$.

10) $(10-6y+e^{-3x})dx - 2dy = 0$.

11) $(x^4+y^4)dx - xy^3dy = 0$.

12) $(x^2-y^2+x)dx + 2xydy = 0$.

In problems 13 and 14 solve the given initial - value problem by finding an appropriate integrating factor.

13) $\left\{ \begin{array}{c} xdx + (x^2y+4y)dy = 0 \\ y(4) = 0 \end{array} \right.$.

14) $\left\{ \begin{array}{c} (x^2+y^2-5)dx = (y+xy)dy \\ y(0) = 1 \end{array} \right.$.

Do the exercise 15 by two methods.

15) $y(8x - 9y)dx + 2x(x - 3y)dy = 0$.

16) Find the value k so that the given differential equation is exact.

$(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$.

17) Prove that $\mu(x, y) = xy$ is an integrating factor of the differential equation

$(-xy \sin x + 2y \cos x)dx + 2x \cos x dy = 0$.

2.6 The General Solution of a Linear Differential equation

We consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

Suppose that P and Q are continuous functions on an interval $a < x < b$, and $x = x_0$ is any number in that interval. If y_0 is an arbitrary real number, there exist a unique solution $y = y(x)$ of the differential equation (1) that satisfies the initial condition

$$y(x_0) = y_0 \, . \tag{2}$$

Moreover, this solution satisfies $Eq.(1)$ throughout the entire interval $a < x < b$. It is easy to see that

$$\mu(x) = e^{\int P(x)dx}. \tag{3}$$

is an integrating factor for the $Eq.(1)$ and the general solution of $Eq.$ (1) is given by

$$y \ \mu(x) = \int \mu(x) \ Q(x) \ dx \ + C \, . \tag{4}$$

Since $\mu(x) \neq 0$ for all $x \in (a, b)$ we can write

$$y(x) = e^{-\int P(x)dx} \left[\int \mu(x) \ Q(x) \ dx \right] + Ce^{-\int P(x)dx}. \tag{5}$$

We can choose the constant C so that $y = y_0$ when $x = x_0$.

Example (1) Find the general solution of the differential equation

$$(1 + x^2)\frac{dy}{dx} + xy + x^3 + x = 0. \tag{6}$$

Example (4) Find the initial value problem (IVP)

$$\begin{cases} (x+1)\frac{dy}{dx}+(x+2)y=2xe^{-x} \quad ; \quad x>-1 \\ y(0)=1 \end{cases} \qquad (15)$$

Solution We have

$$\frac{dy}{dx}+(1+\frac{1}{x+1})y=\frac{2x}{x+1}e^{-x} \quad . \qquad (16)$$

Then

$$\mu(x)=e^{\int(1+\frac{1}{x+1})dx}=e^{x+\ln(x+1)}=(x+1)e^{x}.$$

and the general solution of (16) is

$$\mu(x)y=(x+1)e^{x}y=\int\mu(x)Q(x)dx=\int 2xdx=x^2+C \ ,$$

or

$$y(x)=\frac{x^2}{x+1}e^{-x}+C\frac{1}{x+1}e^{-x} \ . \qquad (17)$$

From the condition $y(0)=1$ we deduce that

$$y(0)=0+C=1 \Longrightarrow C=1$$

Hence the solution of the IVP (15) is

$$y(x)=\frac{x^2}{x+1}e^{-x}+\frac{1}{x+1}e^{-x} \ .$$

2.6 Exercises

In exercises 1 through 9, find the general solution.

1) $(x^5+3y)dx-xdy=0$.
2 $(2xy+x^2+x^4)dx-(1+x^2)dy=0$.
3) $((y-\cos^2(x))dx+\cos x dy=0 \ ; \quad o<x<\frac{\pi}{2}$.
4) $x^2y'+xy=x+1$.
5) $x\frac{dy}{dx}-y=x^2\sin x$.
6) $x^2y'+x(x+2)y=e^x$.
7) $(x+1)\frac{dy}{dx}+(x+2)y=2xe^{-x}$.
8) $\frac{dy}{dx}-\frac{3}{x-1}y=(x-1)^4$.
9) $y'-\frac{x}{1+x^2}=-\frac{x}{1+x^2}y$.

In exercises 10 through 14, solve the initial value problem.

10) $\begin{cases} y' - xy = (1-x^2)e^{\frac{1}{2}x^2} \\ y(0) = 0 \end{cases}$.

11) $\begin{cases} (1-x)\frac{dy}{dx} + xy = x(x-1)^2 \\ y(5) = 24 \end{cases}$.

12) $\begin{cases} (2x+3)y' = y + (2x+3)^{\frac{1}{2}} \\ y(-1) = 0 \end{cases}$.

13) $\begin{cases} (3xy + 3y - 4)dx + (x+1)^2 dy = 0 \\ y(0) = 1 \end{cases}$.

14) $\begin{cases} x(x^2+1)y' + 2y = (x^2+1)^3 \\ y(1) = -1 \end{cases}$.

15) Solve the differential equation $(x+a)y' = bx - ny$; where a, b, and n are constants with $n \neq 0$, $n \neq -1$.

16) Solve the equation of exercise (15) for the exceptional cases $n = 0$ and $n = -1$.

17) In the standard form

$$dy + Pydx = Qdx.$$

put $y = vw$, thus obtaining

$$w(dv + Pvdx) + vdw = Qdx \ .$$

then, by first choosing v so that

$$dv + Pvdx = 0 \ ,$$

and later determining w, show how to complete the solution

$$dy + Pydx = Qdx.$$

2.7 Bernoulli's Equation

The Bernoulli's equation is differential equation, well-known,has the general form

$$y' + P(x)y = Q(x)y^n \ . \qquad (1)$$

where $n \in \mathbb{R}$.

1) If $n = 0$ then the $Eq.(1)$ is a linear first differential equation and we can solve it.

Solution The differential equation in the IVP can be written in the form

$$y' + \frac{1}{x}y = \frac{x^2}{2}y^{-2} . \tag{14}$$

So the $Eq.$ (14) is Bernoulli's equation with $n = -2$ and we suppose that $y \neq 0$ on some interval $I = (a, b)$. From (14) we deduce that

$$y^2 y' + \frac{1}{x}y^3 = \frac{x^2}{2} ,$$

we put

$$u = y^3 \implies u' = 3y^2 y',$$

hence we have

$$\frac{1}{3}u' + \frac{1}{x}u = \frac{x^2}{2}.$$

or

$$u' + \frac{3}{x}u = \frac{3}{2}x^2 , \tag{15}$$

The $Eq.$ (15) is linear which has an integrating factor $\mu(x) = x^3$, then the solution of (15) is

$$ux^3 = \frac{1}{4}x^6 + C,$$

so the solution of the differential equation is

$$y^3 = \frac{1}{4}x^3 + \frac{1}{x^3}C . \tag{16}$$

Now we use the condition $y(1) = 1$, then $C = \frac{3}{4}$, so the solution of the IVP (13) is

$$y^3 = \frac{1}{4}x^3 + \frac{3}{4x^3}. \tag{17}$$

2.7 Exercises

In exercises 1 through 10 , solve the following differential equations .

1) $2x^3y' = y(y^2 + 3x^2)$; $x > 0$.
2) $\frac{dy}{dx} = y(xy^3 - 1)$.
3) $x\frac{dy}{dx} - (1 + x)y = xy^2$; $x > 0$.
4) $y' - 2xy = 4xy^{\frac{1}{2}}$, $y > 0$.
5) $xy' - \frac{1}{2\ln x}y = y^2$.
6) $\frac{dy}{dx} + y = xy^3$.

7) $\frac{dy}{dx} + (\sin x)y = (\sin x)y^2$.
8) $2\frac{dy}{dx} - \frac{y}{x} = -\frac{x}{y^2}$; $x > 0$.
9) $(x+1)y' + (x+2)y = 2e^x y^2$; $x > -1$.
10) $\frac{3}{2}y^{1/2}y' + y^{3/2} = 1$.

In exercises 11 through 15, solve the following initial value problems (IVP)

11) $\begin{cases} xy^{3/2}\, y' + y^{5/2} = x^{-3/2}x \;;\; x > 0 \;,\; y > 0 \\ y(1) = 1 \end{cases}$.

12) $\begin{cases} (\cos x - x\sin x + y^2)dx + 2xydy = 0 \;;\; y \neq 0 \;\&\; x > 0 \\ y(\pi) = 1 \end{cases}$.

13) $\begin{cases} 5xy^2y' + y^3 = 32(1 + \ln x)y^{-2} \;;\; x > 0 \;\&\; y \neq 0. \\ y(1) = 1 \end{cases}$.

14) $\begin{cases} (ye^{-2x} + y^3)dx - e^{-2x}dy = 0 \\ y(0) = -1 \end{cases}$.

15) $\begin{cases} x^2\frac{dy}{dx} - 2xy = 3y^4 \\ y(1) = \frac{1}{2} \end{cases}$.

Review exercises

In problems 1–7 determine a region of xy- plane for which the given differential equation would have a unique solution whose graph through a point (x_0, y_0) in the region .

1) $(4 - y^2)y' = x^3$.
2) $(x^2 + y^2)y' = y^2$.
3) $(1 + y^3)y' = x^4$.
4) $(y - x)y' = yx + x^2$.
5) $\frac{dy}{dx} = \frac{y+x}{y-x}$.
6) $\frac{dy}{dx} = 3 + \sqrt{y - 4x}$.
7) $(9 - y^2)y' = x^3 + 1$.

Fin the largest region of xyplane for the following initial value problem for which the differential equation has unique solution .

8) $(x - y)y' = \tan x$; $y(1) = \frac{1}{2}$.
9) $(x^2 - 4)y' = 4y\ln y$; $y(-3) = 5$,
10) $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$; $y(4) = -2$.

In exercises 11 through 15 , find the solution of the differential equations.

11) $x\cos^2(y)dx + \tan y\; dy = 0$.
12) $dx = t(1 + t^2)\sec^2(x)dt$.

13) $(1+\ln x)dx+(1+\ln y)dy=0$.

14) $\frac{dy}{dx}=\left(\frac{2y+3}{4x+5}\right)^2$.

15) $(e^x+e^{-x})\frac{dy}{dx}=y^2$.

In problems 16 – 19 find an implicit and an explicit solution of the given initial value problem.

16) $\begin{cases} x^2\frac{dy}{dx}=y-xy \\ y(-1)=-1 \end{cases}$.

17) $\begin{cases} \frac{dy}{dx}=x^2y^2+y^2+x^2+1 \\ y(0)=2 \end{cases}$.

18) $\begin{cases} e^y\frac{dy}{dx}=\cos(2x)+2e^y\sin^2(x)-1 \\ y(\frac{\pi}{2})=\ln 2 \end{cases}$.

19) $\begin{cases} yy'=y^2x^3+y^2x \\ y(2)=1 \end{cases}$.

In problems 20 – 23 obtain a family of solutions.

20) $(x^2+2xy-4y^2)dx-(x^2-8xy-4y^2)dy=0$.

21) $(x-y\ln y+y\ln x)dx+x(\ln y-\ln x)dy=0$.

22) $(e^{y/x}+y'-\frac{y}{x}=0$.

23) $(x^3+3y^3)dx-3xy^2dy=0$.

In problems 24– 27 solve the given differential equation by using an appropriate substitution.

24) $\frac{dy}{dx}=\frac{1-x-y}{2x+2y}$.

25) $\frac{dy}{dx}=\sin(x+y)$.

26) $\frac{dy}{dx}=1+e^{y-x+5}$.

27) $\frac{dy}{dx}=\frac{1+(2x+y)^2}{2x+y}$.

Test each of the following equations for the exactness and solve the equation .

28 $(\cos 2y-3x^2y^2)dx+(\cos 2y-2x\sin 2y-2x^3y)dy=0$.

29) $[2x+y\cos(xy)]\,dx+x\cos(xy)dy=0$.

30) $\begin{cases} 3y(x^2-1)dx+(x^3+8y-3x)dy=0 \\ when\ x=0\ ,y=1 \end{cases}$.

31) $(3+y+2y^2\sin^2 x)dx+(x+2xy-y\sin 2x)dy=0$.

32) $\begin{cases} (xy^2+x-2y+3)dx+x^2ydy=2(x+y)dy \\ when\ x=1\ ,y=1 \end{cases}$.

33) $(x^2y^3-\frac{1}{1+9x^2})\frac{dx}{dy}+x^3y^2=0$.

In problems 34 and 35 solve the given initial-value problem by finding an appropriate integrating factor.

34) $xdx + (x^2y + 4y)dy = 0$, $y(4) = 0$.
35) $(x^2 + y^2 - 5)dx = (y + xy)dy$, $y(0) = 1$
36) Find the value k so that the given differential equation is exact .
$(6xy^3 + \cos y)dx + (2kx^2y^2 - x\sin y)dy = 0$.
38) Prove that $\mu(x,y) = (x+y)^{-2}$ is an integrating factor of the differential equation
$(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$, and solve it.
39) *Euler's* Theorem on homogeneous functions that, if F is homogeneous function of degree k in x and y, then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = kF \ .$$

Use *Euler's* Theorem to prove the result that, if M and N are homogeneous functions of the same degree and if $Mx + Ny \neq 0$, then $\frac{1}{M\,x\,+N\,y}$ is an integrating factor for the equation

$$Mdx + Ndy = 0 \ .$$

In exercises 40 through 43 , find the general solution of the differential equations.
40) $e^x dx + x^3dy + 4x^2ydx = 0$.
41) $xy' - 4y = xe^x$.
42) $(1+x)y' - xy = x + x^2$.
43) $(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy$.
In exercises 44 through 48 find the particular solution indicted. Give the largest interval I over which the solution is defined.
44) $xy' - \frac{y}{\ln x} = 0$; $y(e) = -1$.
45) $(1 + x^2(y' + 2xy = -2x$; $y(0) = -1$.
46) $(x-1)y' - 3y = (x-1)^5$; $y(-1) = 16$.
47) $(x+1)\frac{dy}{dx} + y = \ln x$; $y(1) = 10$.
48) $y' + (\tan x)y = \cos^2 x$, $y(0) = -1$.
In exercises 44 through 61, solve the differential equations.
49) $(x + 2y - 1)dx + (2x + 4y - 3)dy = 0$. Solve by two methods.
50) $6y^2dx - x(2x^3 + y)dy = 0$.
51) $2x^3y' = y(y^2 + 3x^2)$. Solve by two methods.
52) $y' = 1 + 6x\exp(x-y)$.
53) $2ydx + x(x^3\ln y - 1)dy = 0$.

54) $\begin{cases} 2xyy' = y^2 - 2x^3 \\ y(1) = 2 \end{cases}$.

55) $\begin{cases} (y^4 - 2xy)dx + 3x^2dy = 0 \\ when \;\; x = 2 \; y = 1 \end{cases}$.

56) $\begin{cases} (2y^3 - x^3)dx + 3xy^2dy = 0 \\ when \;\; x = 1 \;, y = 1 \end{cases}$. Solve by two methods.

57) $\begin{cases} (x^2 + 6y^2)dx - 4xydy = 0 \\ when \; x = 1, y = 1 \end{cases}$. Solve by three methods.

58) $y(\ln x - \ln y)dx = (x \ln x - x \ln y - y)dy$.

59) $\frac{dx}{dy} = -\frac{4y^2+6xy}{3y^2+2x}$.

60) $(2x + y + 1)y' = 1$.

61) $(6x + 1)y^2\frac{dx}{dy} + 3x^2 + 2y^3 = 0$.

The Eq.(32) can be written in the form

$$(-x + y + 2)dx + (x + y + 2)dy = 0,$$

this equation is an exact differential equation can be written in the form

$$(-xdx + ydy) + 2dx + 2dy + (ydx + xdy) = 0,$$

or

$$d\left[\frac{1}{2}(y^2 - x^2) + 2(x + y) + xy\right] = 0.$$

Thus , the general solution of the differential equation (32) is

$$x^2 - 2xy - y^2 - 4x - 4y = c_1. \tag{33}$$

The curves given by Eq.(33) are the orthogonal trajectories of (31) .The both of these families consist of hyperbolas.

Exercises

Find a first order differential equation for the given family of curves.

1. $y(y^2 + x^2) = c$
2. $\ln|xy| = c\,(x^2 + y^2)$
3 $y = e^{x^2} + ce^{-x^2}$
4. $y = sinx + ce^x$

Find the orthogonal Trajectories of the family of curves

5. $y^2 = x - c$
6 $y^2 = c_1x^3$
7. $2x^2 + y^2 = c^2$
8. $y = x + ce^{-x}$
9. $x^2 + 2y^2 = c^2$
10. $xye^{x^2} = c$
11. $c_1x^2 - y^2 = 1$
12. $y^2 = c\,(1 - x^2)$
13. $2x^2 + y^2 = 6cx$
14. $2y + x + c_1e^{2y} = 0$
15. $y = x + 2 - c_1e^y$
16. $x^2 - y^2 = cx$
17.$y^2 = \frac{x^3}{(a-x)}$ (The cissoids)
18. $y(x^2 + c) + 2 = 0.$

19. $y^2 = ax^2(1 - cx)$; with a held fixed .

20. Find the member of the orthogonal trajectories for $x^2 = c$ that passes through $(1, 1)$.

21. Find the member of the orthogonal trajectories for $x^2 + 3y^2 = cy$ that passes through $(1, 2)$.

3.2 Growth and Decay

Many natural processes involve quantities that increase or decrease at a rate proportional to the amount of the quantity present.

If $y = y(t)$ denotes the value of a quantity y at any time t, and if y changes at the rate proportional to amount present.

$$\frac{dy}{dt} = ky \tag{1}$$

where k is constant of proportionality. Eq.(1) describes growth if $k > 0$ or decay if $k < 0$.

Note If at some initial time quantity is known and is $y(x_0) = y_0$, then differential equation

$$\frac{dy}{dt} = ky$$

and initial value $y(x_0) = y_0$ make it initial value problem.

To solve Initial Value Problem

$$\begin{cases} \frac{dy}{dt} = ky, \\ y(x_0) = y_0 \end{cases} \tag{2}$$

We have

$$\begin{aligned} \frac{dy}{dt} &= ky \\ \frac{dy}{y} &= kdt \\ \int \frac{dy}{y} &= \int kdt \\ \ln|y| &= kt + \ln c \\ y &= c_1 e^{kt} \; ; \; c_1 = \mp c \end{aligned}$$

air is 300^0C and the substance cools from 370^0C to 340^0C in 15 minutes .Find when the temperature will be 310^0C.

Solution The temperature of the substance is given by the formula

$$T(t) = T_s + c_1 e^{kt}.$$

But $T_s = 300$, $T(0) = 370$ and $T(15) = 340$. Then

$$T(0) = 300 + c_1 = 370 \implies c_1 = 70,$$

$$T(t) = 300 + 70e^{kt},$$

$$T(15) = 340 = 300 + 70e^{15t} \implies k = \frac{1}{15}\ln(\frac{4}{7}) \simeq -0.037.$$

So

$$T(t) = 300 + 70e^{-0.037t},$$

$$300 + 70e^{-0.037t} = 310 \implies t = \frac{\ln 7}{0.037} \simeq 52.6 \text{ minutes} .$$

Hence the temperature of the substance reaches to 310^0C after 52.6 minutes.

Review exercises

Through exercises 1-9, find the orthogonal trajectories of the given family of curves. Draw a few representative of curves of each family whenever a figure is requested.

1) $x - 4y = c$. Draw the figure.

2) $x^2 - y^2 = c$. Draw the figure.

3) straight lines with slope and y-intercept equal.Draw the figure.

4) $e^x + e^y = c$.

5) $x^3 = 3(y - c)$.Draw the figure.If the liquid cools to 120

6) $y(x^2 + c) = 0$.

7) $y^2 = ax^2(1 - cx)$; with a held fixed.

8) For the family $x^2 + 3y^2 = cy$, find that member of the orthogonal trajectories which passes through (1,2).

9) Find the member of orthogonal trajectories for the family of hyperbolas $xy = c$ which passes through the point (2,4).

10) The population of a town grows at a rate proportional to the population at time t.The initial population of 500 increases by 15% in 10 years.What will be the population in 30 years?.

11) The population of a town grows at a rate proportional to the population at time. The initial population P_0 becomes double after 50 years.When the population becomes $4P_0$?.

12) In a forest the number of rabbits in 2009 was 90. in 2012 the number grows to 180. What will it be in 2014?.

13) Initially there were 10 mg. of radioactive material present. After 2 months the mass decreased by 5%. If the rate of decay is proportional to the amount present at any time , then determine the falf-life of this material.

14) Suppose that $P'(t) = 0.15P(t)$ represent a mathematical model for the growth of a certain cell culture. where $P(t)$ is the size of the culture (measured in millions of cells)at time t.(T is measured in hours). How fast is the culture growing at the time t when the size of the culture reaches 2 million cells?.

15) Bacteria in culture grow at a rate proportional to the number of bacteria present at any time .Initial number of bacteria is 2000, which is increased by 50% in 10 hours.What will be the number of bacteria in 20 hours?..

16) If 0.5% of radium disappears in 12 years .Find what percentage will disappear in 1000 years?.What is the half-life of radium?.

17) A radioactive element has a half-life of 5750 years.If 100 grams of this element is present initially , how much will be left after 1000 years?.

18) A hot iron rod was left in a room where the temperature was 20^0C.After one minute the temperature of the rod was recorded 35^0C.,and after two minutes it was 27.5^0C. What was the initial temperature of the rod?.

19) A liquid with initial temperature 200^0C.is surrounded by air at a constant temperature 80^0C.If the liquid cools to 120^0C.in 30 minutes ,what will the be the temperature after one hour?.

20) The temperature of cup of hot milk is 180^0F. when freshly poured.After 2 minutes in a room of temperature 70^0F, the milk ha cooled to165^0F. Find the temperature of milk at any time and determine the time for the milk has cooled to 120^0F.

21) A small metal bar , whose initial temperature was 20^0C,is dropped in to in to large container of boiling water . How long will it take the bar to reach 90^0C if it is known that its temperature increase 2^0 per second ?.

How will it take the bar to reach 98^0C?.

22) A body initially at 50^0C is put into a 375^0C oven. After 75 minutes it is found that the temperature of the body is 125^0C. How long the body will take to attain the temperature 150^0C?.

22) A cup is filled with tea at temperature 60^0C and is left in a room with air temperature 20^0C. After 1 minute if the temperature of tea is 50^0 , what will be its temperature after 2 minutes?. At what time will the temperature be 30^0C?.

23) Water at 100^0C is allowed cooling in air which remains at the constant temperature 20^0C , and if it is observed that in 10 minutes the water has been cooled to 60^0C. Find the temperature after 13 minutes.

24) A pot of liquid is put on the store to boil. The temperature of the liquid reaches 170^0F and then the pot is taken of the burner and placed on counter in the kitchen where the temperature is 76^0F. After 2 minutes the temperature of the liquid is 123^0F. How long before the temperature of the liquid in the pot will be 84^0F.?

are linearly independent on $(0\ ,\ \infty)$ and we conclude that

$$y_c = c_1 x^3 e^x + c_2 e^x.$$

is the general solution of the differential equation.

Remark (4.1.13) The property of general solution exists only in the homogeneous linear nth -order differential equation (7) but does not exist in the homogeneous non- linear differential equation , for example the differential equation

$$(xy' + 1)(yy' + 1) = 0.$$

is non-linear first order differential equation has not general solution, because it has two family of curves of solutions $y = -\ln|xc_1|$ such that $x \neq 0$ and an arbitrary constant $c_1 \neq 0$, $y^2 + 2x = c_2$ where $y \neq 0$ and c_2 is an arbitrary constant.

Example (20) Given that

$$y = c_1 e^x + c_2 e^{-x}$$

is a two parameters family of solutions of

$$y'' - y = 0 \text{ on } (-\infty, \infty).$$

Find a curve of the family satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution From the Theorem 4.1.3 the initial value problem

$$\begin{cases} y''(x) - y(x) = 0 \\ y(0) = 0\ ,\ y'(0) = 1 \end{cases}$$

has a unique solution of the differential equation.

For $y(0) = 0$ we have $c_1 + c_2 = 0$ and for $y'(0) = 1$ we have $c_1 - c_2 = 1$, hence $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. So the unique solution of the initial value problem is

$$y = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$

Exercises (4.1)

In exercises 1 through 5 determine all intervals on which the equation is normal .

1) $(x-1)y''+xy'+y=\sin(x)$.
2) $(x^2-1)y''+6y=e^x$.
3) $x^3y'''+e^xy=\ln x$.
4) $(\cot x)y'''+y=0$.
5) $(x^2+1)y^{(4)}+(\ln x)y''+3x^2y=x^2+1$.

In problems 6-16 determine whether the given functions are linearly independent or dependent on $(-\infty,\infty)$

6) $f_1(x)=x$, $f_2(x)=x^2$, $f_3(x)=4x-3x^2$.
7) $f_1(x)=0$, $f_2(x)=x$, $f_3(x)=e^x$.
8) $f_1(x)=5$, $f_2(x)=\cos^2(x)$ $f_3(x)=\sin^2(x)$.
9) $f_1(x)=\cos(2x)$, $f_2(x)=3$, $f_3(x)=\cos^2(x)$.
10) $f_1(x)=x$, $f_2(x)=x-1$, $f_2(x)=x+3$.
11) $f_1(x)=x+1$, $f_2(x)=x$, $f_3(x)=x^2$.
12) $f_1(x)=2+x$, $f_2(x)=2+|x|$.
13) $f_1(x)=e^x$, $f_2(x)=e^{-x}$, $f_3(x)=\sinh(x)$.
14) $f_1(x)=1$, $f_2(x)=\sin(x)$, $f_3(x)=\cos(x)$.
15) $f_1(t)=\cos(3t-2)$, $f_2(t)=\cos(3t)$, $f_3(t)=\sin(3t)$.
16) $f_1(x)=x\,|x|$, $f_2(x)=x^2$.

In exercises 17-24 show by computing the *Wronskian* that the given functions are linearly independent or dependent on the indicated interval.

17) $x^{\frac{1}{2}}$, x^2 ; $(0\,,\infty)$.
18) $\sin(x)$, $\csc(x)$; $(0\,,\pi)$.
19) e^x , e^{-x} , e^{4x} $(-\infty,\infty)$.
20) $1+x$, x^3 , $(-\infty,\infty)$.
21) $\tan(x)$, $\cot(x)$; $(0\,,\frac{\pi}{2})$.
22) x , $x\ln(x)$, $x^2\ln(x)$; $(0\,,\infty)$.
23) e^x , xe^x , x^2e^x ; $(0\,,\infty)$.
24) x^3 , x^4 ; $(0\,,\infty)$.

In exercises 25-32 verify that the given functions form a fundamental set solutions of the differential equation on the indicated interval .

25) $y''-y'-12y=0$; e^{-3x} , e^{4x} , $(-\infty\,,\infty)$.
26) $y''-4y'=0$; $\cosh(2x)$, $\sinh(2x)$, $(-\infty\,,\infty)$.
27) $y''-2y'+5y=0$; $e^x\cos(2x)$, $e^x\sin(2x)$ $(-\infty\,,\infty)$.
28) $4y''-4y'+y=0$; $e^{\frac{x}{2}}$, $xe^{\frac{x}{2}}$, $(-\infty\,,\infty)$.
29) $x^2y''+xy'+y=0$; $\cos(\ln x)$, $\sin(\ln x)$, $(0\,,\infty)$.
30) $x^2y''-6xy'+12y=0$; x^3 , x^4 , $(0\,,\infty)$.
31) $x^3y'''+6x^2y''+4xy'-4y=0$; x , x^{-2} , $x^{-2}(\ln x)$, $(0\,,\infty)$.
32) $y^{(4)}+y''=0$; 1 , x , $\cos x$, $\sin x$, $(0\,,\infty)$.

33) Given that $y = c_1e^{4x} + c_2e^{-x}$ is a two - parameters family of solutions of $y'' - 3y' - 4y = 0$ on the interval $(-\infty \ , \infty)$, find a member of the family satisfying the initial conditions

$y(0) = 1 \ , \ y'(0) = 2.$

34) Given that $y = c_1 + c_2 \cos x + c_3 \sin x$ is a three - parameters family of solutions of $y''' + y' = 0$ on the interval $(-\infty \ , \infty)$, find a member of the family satisfying the initial conditions

$y(\pi) = 0 \ , \ y'(\pi) = 2 \ , \ y''(\pi) = -1.$

35) Given that $y = c_1x + c_2x \ln x$ is a two - parameters family of solutions of $x^2y'' - xy' + y = 0$ on the interval $(0 \ , \infty)$, find a member of the family satisfying the initial conditions

$y(1) = 3 \ , \ y'(1) = -1.$

In exercises 36-37 find an interval around $x = 0$ for which the given initial -value problem has a unique solution

36) $\begin{cases} (x-2)y'' + 3y = x \\ y(0) = 0 \ , \ y'(0) = 1 \end{cases}$.

37) $\begin{cases} y'' + (\tan x)y = e^x \\ y(0) = 1 \ , \ y'(0) = 0 \end{cases}$.

38) a) Verify that $y_1 = 1$ and $y_2 = \ln x$ are solutions of the nonlinear differential equation $y'' + (y')^2 = 0$ On the interval $(0 \ , \infty)$.

b) Is $y_1 + y_2$ a solution of the equation ?. Is $c_1y_1 + c_2y_2$; where c_1 and c_2 are arbitrary constants, a solution of the equation ?.

39) Consider the second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \qquad (13)$$

where a_2 , a_1 a_1 and a_0 are continuous functions on an interval I and $a_2(x) \neq 0$. Let y_1 and y_2 be two solutions of (13).

a) If $W(x, y_1, y_2)$ is the Wronskianof y_1 and y_2, show that

$$a_2(x)\frac{dW}{dx} + a_1(x)W = 0.$$

b) Deduce *Abel's* formula

$$W = Ce^{-\int \frac{a_1}{a_2}(x)dx} .$$

where C is an arbitrary constant.

c) Using an alternative for of*Abel's* formula

$$W = W(x_0)e^{-\int_{x_0}^{x} \frac{a_1}{a_2}(x)dx}.$$

where $x_0 \in I$.

From this exercise we conclude that

1) If $W(x_0) \neq 0$ then $W(x) \neq 0$ for all $x \in I$ which is equivalent that y_1 and y_2 are linearly independent on I .

2) If $W(x_0) = 0$ then $W(x) = 0$ for all $x \in I$. In this case y_1 and y_2 are linearly dependent on I .

40) a) Verify that $y_1 = x^3$, $y_2 = x^2 |x|$ are linearly independent solutions of the differential equation

$$x^2y'' - 4xy' + 6y = 0 \text{ on } (-\infty , \infty).$$

b) Show that $W(x , y_1, y_2) = 0$ for all $x \in \mathbb{R}$.

c) Does the result of part (b) violate Theorem (4.1.8).

d) Verify that $f_1(x) = x^3$, $f_2(x) = x^2$ are also linearly independent solutions of the differential equation on $(-\infty , \infty)$.

e) Find a solution of the differential equation satisfying $y(0) = 0$, $y'(0) = 0$.

41) **Astronomy Kopal** obtained a differential equation of the form

$$v\frac{d^2\Psi}{dv^2} + 4\frac{d\Psi}{dv} = 0 \quad , \quad v > 0 .$$

Show that $\Psi = 1$ and $\Psi = v^{-3}$ are linearly independent solutions of this differential equation on $(0 , \infty)$ and obtain the general solution.

42) **Flows Barton and Raynor** in their study of peristaltic flow in tubes, obtained the following linear homogeneous differential equation with variable

coefficients

$$\frac{d^2P}{dv^2} + \frac{1}{v}\frac{dP}{dv} = 0 \quad ; \quad v > 0 .$$

Show that $P(v) = 1$ and $P(v) = \ln v$ are linearly independent solutions of this differential equation on $(0 , \infty)$ and obtain the general solution .

4.2 Reduction of order Method (when one solution is given).

Then the general solution of the equation (7) is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-4x}(x + \frac{1}{7})$$

on the interval $(0, \infty)$.

Exercises (4.2)

In exercises 1 through 12 a differential equation and one of its solutions is given. Apply the reduction of order method to obtain another linearly independent solution.

1) $y'' - 4y' + 4y = 0$; $y_1 = e^{2x}$.
2) $9y'' - 12y' + 4y = 0$; $y_1 = e^{\frac{2x}{3}}$.
3) $y'' + 9y = 0$; $y_1 = \sin(3x)$.
4) $x^2y'' + xy' - y = 0$; $y_1 = x$ and $x > 0$.
5) $x^2y'' + (2x^2 - x)y' - 2xy = 0$; $y_1 = e^{-2x}$ and $x > 0$.
6) $x^3y'' + (5x^3 - x^2)y' + 2(3x^3 - x^2)y = 0$; $y_1 = e^{-2x}$ and $x > 0$.
7) $x^4y'' + 2x^3y' - y = 0$; $y_1 = e^{\frac{1}{x}}$ and $x > 0$.
8) $x^2y'' + x^2y' - (x+2)y = 0$; $y_1 = x^{-1}e^{-x}$; and $x > 0$.
9) $x^2(1 - \ln x)y'' + xy' - y = 0$; $y_1 = x$; and $x > e$.
10) $y'' + \frac{2}{x}y' + \frac{9}{x^4}y = 0$; $y_1 = \cos(\frac{3}{x})$; and $x > 0$.
11) $(1 - x^2)y'' - 7xy' + 7y = 0$; $y_1 = x$; $x > 1$.
12) $(1 - 2x - x^2)y'' + 2(1+x)y'; -2y = 0$; $y_1 = x + 1$; $x > 0$.

In exercises 13 through 21, show that if y_1 is a solution of the differential equation, use the formula (5) to find an expression for a second linearly independent solution.

13) $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$; and $x > 0$.
14) $xy'' + y' = 0$; $y_1 = \ln x$ and $x > 0$.
15) $x^2y'' - xy' + 2y = 0$; $y_1 = x\sin(\ln x)$ and $x > 0$.
16) $(1 + 2x)y'' + 4xy' - 4y = 0$; $y_1 = e^{-2x}$ and $x > -\frac{1}{2}$.
17) $x^2y'' - 20y = 0$; $y_1 = x^{-4}$ and $x > 0$.
18) $x^2y'' - 5xy' + 9y = 0$; $y_1 = x^3 \ln x$ and $x > 0$.
19) $x^2y'' - 4xy' + 6y = 0$; $y_1 = x^2 + x^3$; and $x > 0$.
20) $(3x + 1)y'' - (9x + 6)y' + 9y = 0$; $y_1 = e^{3x}$ and $x > \frac{-1}{3}$.
21) $y'' - 3(\tan x)y' = 0$; $y_1 = 1$ and $0 < x < \frac{\pi}{2}$.
22) Solve the IVP : $xy'' + (1 - 2x)y' + (x - 1)y = 0$; $y(1) = 2e$, $y'(1) = -3e$, Where $y_1 = e^x$ is a particular solution of the differential equation.

In exercises 23 through 26, use the reduction of order method to find the general solution of the differential equation.

23) $y'' + y = \sec x$; $0 < x < \frac{\pi}{2}$, where $y_1 = \sin x$ is a particular solution of

$$y'' + y = 0.$$

24) $y'' - 2y + y = \frac{1}{x}e^x$; $x > 0$ where $y_1 = e^x$ is a particular solution of

$$y'' - 2y' + y = 0.$$

25) $y'' + 4y' + 5y = e^{-2x}\sec x$; $0 < x < \frac{\pi}{2}$ where $y_1 = e^{-2x}\sin x$ is a particular solution of the differential equation

$$y'' + 4y' + 5y = 0.$$

26) $5x^2y'' - 3xy' + 3y = x^{\frac{1}{2}}$; $x > 0$ where $y_1 = e^x$ is a particular solution of the differential equation

$$5x^2y'' - 3xy' + 3y = 0.$$

27) Find the solution of the IVP

$$\begin{cases} (x^2 + 1)y'' - 2xy' + 2y = (x^2 + 1)^2 \\ y(0) = 0, \ y(0) = 1. \end{cases}$$

where $y_1 = x$ is a particular solution of the differential equation

$$(x^2 + 1)y'' - 2xy' + 2y = 0.$$

Quantum Mechanics The study of solutions to the *Schrodinger* wave equation for the hydrogen atom leads to *Laguerr/s* differential equation

$$xy'' + (1 - x)y' + py = 0 \ ; \ x > 0.$$

where p is a constant.

In exercise (28) verify that if y_1 is a given solution of *Laguerr/s* differential equation for given p. Using the formula (5) (*Abel/sformula*) to obtain an integral form for a second linearly independent solution.

28) $p = 1$, $y_1 = 1 - x$.

29) Verify that $y_1 = e^{-x}$ is a solution of the differential equation

$$(x^2 + 2)y'' - 2xy' - \left(x^2 + 2x + 2\right)y = 0.$$

Find the particular solution satisfying the initial conditions : $y(0) = 4$, $y'(0) = 4$.

Hence

$$m = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i.$$

So the general solution of a differential equation is

$$y = c_1 e^{\frac{-x}{2}} \cos(\frac{\sqrt{3}}{2} x) + c_2 e^{\frac{-x}{2}} \sin(\frac{\sqrt{3}}{2} x).$$

from the conditions $y(0) = 1$ and $y'(0) = \sqrt{3}$ we have $c_1 = 1.$ and

$$\frac{-c_1}{2} + c_2 \frac{\sqrt{3}}{2} = \sqrt{3}$$

hence $c_1 = 1$ and $c_2 = 2 + \frac{1}{\sqrt{3}}$. So the solution of the IVP is

$$y = e^{\frac{-x}{2}} \cos(\frac{\sqrt{3}}{2} x) + (2 + \frac{1}{\sqrt{3}}) e^{\frac{-x}{2}} \sin(\frac{\sqrt{3}}{2} x).$$

Example (12) Solve the initial value problem

$$\begin{cases} y''' - 2y'' + 2y' = 0 \\ y(0) = -2 \ , y'(0) = 0 \ , \ y''(0) = 4 \end{cases}$$

Solution The characteristic equation for the differential equation is

$$m^3 - 2m^2 + 2m = 0.$$

hence the roots of this equation are $m = 0$, $m = 1 \mp i$ and the general solution of the differential equation is

$$y = c_1 + c_2 e^x \cos x + c_3 e^x \sin x$$

If we calculate y' , y'' and using the initial conditions we find

$$c_1 + c_2 = -2, \ c_2 + c_3 = 0 \ , 2c_3, = 4.$$

By consequence $c_1 = 0$,$c_2 = -2$ and $c_3 = 2$. Then the solution of the IVP is

$$y = 2e^x (\sin x - \cos x)$$

Exercises (4.3

In exercises 1 through 23, find the general solution of the differential equations.

1) $2y'' + 3y' + y = 0$. 2) $8y'' - 6y' + y =$.

3) $y'' + 6y = 0$.

4) $y'' - 36y = 0$.

5) $y'' - y' - 6y = 0$.

6) $y'' + 8y' + 16y = 0$.

7) $4y'' + y' = 0$.

8) $12y'' - 5y' - 2y = 0$.

9) $3y'' + 2y' + y = 0$.

10) $y''' - 4y'' - 5y' = 0$.

11) $y''' - y = 0$.

12) $y''' + y'' - 2y = 0$.

13) $y''' - 5y'' + 3y' + 9y = 0$.

14) $y''' + 3y'' + 3y' + y = 0$.

15) $y''' - 7y'' + 5y' + y = 0$.

16) $y''' - 6y'' + 12y' - 8y = 0$.

17) $y''' + 3y'' - 4y' - 12 = 0$.

18) $y''' - y'' - 4y = 0$.

19) $y^{(4)} - 2y'' + y = 0$.

20) $y^{(4)} - 7y'' - 18y = 0$.

21) $y^{(5)} - 2y^{(4)} + 17y''' = 0$.

22) $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$.

23) $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$.

In exercises 24 through 31 the factored for the characteristic equation for certain differential equations are given. In each case state the order of the differential equation and give the general form of the solution of the differential equations.

24) $(m-3)(m+2)^2(m^2-2m+5)(m^2+4m+5)^2 = 0$.

25) $(m+1)(m-1)(m^2+1)(m^2+m+1) = 0$.

26) $(2m-3)(5m+4)(m^2+4)^2(m^2-m-1)$.

27) $(m-3)^3(m+2)^4 = 0$.

28) $(m^2+4)(m^2+1)^2 = 0$.

29) $(m+1)(m+3)(m+2)^2(m^2-2n+5)^2(m^2+9) = 0$.

30) $(m^2+3)(m^2+1)^3(m+1)^3 = 0$.

31) $(m+2)(m-5)^2(m-4)^3(m^2+4)^3 = 0$.

In exercises 32 through 35 determine a homogeneous differential equations with constant coefficients having the given solutions

32) $4e^{6x}$, $3e^{-3x}$.
33) $10\cos(4x)$, $-5\sin(4x)$, x .
34) 3 , $2x$, $-x^2$, $-7e^{-7x}$.
35) $8\sinh(3x)$, $10\cosh(3x)$.
36) Use the fact

$$i = (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^2 \text{ and } - \ i = (\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i)^2.$$

to solve the differential equation $y^{(4)} + y = 0$.(Hint : $m^4 + 1 = (m^2 + 1)^2 - 2m = 0$).

In exercises 37 through 50 find the solution of the initial value problems $(IVPs)$.

37) $\begin{cases} y'' + 16y = 0 \\ y(0) = 2 \ , \ y'(0) = -2 \end{cases}$.

38) $\begin{cases} y'' - y = 0 \\ y(0) = y'(0) = 1 \end{cases}$.

39) $\begin{cases} y'' + 6y' + 5y = 0 \\ y(0) = 0 \ , \ y'(0) = 3 \end{cases}$.

40) $\begin{cases} y'' - 8y' + 17y = 0 \\ y(0) = 4 \ , \ y'(0) = -1 \end{cases}$.

41) $\begin{cases} 2y'' - 2y' + y = 0 \\ y(0) = -1 \ , \ y'(0) = 0. \end{cases}$.

42) $\begin{cases} y'' + y' + 2y = 0 \\ y(0) = y'(0) = 0 \end{cases}$.

43) $\begin{cases} y'' - 3y' + 2y = 0 \\ y(1) = 0 \ , \ y'(1) = 1 \end{cases}$.

44) $\begin{cases} y'' + y = 0 \\ y(\frac{\pi}{3}) = 0 \ , \ y'(\frac{\pi}{3}) = 0 \end{cases}$.

45) $\begin{cases} y''' + 12y'' + 36y' = 0 \\ y(0) = 0 \ , \ y'(0) = 1 \ , \ y''(0) = -7. \end{cases}$.

46) $\begin{cases} y''' + 2y'' - 5y' - 6y = 0 \\ y(0) = y'(0) = 0 \ , \ y''(0) = 1 \end{cases}$.

47) $\begin{cases} y''' - 8y = 0 \\ y(0) = 0 \ , \ y'(0) = -1 \ , y''(0) = 0 \end{cases}$.

48) $\begin{cases} y^{(4)} = 0 \\ y(0) = 2 \ , \ y'(0) = 3 \ , y''(0) = 4, \quad y'''(0) = 5 \end{cases}$.

49) $\begin{cases} y^{(4)} - 3y''' + 3y'' - 3y' = 0 \\ y(0) = y'(0) = 0 \ , \ y''(0) = y'''(0) = 1 \end{cases}$.

50) $\begin{cases} y^{(4)} - y = 0 \\ y(0) = y'(0) = y''(0) = 0 \ , \ y'''(0) = 1 \end{cases}$.

51) a) Show that the solution of the IVP

$$\begin{cases} y'' - 2ry' + (r^2 - \frac{\alpha^2}{4})y = 0 \\ y(0) = 0 \ , \ y'(0) = 1. \end{cases}$$

is given by

$$y_\alpha(x) = \frac{1}{\alpha}\left[e^{(r+\frac{\alpha}{2})x} - e^{(r-\frac{\alpha}{2})x}\right],$$

where r and α are real positive .

b) Show that $\lim_{\alpha \to 0} y_\alpha(x) = xe^{rx}$.

4.4 Cauchy- Euler Differential equation

A Cauchy Euler differential equation is the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_1 x y' + a_0 y = 0. \qquad (1)$$

Where a_n ,a_{n-1} ,....., a_1, a_0 are constants and $a_n \neq 0$. Since the leading coefficient $a_n x^n$ should never be zero, the interval of definition of the differential equation (1) is either the open interval $(0, \infty)$ or $(-\infty, 0)$. That is, the differential equation (1) should solved for either $x > 0$ or $x < 0$.

The *Euler* differential equation is probably the simplest type of linear differential equation with variable coefficients. The reason for this is that the change of independent variable

$$x = \begin{cases} e^t \ ; \ x > 0 \\ -e^t ; \ x < 0 \end{cases}.$$

Produce a differential equation with constant coefficients. We illustrate this fact for a second -order case.

Example(1) Show by means of change of independent variable above that the *Euler* differential equation of second order

$$a_2 x^{2'} y'' + a_1 x y' + a_0 y = 0, \qquad (2)$$

with a_0 , a_1 and a_2 are given constants, is reduced to the differential equation

$$a_2 y''(t) + (a_1 - a_2) y'(t) + a_0 y = 0. \qquad (3)$$

Consequently $m = 1$ is a root repeated two times, so the general solution of the differential equation (12) is

$$y = c_1(x-4) + c_2(x-4)\ln(x-4).$$

Example (9) Solve the initial value problem (IVP)

$$\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8 \ , y'(-2) = 0 \end{cases}.$$

Solution From the initial conditions. we should suppose $x < 0$ .The solution of the differential equation is the form $y = x^m$, hence

$$m(m-1) - 4m + 6 = (m-2)(m-3) = 0.$$

So we have two roots $m = 2$ and $m = 3$, so the general solution of the differential equation is

$$y = c_1(-x)^2 + c_2(-x)^3 = c_1x^2 + c_3x^3 \quad \text{where } c_3 = -c_2.$$

From the initial conditions we have

$$c_1 - 2c_3 = 2,$$

and

$$-c_1 + 3c_3 = 0,$$

hence $c_1 = 6$ and $c_3 = 2$. Then the solution of the IVP is

$$y = 6x^2 + 2x^3.$$

Exercises(4.4)

In problems 1 through 20 find the general solution of the following differential equations , where we suppose that $x > 0$.

1) $x^2y'' - y = 0.$
2) $xy'' + y' = 0.$
3) $xy'' - y' = 0.$
4) $4x^2y'' + y = 0.$
5) $x^2y'' + 5xy' + 3y = 0.$
6) $x^2y'' + xy' + 4y = 0.$
7) $x^2y'' - 3xy' - 2y = 0.$
8) $25x^2y'' + 25xy' + y = 0.$

9) $4x^2y'' + 4xy' - y = 0.$
10) $x^2y'' + 5xy' + 4y = 0.$
11) $x^2y'' - xy' + 2y = 0.$
12) $x^2y'' + 8xy' + 6y = 0.$
13) $x^2y'' - 7xy' + 41y = 0.$
14) $3\,x^2y'' + 6xy' + y = 0.$
15) $2x^2y'' + xy' + y = 0.$
16) $x^3y'''' - 6y = 0.$
17) $x^3y'''' + xy' - y = 0.$
18) $x^3y'''' - 2x^2y'' - 2xy' + 8y = 0.$
19) $x^3y'''' - 2x^2y'' + 4xy' - 4y = 0.$
20) $x^3y'''' + 4x^2y'' - 8xy' + 8y = 0.$
21) In problems 5 through 12, solve the given differential equations by the substitution $x = e^t$.

In problems 22 through 30 find the solution of the initial values problems.

22) $\begin{cases} x^2y'' + 3xy' = 0 \\ y(1) = 0\ ,\ y'(1) = 4 \end{cases}$

23) $\begin{cases} x^2y'' + xy' + y = 0 \\ y(1) = 1\ ,\ y'(1) = 2 \end{cases}$

24) $\begin{cases} x^2y'' - 5xy' + 8y = 0 \\ y(2) = 32\ ,\ y'(2) = 0 \end{cases}$

25) $\begin{cases} x^2y'' - 3xy' + 4y = 0 \\ y(1) = 5\ ,\ y'(1) = 3 \end{cases}$

26 $\begin{cases} 4\,x^2y'' + y = 0 \\ y(-1) = 2,\ y'(-1) = 4 \end{cases}$

27) $\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8\ ,\ y'(-2) = 0 \end{cases}$

28) $\begin{cases} x^2y'' - xy' + y = 0 \\ y(-1) = 1\ ,\ y'(-1) = 0 \end{cases}$

29) $\begin{cases} x^2y'' + \frac{7}{2}xy' - \frac{3}{2}y = 0 \\ y(-4) = 1\ ,\ y'(-4) = 0 \end{cases}$

30) $\begin{cases} x^3y'''' + 4x^2y'' - 8xy' + 8y = 0 \\ y(1) = 0\ ,\ y'(1) = 1\ ,\ y''(1) = 0 \end{cases}$

The equation (8) is called also the characteristic equation corresponding to the *Euler* differential equation (1). Now in exercises 31 through 38, write only the characteristic equation associated with the *Euler* differential equation.

31) $3x^3y'''' - x^2y'' + 4xy' - 4y = 0 \quad , \quad x > 0.$

32) $x^4y^{(4)} - 5x^3y'''' + 3x^2y'' - 6xy' + 6y = 0 \quad , \quad x > 0.$

33) $2\,x^4y^{(4)} + 3x^3y'''' - 4x^2y'' + 8xy' - 8y = 0 \quad , \quad x < 0.$

34) $2x^3y'''' + x^2y'' - 12xy' - 2y = 0 \quad , \quad x < 0.$

35) $x^5y^{(5)} - 2x^3y'''' + 4x^2y'' = 0 \quad , \quad x < 0.$

36) $7\,x^4y^{(4)} - 2x^3y'''' + 3x^2y'' - 6xy' + 6y = 0 \quad , \quad x > 0.$

37) $x^5y^{(5)} + 2x^3y'''' - 9x^2y'' + 18x\,y' - 18y = 0 \, , \quad x > 0.$

38) $x^6y^{(6)} - 12x^4y^{(4)} = 0 \quad , \quad x > 0.$

39) Astronomy *Kopal* obtained a differential equation of the form

$$r\psi''(r) + \psi'(r) = 0 \quad , \quad r > 0.$$

Solve the differential equation (Multiplication of the *D.E.* by r reduces an *Euler* differential equation).

40) FLOWS Borton and Rayor in the stady of peristaltic flow in tubes, obtained the following linear homogeneous differential equation with the form

$$P''(r) + \frac{1}{r}P'(r) = 0 \quad , \; r > 0.$$

Solve this differential equation (Multiplication of the *D.E.* by r^2 produces an *Euler* differential equation)

In Exercises 41 through 45 , find the general solutions of the differential equations .

41) $(x-1)^2y'' + 5(x-1)y' + 4y = 0 \quad ; \quad x > 1.$

42) $(x+3)y'' + 3(x+3)y' + 5y = 0 \quad ; \quad x < -3.$

43) $(x-2)^2y'' - (x-2)y' + y = 0 \quad ; \quad x > 2.$

44) $(3x+4)y'' + 10(3x+4)y' + 9y = 0 \quad ; \quad 3x + 4 > 0.$

45) $(x+2)^2y'' + (x+2)y' + y = 0 \quad ; \quad x < -2 \, .$

4.5 General solution of Nonhomogeneous linear differential equations

Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots\ldots. + a(x)y' + a_0(x)y = g(x). \qquad (1)$$

be a nonhomogeneous linear differential equation, where a_n , a_{n-1}, ...,a_1 , a_0 and g are continuous functions defined on an interval $I = (a\ , b)$ such that $a_n(x) \neq 0$ for all $x \in I$ and g is not identically zero on I .

3) The general solution of

$$y'' + 4y = 0 \; ; x > \frac{\pi}{2} \tag{32}$$

$$y = c_3 \cos(2x) + c_4 \sin(2x) \tag{33}$$

So the function

$$y = f(x) = \begin{cases} \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin x & ; \; 0 \le x \le \frac{\pi}{2} \\ c_3 \cos(2x) + c_4 \sin(2x) \; . & ; \; x > \frac{\pi}{2} \end{cases}$$

is a solution of the IVP (29)) and (32) such that y and y' are continuous at $x = \frac{\pi}{2}$. For this we should have

$$\lim_{x \to (\frac{\pi}{2})^+} y(x) = \lim_{x \to (\frac{\pi}{2})^-} y(x) \; ,$$

which implies

$$-1 + \frac{1}{3} = -c_3 \; or \; c_3 = \frac{2}{3}.$$

But

$$y'(x) = f'(x) = \begin{cases} -2\sin(2x) + \frac{5}{3}\cos(2x) + \frac{1}{3}\cos x & ; \; 0 \le x \le \frac{\pi}{2} \\ -2c_3 \sin(2x) + 2c_4 \cos(2x) & ; \; x > \frac{\pi}{2} \end{cases}$$

we should have

$$\lim_{x \to (\frac{\pi}{2})^+} y'(x) = \lim_{x \to (\frac{\pi}{2})^-} y'(x) = y'(\frac{\pi}{2}) \; ,$$

which implies

$$-2c_4 = -\frac{5}{3} \; or \; c_4 = \frac{5}{6} \; .$$

Finally the solution of the IVP (29) ; (30) and (32) is

$$y = f(x) = \begin{cases} \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin x & ; \; 0 \le x \le \frac{\pi}{2} \\ \frac{2}{3}\cos(2x) + \frac{5}{6}\sin(2x) & ; \; x > \frac{\pi}{2} \; . \end{cases}$$

Exercises (4.5)

In problems 1 through 20 find the general solution of the given differential equations by using undetermined coefficients method .

1) $y'' + 3y' + 2y = 0.$

2) $y'' - 10y' + 25y = 30x + 3$.

3) $y'' + y' - 6y = 2x$.

4) $\frac{1}{4}y'' + y' + y = x^2 - 2x$.

5) $y'' - 8y' + 20y = 3x^2 - 4xe^x$.

6) $4\,y'' - 4y' - 3y = \cos(2x)$.

7 $y'' + 3y' = -48x^2e^{3x}$.

8) $y'' + 2y' = 2x + 5 - e^{-2x}$.

9) $y'' - y' - 2y = 2xe^{-x} + x^2$.

10) $y'' - y = 4\cosh(x) = 2(e^x + e^{-x})$.

11) $y'' - 7y' - 8y = e^x(x^2 + 2)$.

12) $y'' - 5y' + 4y = e^{2x}(\cos x + \sin x)$.

13) $y''' - 3y'' + 3y' - y = x^2 + 5e^x$.

14) $y''' + y = x + xe^x$.

15) $y''' - 6y'' = 3 - \cos x$.

16) $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$.

17) $y''' - 3y'' + 3y' - y = x - 4e^x$.

18) $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$.

19) $y^{(4)} + 2y'' + y = (x-1)^2$.

20) $y^{(4)} - y'' = 4x + 2xe^{-x}$.

In problems 21 through 22 use trigonometric identity as an aid in finding a particular solution of the given differential equation.

21) $y'' + 4y = 8\sin^2(x)$.

22) $y'' + y = \sin x \cos(2x)$.

In problems 23 through 30 find only the form of the particular solution of the given differential equation by using undetermined coefficients method.

23) $y'' - y = e^x + \sin x$.

24) $y'' - 4y' + 4y = e^{2x}$.

25) $y''' + y' = x^2 - 3x + 1$.

26) $y'' - y = x^2e^x$.

27) $y^{(6)} - 3y^{(3)} = 3x + 1$.

28) $y''' - y' = x^5 + \cos x$.

29) $y''' + 3y'' - 4y = e^{-2x}$.

30) $y'' + 4y = 4x^3 - 8x^2 - 14x + 7$.

In exercises 31 through 35 answer true or false.

31) A particular solution of $y'' + 3y' + 2y = e^x$ is the form Axe^x.

32) A particular solution of $y'' - 3y' + 2y = e^x$ is the form Axe^x.

33) A particular solution of $y'' + y = \frac{1}{x}$ cannot be found by the method of undetermined coefficients .

34) A particular solution of $y''+y = \cos x$ is the form $Ax\cos x+Bx\sin x$.

35) A particular solution of $y'' - 3y = x\ln x$ can be found by the method of undetermined coefficients .

In exercises 36 through 50 solve the initial value problems (IVP).

36) $\begin{cases} y''+y=-2 \\ y(\frac{\pi}{4})=\frac{1}{2} \ , \ y'(\frac{\pi}{4})=-1 \end{cases}$.

37) $\begin{cases} 2y''+3y'-2y=14x^2-4x-11 \\ y(0)=y'(0)=0 \end{cases}$.

38) $\begin{cases} 5y''+y'=-6x \\ y(0)=0, \ y'(0)=-10 \end{cases}$.

39) $\begin{cases} y''+4y'+4y=(3+x)e^{-2x} \\ y(0)=2 \ , y'(0)=5 \end{cases}$.

40) $\begin{cases} y''+4y'+5y=35e^{-5x} \\ y(0)=-3 \ , \ y'(0)=1 \end{cases}$.

41) $\begin{cases} y''-y=\frac{1}{2}(e^x+e^{-x}) \\ y(0)=2 \ , y'(0)=12 \end{cases}$.

42) $\begin{cases} y''+y=\cos x-\sin(2x) \\ y(\frac{\pi}{2})=0 \ , y'(\frac{\pi}{2})=0 \end{cases}$.

43) $\begin{cases} y''-2y'-3y=2\cos^2(x) \\ y(0)=-\frac{1}{3} \ , \ y'(0)=0 \end{cases}$

44) $\begin{cases} y'''+y'=x \\ y(0)=0 \ , \ y'(0)=1 \ \ y''(0)=0 \end{cases}$

45) $\begin{cases} y'''-2y''+y'=2-24e^x+40e^{5x} \\ y(0)=\frac{1}{2} \ ; \ y'(0)=\frac{5}{2} \ \ y''(0)=-\frac{9}{2} \end{cases}$.

46) $\begin{cases} y^{(4)}-y'''=x+e^x \\ y(0)= \ \ y'(0)=0 \ , \ \ y''(0)=y'''(0)=0 \end{cases}$.

47) $\begin{cases} y^{(4)}-y=x^2+1 \\ y(0)= \ \ y'(0)=1 \ \ , \ y''(0)=y'''(0)=0 \end{cases}$.

48) $\begin{cases} y'''-3y''+3y'-1=0 \\ y(0)=1 \ , \ y'(0)=-1 \ , y''(0)=0 \end{cases}$.

49) $\begin{cases} y^{(5)}-6y^{(4)}+9y'''=x \\ y(0)= \ \ y'(0)=0, \ y''(0)=y'''(0)=1, \ y^{(4)}(0)=-1 \end{cases}$.

50) $\begin{cases} y^{(4)}+2y'''+3y''+2y'+1=x^2+1 \\ y(0)= \ \ y'(0)=-1, \ \ y''(0)=y'''(0)=0 \end{cases}$.

4.6 Variation of parameters

The method of variation of parameter, likes the method of undetermined coefficients, is used to compute a particular solution of which implies $c_1 = \frac{9}{10}$ and $c_2 = \frac{2}{45}$ .Thus the solution of the IVP is

$$y = \frac{9}{10}x^{-1} + \frac{2}{45}x^{\frac{3}{2}} + \frac{1}{18}x^{-3} \quad .$$

Exercises (4.6)

In exercises 1 through 26 use the variation of parameters method to compute the general solution of the nonhomogeneous differential equations.

1) $y'' + y = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}.$

2) $y'' + y = \sec x \quad ; \quad 0 < x < \frac{\pi}{2}.$

3) $y'' - 2y' + y = \frac{1}{x}e^{x} \quad ; \quad x > 0.$

4) $y'' + 10y' + 25y = e^{-5x}\frac{\ln x}{x^2} \quad ; \quad x > 0.$

5) $y'' + 6y' + 9y = \frac{1}{x^3}e^{-3x} \quad ; \quad x > 0.$

6) $y'' - 12y' + 36y = e^{6x}\ln x \quad ; \quad x > 0.$

7) $y'' + y = \csc x.\cot x \quad ; \quad 0 < x < \frac{\pi}{2}.$

8) $y'' + 4y' + 5y = e^{-2x}\sec x \quad ; \quad 0 < x < \frac{\pi}{2}.$

9) $y'' + y = \sec^3(x) \quad ; \quad 0 < x < \frac{\pi}{2}.$

10) $y'' - 4y' + 4y = e^{2x}x^{-4} \quad ; \quad x > 0.$

11) $y'' + 2y' + y = x^{-2}e^{-x}\ln x \quad ; \quad x > 0.$

12) $y'' - 2y' + y = \frac{e^x}{(e^x+1)^2}.$

13) $y'' + 2y' + 2y = e^{-x}\csc x \quad ; \quad 0 < x < \frac{\pi}{2}.$

14) $y'' + y = \tan^2(x) \quad ; \quad 0 < x < \frac{\pi}{2}.$

15) $y'' + y = \sec^2(x).\csc x \quad ; \quad 0 < x < \frac{\pi}{2}.$

16) $y'' - 3y' + 2y = \cos(e^{-x}).$

17) $y'' - y = \frac{2}{\sqrt{1-e^{-2x}}} \quad ; \quad x > 0.$

18) $y'' - y = e^{-2x}\sin(e^{-x}).$

19) $5x^2y'' - 3xy' + 3y = x^{\frac{1}{2}} \quad ; \quad x > 0.$

20) $x^2y'' + 4xy' - 4y = x^{\frac{1}{4}}\ln x \quad ; \quad x > 0.$

21) $2x^2y'' + xy' - 3y = x^{-3} \quad ; \quad x < 0.$

22) $2x^2y'' + 7xy' - 3y = x^{-2}\ln x \quad ; \quad x > 0.$

24) $y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}.$

25) $y''' + 4y' = \sec(2x) \quad ; \quad 0 < x < \frac{\pi}{4}.$

26) $2y''' - 6y'' = x^2.$

In exercises 27 through 34 solve the initial value problems (IVP).

27) $\begin{cases} y'' + y = \csc x \\ y(\frac{\pi}{4}) = 0 \ , \ y'(\frac{\pi}{4}) = 1 \end{cases}.$

28) $\begin{cases} y'' + y = \tan x \\ y(\frac{\pi}{3}) = 1 \ , \ y'(\frac{\pi}{3}) = 0 \end{cases}$.

29) $\begin{cases} y'' - 2y' + y = \frac{e^x}{x} \\ y(1) = e \ , \ y'(1) = 0 \end{cases}$.

30) $\begin{cases} y'' + 6y' + 9y = x^{-3}e^{-3x} \\ y(1) = 4e^{-3} \ , \ y'(1) = -2e^{-3} \end{cases}$.

31) $\begin{cases} y'' + y = \sec^3(x) \\ y(0) = 1 \ , \ y'(0) = 1 \end{cases}$.

32) $\begin{cases} y'' - 4y' + 4y = x^{-4}e^{2x} \\ y(1) = 0 \ , \ y'(1) = e^2 \end{cases}$.

33) $\begin{cases} y'' - 2y' + y = \frac{e^{2x}}{(e^x+1)^2} \\ y(0) = 3 \ , \ y'(0) = \frac{5}{2} \end{cases}$.

34) Given that $y_1 = x^{-\frac{1}{2}} \cos x$ and $y_2 = x^{-\frac{1}{2}} \sin x$; $x > 0$, form a fundamental set of solutions of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \ \ on \ (0, \infty).$$

Find the general solution of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}} \ ; \quad x > 0.$$

35) Find the general solution of the differential equation

$$y''' - 5y'' + 6y' = 2\sin x + 8 \quad .$$

REVIEW EXERCISES

In exercises 1 through 25, they are multiple choice, each exercise has four solutions and there is only one correct answer, find it.

1) The initial value problem

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0 \ , \ y'(0) = 0 \end{cases}$$

a) has a unique solution.
b) may have no solution.
c) may have many solution.

d) b) and c) are possible.

2) A fundamental set of solutions of a homogeneous linear differential equations of order n comprises

a) any set of solutions.

b) any set of n solutions.

c) any set of n solutions linearly independent.

d) any set of n homogeneous solutions linearly independent.

3) To obtain the general solution of a homogeneous linear differential equation of order n, we must construct a linear combination of

a) any set of solutions.

b) any set of n solutions.

c) any set of n solutions linearly independent.

d) any set of n homogeneous solutions linearly independent.

4) If $y_1(x)$, $y_2(x)$,......, $y_n(x)$ form a fundamental set of solutions of a homogeneous linear differential equations of order n non an interval I , then the *Wronskian* $W(y_1(x)$, $y_2(x)$,......, $y_n(x))$

a) is zero at every $x \in I$.

b) may be zero at infinity many $x \in I$.

c) is not zero at any $x \in I$.

d) is not zero at any $x \in I$ except at $x = 0$.

5) If a function

$$y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x),$$

is a solution of a homogeneous linear differential equations of order 2 on an interval I , then $y_1(x)$, $y_2(x)$ and $y_3(x)$

a) form fundamental set of solutions of the differential equation on I .

b) are linearly independent on I.

c) are linearly dependent on I.

d) need not all be solutions of the differential equation on I .

6) If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the same second order differential equation, then $\frac{y_1}{y_2}$

a) is a function of x.

b) is a constant,

c) could be a) or b).

d) none of these.

7) If $y_1 = e^{2x}$ is a solution of the given differential equation, use the reduction of order to find a second solution y_2 of

$$y'' - 5y' + 6y = 0$$

a) $y_2 = e^{-2x}$, b) $y_2 = xe^{2x}$, c) $y_2 = e^{-3x}$, d) $y_2 = e^{3x}$.

8) If $y_1 = x^3$ is a solution of the given differential equation, use the reduction of order to find a second solution y_2 of

a) $y_2 = x^3 \ln x$, b) $y_2 = x^{-1}$, c) $y_2 = x^{-1} \ln x$, d) $y_2 = x^4$.

9) If an initial value problem comprises a 4th-order homogeneous linear differential equations , we would expect it to have

a) one initial condition.

b) two initial conditions.

c) three initial conditions.

d) eight initial conditions.

10) If $y = xe^{-3x}$ is a solution of homogeneous linear differential equation , then another solution must be

a) $y = x$, b) $y = e^{-3x}$, c) $y = xe^{3x}$, d) $y = x^2e^{-3x}$.

11) If $y = x\cos(2x)$ is a solution of homogeneous linear differential equation , then another solution must be

a) $y = x\sin(2x)$, b) $y = x\sin x$, c) $y = \sin(2x)$, d) all of the above .

12) If one root of the characteristic (auxiliary) equation of a homogeneous linear differential equation is $3 + 7i$, then another root must be

13) If the characteristic equation of a homogeneous linear differential equation has a factor $(m-4)^3$, then the solutions of the differential equation must include

a) $3 + 7i$, b) $-3 + 7i$, c) $3 - 7i$, d) $-3 - 7i$.

a) $y = e^{4x}$, b) $y = e^{4x}$ and $y = xe^{4x}$, c) $y = xe^{4x}$ and $y = x^2e^{4x}$, d) $y = e^{4x}$, $y = xe^{4x}$ and $y = x^2e^{4x}$.

14) It is possible for the characteristic equation of 3rd-order homogeneous linear differential equation to have

a) two real roots and one complex root.

b) one real and two complex roots.

c) three complex roots.

d) a) or c).

15) undetermined coefficients cannot be used if the input function contains what term ?

a) e^{3x} , b) $\ln(4x)$, c) x^2 , d) $x\cos(4x)$.

16) undetermined coefficients cannot be used if the input function contains what term ?

a) $\sqrt{x}$, b) x^2e^{-3x} , c) $x^2\cos(2x)$, d) $x^3\sin(4x)$.

17) Without solving the differential equation, apply undetermined coefficients the simplest form of a particular solution of the differential

$$y'' - 16y = e^{4x}.$$

a) $y_p = ae^{4x}$, b) $y_p = axe^{4x}$, c) $y_p = a\sin(4x)$, d) $y_p = a\cos(4x)$.

18) Without solving the differential equation, apply undetermined coefficients the simplest form of a particular solution of the differential equation

$$y'' + 9y = 4\cos(3x).$$

a) $y_p = a\cos(3x) + b\sin(3x)$.

b) $y_p = ax\cos(3x) + b\sin(3x)$.

c) $y_p = ax\cos(3x) + bx\sin(3x)$.

d) $y_p = a\cos(3x) + bx\sin(3x)$.

19) We use the variation of parameters to obtain

a) a particular solution of a nonhomogeneous differential equation.

b) the complementary solution 0f linear differential equation.

c) both a) and b).

d) neither a) nor b).

20) When using variation of parameters to solve a 2nd-order linear differential equation, we must evaluate

a) 1 integral , b) 2 integrals , c) 3 integrals , d) 4 integrals.

21) Solving a *Euler's* equation equivalent to using the substitution , where $x > 0$.

a) $y = m^x$, b) $y = \ln x$, c) $y = x^m$, d) $y = e^{mx}$.

22) We may transform a *Euler* differential equation to a *Euler* differential equation with constant coefficients by using the variable substituting , where $x > 0$.

a) $x = e^t$, b) $x = t^m$, c) $t = e^x$, d) $t = x^m$.

23) If the characteristic equation of a homogeneous *Euler* differential equation is $(m-3)^2 = 0$, then linearly independent solutions of the differential equation are (where $x > 0$)

a) $y_1 = x^3$ and $y_2 = x^4$.

b) $y_1 = x^3$ and $y_2 = x^3\ln x$.

c) $y_1 = e^{3x}$ and $y_2 = xe^{3x}$.
d) $y_1 = e^{3x}$ and $y_2 = e^{3x}\ln x$.

24) If the characteristic equation of a homogeneous *Euler* differential equation is $m^2(m-4) = 0$, then linearly independent solutions of the differential equation are (where
a) $y_1 = 1$, $y_2 = x$, and $y_3 = e^{4x}$.
b) $y_1 = 1$, $y_2 = \ln x$, and $y_3 = e^{4x}$.
c) $y_1 = 1$, $y_2 = x$ and $y_3 = x^4$.
d) $y_1 = 1$, $y_2 = \ln x$ and $y_3 = x^4$.

25) To obtain a solution of the *Euler* differential equation that is valid on the interval $(-\infty, 0)$ we must use the substitution
a) $-x = \ln t$, b) $x = t^m$, c) $x = t^{-m}$, d) $t = \ln(-x)$.

Complete the statement of the exercises 26 through 33.

26) The only solution of $y'' + x^2y = 0$, $y(0) = 0$, $y'(0) = 0$ is ________________.

27) If two differential functions f_1 and f_2 are linearly independent on an interval I , then $W(f_1(x), f_2(x) \neq 0$ for at least one point in the interval I ________.

28) The functions $f_1(x) = x^2$, $f_2(x) = 1 - x^2$ and $f_3(x) = 2 + x^2$ are linearly on the interval ________________.

29) The functions $f_1(x) = x^2$ and , $f_2(x) = x\,|x|$ linearly independent on the interval ____________ , whereas they are linearly dependent on the interval ________________.

30) Two solutions y_1 and y_2 of $y'' + y' + y = 0$ are linearly dependent if $W(y_1, y_2) = 0$ for every real value x ______________.

31) A constant multiple of a solution of a differential equation is also a solution ____________________.

32) A fundamental set of solutions of $(x-2)y'' + y = 0$ exist on any interval not containing the point ______________.

33) For the method of undetermined coefficients , the assumed form of the particular solution y_p for $y'' - y = 1 + e^x$ is ____________.

In exercises 34 through 35, write the *Wronskian* of two linearly dependent solutions without finding the solutions them self.

34) $2y'' + \frac{7}{x}y' - y = 0$.

35) $y'' - 8y' + 8e^xy = 0$.

36) Show that $W(5, \sin^2 x, \cos(2x)) = 0$ for all x .Can you establish this result directly without evaluating the *Wronskian* ?.

37) Verify that there exist a unique solution to the initial value problem

$$\begin{cases} (2-x)y'' + (\cos x)y' + (\tan x)y = 0 \quad ; \quad -1 < x < 1 \\ y(0) = 0 \quad , \quad y'(0) = 1 \end{cases}.$$

38) Find the second solution for the differential equation given that $y_1(x)$ is known solution.

$$xy'' - 2(x+1)y' + (x+2)y = 0 \quad ; \quad y_1 = e^x.$$

39) A certain homogeneous linear differential equation with constant coefficients has the characteristic equation

$$(m-7)^3(m^2-4)(m^2+2m+2) = 0.$$

Write the general solution of this differential equation and state its order.

In problems 40 through 65 find the general solution of each differential equation.

40) $y'' - 2y' - 2y = 0.$

41) $2y'' + 2y' + 3y = 0.$

42) $y''' + 10y'' + 25y' = 0.$

43) $3y''' + 10y'' + 15y' + 4y = 0.$

44) $2y''' + 9y'' + 12y' + 5y = 0.$

45) $y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0.$

46) $y'' + 17y' + 16y = e^x + 4e^{-x}.$

47) $(x^2-2x)y'' + (2-x^2)y' + 2(x-1)y = 0 \quad ; \quad 0 < x < 1$, where $y_1 = x^2$ is a particular solution.

48) $3x^2y'' - 2xy' - 12y = 0 \quad ; \quad x > 0.$

49) $y'' + y = \csc x + 1 \quad ; \quad 0 < x < 1.$

50) $y'' - y = x^3e^x.$

51) $x^2y'' + 7xy' + 9y = 0 \quad ; \quad x < 0.$

52) $y''' + 27y = x$.

53) $9y'' + 48y' + 64y = \sin(2x).$

54) $y'' + y = \cot x \quad ; \quad 0 < x < \pi.$

55) $(x\cos x - \sin x)y'' + (x\sin x)y' - (\sin x)y = 0 \quad ; \quad \frac{\pi}{4} < x < \frac{\pi}{2}$, where $y_1 = \sin x$ is a particular solution.

56) $y'' + y = x\sin x.$

57) $y^{(4)} - 2y''' + 2y'' - 2y' + y = 0.$

58) $x^2y'' + 9xy' + 17y = 0 \quad ; \quad x > 0.$

59) $3\,x^2y'' - 2xy' - 12y = x^{-2}$; $x > 0.$

60) $y'' - 3y' + 5y = 4x^3 - 2x$.

61) $y''' - 5y'' + 6y' = 2\sin x + 8.$

62) $y'' - 2y' + y = x^2e^{2x}.$

63) $y''' - y'' = 6.$

64) $y'' - 2y' + 2y = e^x \tan x$; $0 < x < \frac{\pi}{2}.$

65) $y'' - y = \frac{2e^x}{e^x + e^{-x}}.$

In problems 66 through 77 , solve the given differential equation subject to the indicate conditions .

66) $\begin{cases} y'' - 2y' + 2y = 0 \\ y(\frac{\pi}{2}) = 0\ , \quad y'(\frac{\pi}{2}) = -1 \end{cases}$.

67) $\begin{cases} y'' - y = x + \sin x \\ y(0) = 2\ ,\ y'(0) = 3 \end{cases}$.

68) $\begin{cases} y'' + y = \sec^3(x) \\ y(0) = 1\ , \quad y'(0) = \frac{1}{2} \end{cases}$.

69) $\begin{cases} y''' - 4y' = 0 \\ y(0) = 0, \quad y'(0) = 0,\ y''(0) = 2 \end{cases}$.

70) $\begin{cases} y^{(4)} + 2y'' + y = 3x + 4 \\ y(0) = y'(0) = 0\ ,\ y''(0) = y'''(0) = 1 \end{cases}$.

71) $\begin{cases} y''' - 3y'' + 2y' = x + e^x \\ y(0) = 1,\ y'(0) = -\frac{1}{4}\ ,\ y''(0) = -\frac{3}{2} \end{cases}$.

72) $\begin{cases} x^2y'' + 4xy' = 0 \\ y(1) = 0\ ,\ y'(1) = 6 \end{cases}$.

73) $\begin{cases} x^2y'' - 5xy' + 8y = 0 \\ y(2) = 32\ ,\ y'(2) = 0 \end{cases}$.

74) $\begin{cases} xy'' + y' = x \\ y(1) = 1\ ,\ y'(1) = -\frac{1}{2} \end{cases}$

75) $\begin{cases} x^2y'' - 5xy' + 8y = 8x^6 \\ y(\frac{1}{2}) = 0\ ,\ y'(\frac{1}{2}) = 0 \end{cases}$.

76) $\begin{cases} 4x^2 + y = 0 \\ y(-1) = 2\ ,\ y'(-1) = 4 \end{cases}$.

77) $\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8\ ,\ y'(-2) = 0 \end{cases}$.

78) Show that the general solution of the differential equation $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos x + c_2 \sin x + c_3 \cosh x + c_4 \sinh x\ .$$

79) Find the particular solution of

$$y''' - 4y' = x + 3\cos x + e^{-2x} \quad .$$

80) Find the general solution of the differential equation

$$(x+2)^2 y'' + (x+2)y' + y = 0.$$

and

$$(x-x_0)^2 a_1(x)/a_2(x) = \frac{(x+1)^2}{x-1}.$$

Since both of these functions are analytic at $x_0 = -1$, we conclude that $x_0 = -1$ is a regular singular point for the differential equation (4).

Example(4) Locate the ordinary points, regular singular points and irregular singular points of the differential equation

$$(x-1)^2 y'' - (x^2 - x)y' + y = 0 \tag{7}$$

Solution. Here the functions

$$a_1(x)/a_2(x) = -\frac{x(x-1)}{(x-1)^2} = \frac{-x}{x-1}, \tag{8}$$

and

$$a_0(x)/a_2(x) = \frac{1}{(x-1)^2} \tag{9}$$

are analytic at any real number except $x_0 = 1$, so every real number x_0 is an ordinary point of (7) $x_0 = 1$ which is a singular point. We now see whether $x_0 = 1$ is regular or irregular. We have

$$(x-x_0)a_1(x)/a_2(x) = -x,$$

and

$$(x-x_0)^2 a_0(x)/a_2(x) = 1.$$

These functions are analytic at $x_0 = 1$, so $x_0 = 1$ is a regular singular point.

Exercises. In the following exercise answer true or false

1. The point $x_0 = -1$ is a regular singular point for the differential equation

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

2. The point $x_0 = 0$ is an ordinary point for the differential equation

$$xy'' + (1-x)y' + 2y = 0.$$

3. The point $x_0 = 0$ is a singular point for the differential equation

$$(1+x)y'' - 2y' + 2xy = 0.$$

given by

$$y_1 = 1 + 2x^2 + 2x^4 + \frac{4}{3}x^6....$$

$$y_2 = x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + ...$$

and y_p is a particular solution of the given non-homogeneous and is given by

$$y_p = \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{13}{4!}x^4 + \frac{17}{5!}x^5....$$

Exercises

In exercises 1 through 9, locate the ordinary points, regular singular points and irregular singular points of the given differential equation

1) $xy'' - (2x+1)y' + y = 0.$
2) $(1-x)y'' - y' + xy = 0.$
3) $x^3(1-x^2)y'' + (2x-3)y' + xy = 0.$
4) $(1-x)^4y'' - xy = 0.$
5) $2x^2y'' + (x-x^2)y' - y = 0.$
6) $x^2(x^2-9)y'' - (x^2-9)y' + xy = 0.$
7) $x^4 - 16)y'' + 2y = 0.$
8) $x(x^2+1)^3y'' + y' - 8xy = 0.$
9) $x^3 - 8)^3y'' - 2xy' + y = 0.$

In exercises 10 through 13 verify that all singular points of the differential equation are regular singular points

10) $x^2y'' + xy' + (x^2 - \nu^2)y = 0.$ (Bessel equation)
11) $(1-x^2)y'' - xy' + \nu^2 y = 0.$ (Chebyshev equation)
12) $xy'' + (1-x)y' + \nu y = 0.$ (Laguerre equation)
13) $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$ (Legendre equation)

For the following equations, specify an interval around $x_0 = 0$ for which a power series solution converges

14) $y'' - xy' + 6y = 0.$
15) $(x^2-4)y'' - 2xy' + 9y = 0.$

In exercises 16 through 22 solve the initial value problems by using the method of power series about the given initial point x_0

16) $\begin{cases} (1-x^2)y'' - 2xy' + 6y = 0 \\ y(0) = 1,\ y'(0) = 0, \end{cases}$

17) $\begin{cases} y'' - 2(x+2)y' + 4y = 0 \\ y(-2) = 1,\ y'(-2) = 0, \end{cases}$

18) $\begin{cases} (-x^2+4x-3)y''-2(x-2)y'+6y=0 \\ y(2)=1,\ y'(2)=0, \end{cases}$

19) $\begin{cases} (1-x^2)y''-2xy'+2y=0 \\ y(0)=0,\ y'(0)=-1, \end{cases}$

20) $\begin{cases} y''-2(x-1)y'+2y=0 \\ y(1)=0,\ y'(1)=1, \end{cases}$

21) $\begin{cases} (x^2+4x+3)y''-2(x+2)y'-2y=0 \\ y(-2)=0,\ y'(-2)=-1. \end{cases}$

22) $\begin{cases} y''+(x-1)^2y'-4(x-1)y=0 \\ y(1)=1,\ y'(1)=-1, \end{cases}$

In exercises 23 through 27, compute the first four coefficients of power series solution about the given initial point

23) $\begin{cases} y''-xy=0 \\ y(0)=0,\ y'(0)=1, \end{cases}$

24) $\begin{cases} (x^2+2)y''-3y'+(x-1)y=0 \\ y(1)=-20,\ y'(1)=-2, \end{cases}$

25) $\begin{cases} (x-1)y''-xy'+y=0 \\ y(0)=0,\ y'(0)=1, \end{cases}$

26) $\begin{cases} y''-2(x-1)y'+2y=0 \\ y(1)=1,\ y'(1)=0. \end{cases}$

27) $\begin{cases} x^2y''+xy'+2y=0 \\ y(1)=1,\ y'(1)=0. \end{cases}$

28) Derive the Taylor series for the function $\cos x$ by solving the initial value problem

$$\begin{cases} y''+y=0 \\ y(0)=1,\ y'(0)=0 \end{cases}$$

Solve the following equations in power series

29) $(3-x^2)y''-4xy'-7y=0.$

30) $(1-x^2)y''-3xy'+y=0.$

that is

$$y(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{1}{15}e^{2t} + \frac{1}{16}\cos 2t. \qquad (40)$$

It follows from the second equation in (37) and (40) that

$$x(t) = -e^{2t} + \frac{1}{4}\cos 2t + c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t + \frac{2}{15}e^{2t} - \frac{1}{8}\sin 2t.$$

Exercises

In each problems 1 through 4 solve the given systems of equations

1) $\begin{cases} x' + y - x = -t^2 \\ y' - x - 3y = 2t, \end{cases}$

2) $\begin{cases} x' - 4x + 2y = 2t \\ y' - 8x + 4y = 1, \end{cases}$

3) $\begin{cases} x' + 2y' - 3x = -e^{-t}\sin t \\ y' - 4x + y = e^{-t}\cos t, \end{cases}$

4) $\begin{cases} x' = x - 5y, \quad x(0) = 1 \\ y' = 2x - 5y, \quad y(0) = 0. \end{cases}$

In each problems 5 through 8 solve the given system of equations

5) $\begin{cases} x'' - 3x' + y' + 2x - y = 0 \\ y' + x' - 2x + y = 0, \end{cases}$

6) $\begin{cases} y'' + 3x' - 4y' + 4y = 1 \\ y' + x' + 2x - 2y = 0, \end{cases}$

7) $\begin{cases} x'' + y'' - 4x' + 2y' + 4x = 0 \\ x'' + y'' + 4y' - 2x' + 4y = 0, \end{cases}$

8) $\begin{cases} y'' + x' + x = 0 \\ y' - y + x = \sin t. \end{cases}$

Chapter 7

7. Fourier Series

7.1 Orthogonal functions

We will use a new tool called inner product to define orthogonal functions and sets of orthogonal functions.

So the given set of functions is orthogonal on $[-2, 2]$.To find the orthonormal set, we have to compute

$$\left\|\cos\frac{m\pi x}{2}\right\|^2 = \int_{-2}^{2}(\cos\frac{m\pi x}{2})^2 dx = \int_{-2}^{2}\frac{1+\cos m\pi x}{2}dx = 2,$$

$$\left\|\sin\frac{m\pi x}{2}\right\|^2 = \int_{-\pi}^{\pi}(\sin\frac{m\pi x}{2})^2 dx = \int_{-2}^{2}\frac{1-\cos m\pi x}{2}dx = 2.$$

Hence the orthonormal set

$$\left\{\frac{\cos\frac{n\pi x}{2}}{\sqrt{2}}, \frac{\sin\frac{n\pi x}{2}}{\sqrt{2}}\right\}_{n\geq 1}.$$

Example (5) Show that the functions

$$f(x) = 1, g(x) = 2x, h(x) = 4x^2 - 2$$

are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.

$$(1, 2x)_{w(x)} = \int_{-\infty}^{\infty} 2xe^{-x^2}dx = -2\int_{-\infty}^{\infty} d(e^{-x^2}) = -2\, e^{-x^2}\Big|_{-\infty}^{\infty} = 0,$$

$$\begin{aligned}(1, 4x^2-2)_{w(x)} &= \int_{-\infty}^{\infty}(4x^2-2)e^{-x^2}dx = -\int_{-\infty}^{\infty} 2xd(e^{-x^2}) - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= -2xe^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} e^{-x^2}dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx = 0.\end{aligned}$$

In the same way and by integration by parts, we find that

$$(2x, 4x^2-2)_{w(x)} = 0.$$

Exercises.

1. Show that the following functions are orthogonal on the given intervals.

a) $f(x) = \sin^2 x, \quad g(x) = \cos x, \quad [0, \pi]$
b) $f(x) = \cos x, \quad g(x) = e^{-x}, \quad [\frac{\pi}{4}, \frac{5\pi}{4}]$
c) $f(x) = e^x, \quad g(x) = xe^{-x} - e^{-x}, \quad [0, 2]$.

2. Show that the following functions are orthogonal with respect to the given weight function on the indicated interval
a) $f(x) = 4x^2 - 2,\ g(x) = 2x; \quad w(x) = e^{-x^2},\ (-\infty, \infty)$
b) $f(x) = 1 - x,\ g(x) = \frac{1}{2}x^2 - 2x + 1;\ w(x) = e^{-x},\ [0, \infty)$

3) Verify that the functions $f(x) = 3x^2$, $g(x) = 2x^3$ are orthogonal on $[-1, 1]$. Determine the constants α and β, so that the function

$$h(x) = \alpha x^2 + \beta x + 1$$

will be orthogonal to $f(x)$ and $g(x)$ on the interval $[-1, 1]$.

4) Determine the constants λ and δ so that the functions

$$f(x) = \lambda x + 2, g(x) = \delta x^2 - 6x + 1$$

and

$$h(x) = x - 1$$

are mutually orthogonal on $[0, 1]$ and then obtain the corresponding orthonormal set.

5) Show that the following set of functions are orthogonal on the given intervals and find the norm of each function
a) $\{\sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \sin \frac{3\pi x}{L}, \sin \frac{4\pi x}{L}, \sin \frac{5\pi x}{L}, ...\}$; $[0, L]$
b) $\{1, \cos \frac{n\pi x}{L}, \sin \frac{m\pi x}{L}\}$; $m, n = 1, 2, 3, 4...;\ [0, L]$
c) $\{\sin(2n + 1)\}$; $n = 0, 1, 2, 3, 4...;\ [0, \frac{\pi}{2}]$.

7.2 Trigonometric series

Definition (7.2.1) A trigonometric series is a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

where the coefficients a_n and b_n are constants. If the coefficients a_n and b_n satisfy certain conditions which will be specified later on, then the series is called Fourier series. almost all trigonometric series encountered in physical problems are of Fourier type.

where

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin x dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \sin nx dx.$$

We let $u = nx$ and integrate by parts, we get

$$b_n = \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \begin{cases} 0, & n \text{ even} \\ \frac{4(-1)^{(n-1)/2}}{\pi n^2}, & n \text{ odd} \end{cases}$$

Hence the Fourier series

$$\begin{aligned} f(x) &= \begin{cases} x, & 0 \le x \le \pi/2, \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases} \\ &= \frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + ...\right) \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin(2n+1)x. \end{aligned}$$

At $x = \pi$ and $x = -\pi$, the Fourier series converges to 0.

Exercises

1. Compute the Fourier series for the following functions on the given intervals

a) $f(x) = \begin{cases} x + \pi, & -\pi < x \le 0 \\ 0, & 0 \le x < \pi. \end{cases}$

b) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi. \end{cases}$

c) $f(x) = |x| - x, \quad -3 < x < 3$

d) $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi. \end{cases}$

In b) deduce that $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{\pi}{4}$.

e) $f(x) = \begin{cases} x, & 0 < x < \pi \\ x + \pi, & -\pi < x < 0 \end{cases}$

2. Given the function

$$f(x) = \begin{cases} 1, & -2 < x < -1 \\ x^2 - 1, & -1 < x < 0 \\ 1 - x^2, & 0 < x < 1 \\ -1, & 1 < x < 2. \end{cases}$$

a) Determine whether the function f is even or odd
b) Compute the Fourier series
c) Discuss the convergence of the obtained Fourier series at the points: $x = -1$, $x = 0$, $x = 1$.

3. a) Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \leq x < \pi. \end{cases}$$

b) Use a) to show that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12}$$

c) Use b) to obtain a numerical series which represents the value $\frac{\pi^2}{8}$.

4. **2.** Compute the Fourier series fo the function

$$f(x) = |\sin x|, \quad |x| \leq \pi.$$

and deduce that

$$\sum_{n=0}^{\infty} \frac{1}{1-(2n)^2} = 1.$$

4. Test whether each of the following given functions is odd or even, then expand it in a cosine or sine series

a) $f(x) = |\cos x|, \ |x| < \pi$

b) $g(x) = x \cos x, \ |x| < \pi$

c) $h(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2. \end{cases}$

d) $k(x) = x^2 |x|, \ |x| < 1$

e) $M(x) = \begin{cases} x+1, & -3 < x < 0 \\ -x+1, & 0 \leq x < 3 \end{cases}$

5. Find the half range sine series for the functions

a) $f(x) = \begin{cases} x, & 0 \leq x \leq \pi l2 \\ \pi - x, & \pi/2 \leq x \leq \pi. \end{cases}$

b) $g(x) = x - x^2, \ 0 < x < 1$

c) $h(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < 2. \end{cases}$

6. Find the half range cosine series for the functions

a) $f(x) = \frac{1}{3}x(1-x), 0 < x < 1$

b) $g(x) = \begin{cases} -x, & 0 < x \le \pi l2 \\ \pi - x, & \pi/2 \le x < \pi. \end{cases}$

c) $h(x) = 1 - \cos x, \ 0 < x < \pi.$

7.8 Complex form of a Fourier series

We have seen that the Fourier series in the interval $(-T, T)$ of a function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}). \tag{1}$$

From *Euler's* formula we have

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}. \tag{2}$$

Substitution of (2) in (1) leads to

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{\frac{in\pi x}{T}} + e^{\frac{in\pi x}{T}}}{2}\right) + b_n \left(\frac{e^{\frac{in\pi x}{T}} - e^{-\frac{in\pi x}{T}}}{2i}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{\frac{in\pi x}{T}} + \left(\frac{a_n + ib_n}{2}\right) e^{-\frac{in\pi x}{T}} \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{T}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{T}}, \end{aligned}$$

where

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n \ge 1.$$

Hence the complex form of the Fourier series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}},$$

where

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(x) e^{-\frac{in\pi x}{T}}\,dx.$$

By equating the real part, we have

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}$$

Exercises

1. Find the complex form of the Fourier series for:

a) $f(x) = e^x, \ -\pi < x < \pi$

b) $g(x) = e^{-x}, \ \ -1 < x < 1$

2. Compute the and find the complex form of the Fourier series for the function:

$$f(x) = \cos\theta x, \ \ -\pi < x < \pi.$$

Chapter 8

8. Fourier Integral

We have seen that a periodic function defined on a finite interval $(-L, L)$ or $(0, L)$ can be represented by a Fourier series which converges to the periodic extension of the function outside the interval. We can say that Fourier series are associated only with periodic functions. We now try to represent a given non periodic function defined either on an infinite interval $(-\infty, \infty)$ or semi-infinite interval $(0, \infty)$.

8.1 Fourier series to Fourier integral

If f is a function defined on $(-L, L)$, then formulas (11) – (14) give

$$\begin{aligned} f(x) &= \frac{1}{2T}\int_{-T}^{T} f(x)dx + \frac{1}{T}\sum_{n=1}^{\infty}\left(\int_{-T}^{T} f(t)\cos\frac{n\pi t}{T}dt\right)\cos\frac{n\pi x}{T} \\ &+\frac{1}{T}\sum_{n=1}^{\infty}\left(\int_{-T}^{T} f(t)\sin\frac{n\pi t}{T}dt\right)\sin\frac{n\pi x}{T}. \qquad (1) \end{aligned}$$

By letting

$$\lambda_n = \frac{n\pi}{T}, \Delta\lambda = \lambda_{n+1} - \lambda_n = \frac{\pi}{T},$$

where

$$\beta(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx.$$

Example 4 Find the complex form of the Fourier integral for the function
$f(x) = \begin{cases} e^x, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$.

We have

$$\begin{aligned} \beta(\lambda) &= \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx. = \int_{-1}^{1} e^{(i\lambda+1)x} dx \\ &= \frac{1}{(i\lambda+1)} e^{(i\lambda+1)x}\Big|_{-1}^{1} = \frac{1}{(i\lambda+1)} \left(e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right) \\ &= \frac{1-i\lambda}{1+\lambda^2} \left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right]. \end{aligned}$$

Hence

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2} \left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right] e^{-i\lambda x} d\lambda.$$

Exercises.

1. Find the Fourier integral for the following functions:

a) $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$.

b) $g(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$

c) $h(x) = e^{-|x|} \cos x$,

d) $k(x) = e^{-|x|} \sin x$

e) $M(x) = \begin{cases} 0, & -\infty < x < -1 \\ 2x, & -1 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$

f) $N(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$

3. a) Using the Fourier sine integral to show that

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda d\lambda = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

b) Find the Fourier cosine of the functions:
i) $f(x) = e^{-x}\cos x,\ x > 0,$ ii) $g(x) = xe^{-2x}$,
ii) $h(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$.

4. Use Fourier integral to show that

$$\int_0^\infty \frac{\sin\pi\lambda}{1-\lambda^2}\sin x\lambda d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases} \qquad \text{(a)}$$

$$\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}d\lambda = \frac{\pi}{2}e^{-x}, \quad x \ge 0. \qquad \text{(b)}$$

$$\int_0^\infty \frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda = \frac{\pi}{2}e^{-\beta x}, \quad x > 0,\ \beta > 0. \qquad \text{(c)}$$