Solutions to the second Mid-Term exam 244 Math - Semester 452

Question 1:

a) i) False: $(1,1,1)$ is in the set while $(-1)(1,1,1)=(-1,-1,-1)$ is not in the set. So it is not a subspace of $\mathbb{R}^3$.

ii) True: Call this set $Z(Y)$. $Z(Y) \subseteq M_n(\mathbb{R})$ which is already a vector space.

① Because $0Y = 0 = Y0$; $0 \in Z(Y)$

② Let $A, B \in Z(Y)$; $\alpha, \beta \in \mathbb{R}$

Because $AY = YA$ and $BY = YB$, we have

$(\alpha A + \beta B)Y = \alpha AY + \beta BY = \alpha YA + \beta YB = Y(\alpha A + \beta B)$

Hence; $\alpha A + \beta B \in Z(Y)$.

Thus $Z(Y)$ is a vector subspace of $M_n(\mathbb{R})$.

iii) True: The set of 2×2 matrices is a vector space of dimension 4. Because $5>4$; any subset of at least 5 2×2 matrices is linearly dependent.

b) i) Consider $p(x) = x$. Because $\deg p(x) = 1 \leq 2$, $p(x) \in \mathcal{P}_2$. Because $p(1) = 1 \neq p(2) = 2$, $p(x) \notin W$. Hence $\mathcal{P}_2 \setminus W \neq \emptyset$.

ii) Because the polynomials $1, (x-1)(x-2)$ are not proportional they form a linearly independent subset. Both polynomials are in W, so $\dim W \geq 2$. Using (i), we have $\dim W < \dim \mathcal{P}_2 = 3$. Hence $\dim W = 2$ and the polynomials form a basis of W

Question 2:

i) Because $\dim \mathbb{R}^2 = 2$ and the vectors in B are not proportional and in C are not proportional, they are linearly independent and form therefore bases for $\mathbb{R}^2$.

ii) We have ${}_C P_B = P_{C \leftarrow B} = P_{C \leftarrow A} P_{A \leftarrow B}$

$= P_{A \leftarrow C}^{-1} P_{A \leftarrow B}$

where $A = \{(1, 0); (0,1)\}$ is the standard basis of $\mathbb{R}^2$.

Therefore

$${}_C P_B = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 17 & 41 \\ 5 & 12 \end{pmatrix}$$

iii) We have ${}_B P_C = {}_C P_B^{-1} = \begin{pmatrix} 17 & 41 \\ 5 & 12 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} 12 & -41 \\ -5 & 17 \end{pmatrix}$

Therefore ${}_B P_C = \begin{pmatrix} -12 & 41 \\ 5 & -17 \end{pmatrix}$.

iv) We have $[u_1]_C = {}_C P_B [u_1]_B = {}_C P_B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 17 \\ 5 \end{pmatrix}$

and $[v_2]_B = {}_B P_C [v_2]_C = {}_B P_C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 41 \\ -17 \end{pmatrix}$

Question 3:

i) Because the reduced row echelon form of the matrix A has 2 non-zero rows, rank A = dim row A = 2.

Because rank A + nullity A = 3 (A is a 4×3 matrix), we deduce nullity A = 1.

ii) → A basis of row A is given by the non-zero rows of RREF(A):

$$((1,0,1);(0,1,1))$$

→ A basis of col A is given by the non-zero rows of RREF(A^T):

$$((1,0,1,2)^T;(0,1,1,1)^T)$$

→ To get a basis of Nul A, we solve

$$RREF(A)\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \iff \begin{cases}x+z=0\\y+z=0\end{cases} \iff (x,y,z)=z(-1,-1,1)$$

Hence $((-1,-1,1))$ is a basis of Nul(A).

iii) Notice that

→ $x(1,0,1)+y(0,1,1)=(x,y,x+y)$

Hence $(1,1,1)$ cannot be written in this form; so $(1,1,1)\in\mathbb{R}^3\setminus \text{row}(A)$.

→ $x(1,0,1,2)^T+y(0,1,1,1)^T=(x,y,x+y,2x+y)^T$

so $(1,1,0,0)^T$ cannot be written in this form and $(1,1,0,0)^T\in\mathbb{R}^4\setminus \text{col}\,A$.

not proportional to … in