King Saud University: Mathematics Department Math-254
Second Semester 1444 H Solution of Midterm Examination
Time: 120 Mins. Marks: 30

Question 1: Show that one of the possible iterative formula for finding the approximation of the root of the nonlinear equation $1 - \sin x = x$ is

$$x_{n+1} = \frac{\lambda x_n + 1 - \sin x_n}{\lambda + 1}, \qquad \lambda \neq -1, \qquad n \geq 0.$$

Find a constant λ so that this iterative formula will give at least quadratic convergence for finding the root near $\alpha = 0.5$ of the nonlinear equation $1 - \sin x = x$. Find x_2 if $x_0 = 0.45$. (6)

Solution. Given the nonlinear equation $1 - x - \sin x = 0$, which can be written as

$$x = 1 - \sin x \quad \text{or} \quad x + \lambda x - \lambda x = 1 - \sin x \quad \text{or} \quad x(1+\lambda) = \lambda x + 1 - \sin x,$$

and from this we have

$$x = \frac{\lambda x + 1 - \sin x}{1 + \lambda} = g(x).$$

Taking the derivative of $g(x)$, gives

$$g'(x) = \frac{\lambda - \cos x}{1 + \lambda}.$$

Hence for rapid convergence,

$$g'(\alpha) = \frac{\lambda - \cos \alpha}{1 + \lambda} = 0,$$

we obtain, $\lambda = \cos \alpha = \cos 0.5 = 0.8776$.
Now using $x_0 = 0.45$, we have

$$x_1 = \frac{0.8776 x_0 + 1 - \sin x_0}{1.8776} = 0.5113,$$

$$x_2 = \frac{0.8776 x_1 + 1 - \sin x_1}{1.8776} = 0.5110,$$

the second approximation which is corrected up to 3 decimal places. •

Question 2: Successive approximations x_n to the desired root are generated by the scheme

$$x_{n+1} = \frac{e^{x_n}(x_n + 1) + 2x_n^2}{e^{x_n} + 3x_n}, \qquad n \geq 1.$$

Find $f(x) = 0$ and then use secant method to find the second approximation x_3 of the root $\alpha = -0.7035$, starting with $x_0 = -0.5$ and $x_1 = -0.25$. Compute the relative error. (6)

Solution. Given

$$x_{n+1} = \frac{e^{x_n}(x_n + 1) + 2x_n^2}{e^{x_n} + 3x_n} = g(x_n), \qquad n \geq 1.$$

$$x = \frac{e^{x}(x + 1) + 2x^2}{e^{x} + 3x} = g(x),$$

$$g(x) - x = \frac{e^x(x+1) + 2x^2}{e^x + 3x} - x = 0,$$

$$g(x) - x = \frac{e^x(x+1) + 2x^2 - x(e^x + 3x)}{e^x + 3x} = 0,$$

and after simplifying, we obtained

$$g(x) - x = \frac{(xe^x + e^x + 2x^2 - xe^x - 3x^2)}{e^x + 3x} = \frac{(e^x - x^2)}{e^x + 3x} = -\frac{(x^2 - e^x)}{e^x + 3x} = x^2 - e^x = 0.$$

Thus

$$f(x) = g(x) - x = x^2 - e^x = 0,$$

and we can check

$$f(-1) = 0.6321, \qquad f(0) = -1, \qquad f(-1)f(0) = -0.6321 < 0,$$

so $f(x)$ has a zero in $[-1, 0]$. Applying secant iterative formula to find the approximation of this zero, we use the formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^2 - e^{x_n})}{(x_n^2 - e^{x_n}) - (x_{n-1}^2 - e^{x_{n-1}})}, \qquad n \geq 1.$$

Finding the second approximation using the initial approximations $x_0 = -0.5$ and $x_1 = -0.25$, we get

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^2 - e^{x_1})}{(x_1^2 - e^{x_1}) - (x_0^2 - e^{x_0})} = -0.7477,$$

$$x_3 = x_2 - \frac{(x_2 - x_1)(x_2^2 - e^{x_2})}{(x_2^2 - e^{x_2}) - (x_1^2 - e^{x_1})} = -0.6946.$$

The relative error is,

$$\frac{|\alpha - x_3|}{|\alpha|} = \frac{|-0.7035 - (-0.6946)|}{|-0.7035|} = 0.0127.$$

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Question 3 Show that Newton's iterative formula for finding the approximation of the root $\alpha = \pi$ of the nonlinear equation $f(x) = \tan x = 0$ is

$$x_{n+1} = x_n - \sin(x_n)\cos(x_n), \qquad n \geq 0.$$

Find the absolute error $|\alpha - x_2|$ using $x_0 = 3.0$. Find the rate of convergence of the developed iterative formula. (6)

Solution. As $f(x) = \tan x$ and so $f'(x) = \sec^2 x$, and

$$f(\pi) = \tan(\pi) = 0, \qquad f'(\pi) = \sec^2(\pi) \neq 0,$$

therefore, the root is the simple root of the given nonlinear equation and the best numerical method is Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)} = x_n - \sin(x_n)\cos(x_n), \quad n \geq 0.$$

To find the second approximation to the root by using above scheme using $x_0 = 3.0$ and $n = 1, 2$, we obtain

$$\begin{aligned} x_1 &= x_0 - \sin(x_0)\cos(x_0) = 3.1397, \\ x_2 &= x_1 - \sin(x_1)\cos(x_1) = 3.1416, \end{aligned}$$

which is the required second approximation and

$$|\pi - x_2| = |3.1416 - 3.1416| = 0, \qquad \text{(up to 4 decimal places)},$$

gives the required absolute error in the solution.
Since the fixed-point form of the Newton's method is

$$g(x) = x - \frac{\tan x}{\sec^2 x} = x - \sin x \cos x,$$

therefore,

$$\begin{aligned} g(x) &= x - \sin x \cos x, \quad g(\pi) = \pi - \sin(\pi)\cos(\pi) = \pi, \\ g'(x) &= 1 + \sin^2(x) - \cos^2(x) = 0, \quad g'(\pi) = 1 + \sin^2(\pi) - \cos^2(\pi) = 1 - 1 = 0, \\ g''(x) &= 4\sin(x)\cos(x), \quad g''(x) = 4\sin(\pi)\cos(\pi) = 4(0)(-1) = 0, \\ g'''(x) &= 4\cos^2(x) - 4\sin^2(x), \quad g'''(x) = 4\cos^2(\pi) - 4\sin^2(\pi) = 4(-1)^2 - 0 = 4 \neq 0. \end{aligned}$$

Hence the rate of convergence of the Newton's method is cubic. •

Question 4: Use LU decomposition by Dollittle's method to find the value(s) of nonzero α for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix},$$

is inconsistent and consistent. Solve the consistent system. (6)

Solution. Since we know that

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = \dfrac{2\alpha}{\alpha} = 2 = l_{21}$, $m_{31} = \dfrac{1}{\alpha} = l_{31}$, and $m_{32} = \dfrac{(3\alpha - 4)}{(-9\alpha)} = l_{32} (\alpha \neq 0)$, gives

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & \frac{(3\alpha-4)}{\alpha} & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix},$$

which is the required decomposition of A. The given linear system has no solution or infinitely many solution if

$$det(A) = det(U) = \frac{-9\alpha(\alpha^2-1)}{\alpha} = -9(\alpha^2-1) = (\alpha^2-1) = 0,$$

which gives, $\alpha = -1$ or $\alpha = 1$.
To find the solution of the given system when $\alpha = -1$ and it gives

$$\begin{pmatrix} -1 & 4 & 1 \\ -2 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{7}{9} & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{7}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 4]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 4 \end{pmatrix}.$$

Last row gives, $0x_1 + 0x_2 + 0x_3 = 4$, which is not possible, and so no solution. To find the solution of the given system when $\alpha = 1$ and it gives

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{9} & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 0]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 0 \end{pmatrix}.$$

Last row gives, $0x_1 + 0x_2 + 0x_3 = 0$, which means we have many solutions. Performing backward substitution and using $x_3 = t$, yields

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 6 \\ & & -9x_2 & & & = & -9 \end{array}$$

and it gives, $[x_1, x_2, x_3]^T = [2 - t, 1, t]^T$, for any nonzero t. •

Question 5: Consider the following linear system of equations

$$\begin{array}{rcrcrcr} 6x_1 & + & 2x_2 & & & = & 8 \\ x_1 & + & 7x_2 & - & 2x_3 & = & 6 \\ 3x_1 & - & 2x_2 & + & 9x_3 & = & 10 \end{array}$$

Find the matrix form of the Gauss-Seidel iterative method and use it to compute the second approximation $\mathbf{x^{(2)}}$ using the initial solution $\mathbf{x^{(0)}} = [0.5, 0.5, 0.5]^T$. Compute the error bound $\|\mathbf{x} - \mathbf{x}^{(5)}\|_\infty$. (6)

To find the matrix form of the Gauss-Seidel iterative method, firstly we compute the Gauss-Seidel iteration matrix T_G and the vector $\mathbf{c_G}$ as follows:

$$T_G = -(D+L)^{-1}U = \begin{pmatrix} 0 & -1/3 & 0 \\ 0 & 1/21 & 2/7 \\ 0 & 23/189 & 4/63 \end{pmatrix} \quad \text{and} \quad \mathbf{c_G} = (D+L)^{-1}\mathbf{b} = \begin{pmatrix} 4/3 \\ 2/3 \\ 22/27 \end{pmatrix}.$$

Thus the matrix form of Gauss-Seidel iterative method is

$$\mathbf{x^{(k+1)}} = \begin{pmatrix} 0 & -1/3 & 0 \\ 0 & 1/21 & 2/7 \\ 0 & 23/189 & 4/63 \end{pmatrix} \mathbf{x^{(k)}} + \begin{pmatrix} 4/3 \\ 2/3 \\ 22/27 \end{pmatrix}, \quad k = 0, 1, 2,$$

or

$$\mathbf{x^{(k+1)}} = \begin{pmatrix} 0 & -0.3333 & 0 \\ 0 & 0.0476 & 0.2857 \\ 0 & 0.1217 & 0.0635 \end{pmatrix} \mathbf{x^{(k)}} + \begin{pmatrix} 1.3333 \\ 0.6667 \\ 0.8148 \end{pmatrix}, \quad k = 0, 1, 2.$$

Now using above matrix form to compute the second approximation $\mathbf{x^{(2)}}$ using the initial solution $\mathbf{x^{(0)}} = [0.5, 0.5, 0.5]^T$ and $k = 0$, we obtain

$$\mathbf{x^{(1)}} = \begin{pmatrix} 0 & -0.3333 & 0 \\ 0 & 0.0476 & 0.2857 \\ 0 & 0.1217 & 0.0635 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 1.3333 \\ 0.6667 \\ 0.8148 \end{pmatrix} = \begin{pmatrix} 1.1667 \\ 0.8333 \\ 0.9074 \end{pmatrix},$$

and for $k = 1$, gives

$$\mathbf{x^{(2)}} = \begin{pmatrix} 0 & -0.3333 & 0 \\ 0 & 0.0476 & 0.2857 \\ 0 & 0.1217 & 0.0635 \end{pmatrix} \begin{pmatrix} 1.1667 \\ 0.8333 \\ 0.9074 \end{pmatrix} + \begin{pmatrix} 1.3333 \\ 0.6667 \\ 0.8148 \end{pmatrix} = \begin{pmatrix} 1.0556 \\ 0.9656 \\ 0.9738 \end{pmatrix},$$

the required second approximation of the exact root $\mathbf{x} = [1, 1, 1]^T$.
Since the l_∞ norm of the matrix T_G is

$$\|T_G\|_\infty = \max\left\{\frac{1}{3}, \frac{7}{21}, \frac{35}{189}\right\} = \max\{0.3333, 0.3333, 0.1852\} = 0.3333 < 1,$$

and

$$\mathbf{x}^{(1)} - \mathbf{x}^{(0)} = \begin{pmatrix} 1.1667 \\ 0.8333 \\ 0.9074 \end{pmatrix} - \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6667 \\ 0.3333 \\ 0.4074 \end{pmatrix}.$$

Thus using error bound formula,

$$\|\mathbf{x} - \mathbf{x^{(k)}}\| \leq \frac{\|T_G\|^k}{1 - \|T_G\|} \|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\|,$$

and using $\|T_G\| = 0.3333$, $\|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\| = 0.6667$, $k = 5$, we obtain

$$\|\mathbf{x} - \mathbf{x^{(5)}}\| \leq \frac{(0.3333)^5}{(1 - 0.3333)}(0.6667) = 0.0041,$$

the required error bound. •