Solution to the first mid-term exam — Math 244
Semester 452

Q1 a) We have $A^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9I$

Therefore $\left(\frac{1}{9}A\right)A = I$. But A^{-1} is the only matrix satisfying $A^{-1}A = I$. We deduce that $A^{-1} = \frac{1}{9}A = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$

Q1 b) We use row elimination method to find at the same time A^{-1} and $\det A$. We have

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1+2R_4 \\ R_2+2R_4 \\ R_3-R_4 \\ \end{array} \sim \left[\begin{array}{cccc|cccc} -3 & 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ -3 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+R_3 \\ \sim \end{array} \left[\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ -3 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1-R_2 \\ \sim \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -3 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2+3R_1 \\ R_3-R_1 \\ R_4+2R_1 \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right] = [I \mid A^{-1}]$$

Hence $A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & -2 & 3 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 \end{bmatrix}$. Moreover, because none of the performed row operations modify the determinant, we have $\det A = \det I = 1$.

But $A^{-1} = \frac{1}{\det A} \operatorname{Adj} A = \operatorname{Adj} A$, we obtain

$$\operatorname{Adj} A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & -2 & 3 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 \end{bmatrix}.$$

Q1c) We have $A \operatorname{Adj} A = \det A \, I = 2I$.
Because the size of the matrices here is 3×3
we have

$$8 = \det(2I) = \det(A \operatorname{Adj} A) = \det A \det(\operatorname{Adj} A)$$
$$= 2\det(\operatorname{Adj} A).$$

We deduce that $\det(\operatorname{Adj} A) = 4$.

Q2a) The relation $AM = MA$ can be written

$$\begin{pmatrix} 2x+z & 2y+t \\ x-z & y-t \end{pmatrix} = \begin{pmatrix} 2x+y & x-y \\ 2z+t & z-t \end{pmatrix}$$ which is equivalent

to $\begin{cases} z = y \\ x-3y-t=0 \\ x-3z-t=0 \\ y=z \end{cases} \iff \begin{cases} z=y \\ x=3y+t \end{cases}$ This means that y, t can take any values and x, z are deduced from y, t.

This gives $M = \begin{pmatrix} 3y+t & y \\ y & t \end{pmatrix} = y\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

for all $y, t \in \mathbb{R}$.

Q2b) Asking that $(x,y,z)=(1,-2,3)$ is a solution of the system is equivalent to solving the system

$$\begin{cases} a-4b+3c=6 \\ a-12b+3c=-2 \\ 3a-8b+3c=-8 \end{cases}$$ where a,b,c becomes the unknowns.

This system is equivalent to: $(Eq_2 \leftarrow Eq_2 - Eq_1 \; ; \; Eq_3 \leftarrow Eq_3 - Eq_1)$

$$\begin{cases} a-4b+3c=6 \\ -8b=-8 \\ 2a-4b=-14 \end{cases}$$

Therefore, $b=1$. By substitution in Eq_3: $2a-4=-14 \iff a=-5$

By substitution in Eq_1: $-5-4+3c=6 \iff c=5$.

Hence the only values of (a,b,c) so that $(1,-2,3)$ is a solution of the system are $(-5,1,5)$.

Q3a) We use Gauss-Jordan elimination method:

$$\left[\begin{array}{cccc|c} 2 & -1 & -4 & 3 & 1 \\ 3 & -2 & -5 & 4 & 1 \\ 3 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array} \sim \left[\begin{array}{cccc|c} 2 & -1 & -4 & 3 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & -1 & -4 & 3 & 1 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & -1 & 3 & -4 & 0 \end{array}\right] \underset{R_3+R_2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right] \begin{array}{l} R_1+R_3 \\ R_2+2R_3 \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right] \overset{R_1+R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 4 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right]$$

This matrix is in a reduced row echelon form.

We introduce a parameter $z=t$. The system becomes:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 4 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & t \end{array}\right] \begin{array}{l} R_1+7R_4 \\ R_2+5R_4 \\ R_3+3R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4+7t \\ 0 & 1 & 0 & 0 & 3+5t \\ 0 & 0 & 1 & 0 & 1+3t \\ 0 & 0 & 0 & 1 & t \end{array}\right]$$

Therefore, the system has infinitely many solutions
$(4+7t, 3+5t, 1+3t, t)$; $t \in \mathbb{R}$.

Q3b) We have

$$[A|0] = \left[\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \sim \\ \\ R_4 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \frac{1}{2}R_2 \\ R_3 - 2R_1 \\ R_4 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & -9 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \\ -\frac{1}{9}R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + 3R_4 \\ R_2 - 2R_4 \\ R_3 + 2R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_3 \leftrightarrow R_4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We introduce a parameter $y = t$. This gives

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & -t \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, the system has infinitely many solutions $(t, -t, t, 0)$, $t \in \mathbb{R}$.