Question 1 [3+1]

1. Prove that for every real number, there exists an integer n such that $n-1 \leq x < n$. Find such n if $x = -\frac{17}{5}$.
2. Determine $\sup(A)$ and $\inf(A)$ where $A = \{x \in \mathbb{R} : x^2 - 9 < 0\}$, and justify your answer.

for $x \geq 0$, the set $A = \{m \in \mathbb{N} : x < m\} \neq \phi$, hence by well ordering property, $\exists$ a smallest element $n \in \mathbb{N}$, $\therefore x < n$, and $n-1 \notin A$ $n \in A$
hence $n-1 \leq x < n$

if $x < 0$ $\therefore$ let $A = \{m \in \mathbb{N} : -x \leq m\}$, $\therefore \exists$ a smallest element $n \in A$, $\therefore$ $-x \leq n$ but $-x > n-1$

$(-n)-1 \leq x < -n$

0.5 $-4 < -\frac{17}{5} \leq -3$, $\therefore$ $n = -3$

1+1=2 2) $A = \{x : x^2 < 9\} \iff |x| < 3 \iff -3 < x < 3$

$\text{Inf } A = -3$, $\text{Sup } A = 3$

if $\exists$ a lower bd c of A $\ni$, $c > -3$, then
$-3 < x = \frac{c+(-3)}{2} < c$

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Question 2 2 + 2 + 2

Use the definition of the limit to find the following if they exists.

1. $\lim_{n\to\infty} \frac{n^3}{2n^4+1}$.
2. $\lim_{n\to\infty} c^{\frac{1}{n}}$, where $c > 1$.
3. $\lim_{n\to\infty} na^n = 0$, where $0 < a < 1$.

① $0 \leq \frac{n^3}{2n^4+1} \leq \frac{n^3}{n^4} = \frac{1}{n}$ By Sandwich theo

$2n^4+1 \geq n^4$

$0 \to 0$, $\frac{1}{n} \to 0$

$\therefore \lim \frac{n^3}{2n^4+1} = 0$

$\forall \varepsilon > 0, \exists N \in \mathbb{N} \ni \left|\frac{n^3}{2n^4+1} - 0\right| \leq \frac{1}{n} < \varepsilon \quad \forall n \geq N > \frac{1}{\varepsilon}$

$N > \frac{1}{\varepsilon}$ $\quad \Leftrightarrow n > \frac{1}{\varepsilon}$

② $\lim c^{\frac{1}{n}} = 1$

$c^{\frac{1}{n}} = 1 + d_n \Rightarrow c = (1+d_n)^n > n d_n$ by the binomial Theorem

$0 < d_n < \frac{c}{n}$, $\because \frac{c}{n} \to 0$, $\therefore$ By Sandwich th.

$\therefore \lim d_n = 0 \therefore \lim c^{\frac{1}{n}} = \lim 1 + d_n = 1$

③ $\lim n a^n = 0$

$0 < a < 1$, $\therefore \exists b \ni a = \frac{1}{1+b}$, $b > 0$

$\therefore (1+b)^n = 1 + nb + \ldots > nb$

$\therefore a^n = \frac{1}{(1+b)^n} < \frac{1}{nb}$, $\therefore \lim a^n = 0$

$\therefore \lim n a^n = \lim n e^{\ln a^n} = \lim n e^{\boxed{n \ln a}}$ $\because \ln a < 0 \therefore \infty \cdot 0$ $\Rightarrow \lim \frac{n}{e^{-n\ln a}} \overset{\infty/\infty}{=} \lim \frac{1}{\ldots}$

$n \cdot 0$

Question 3 $3+3$

Discuss the convergence of the following series:

3 (i) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n^2+1}$

3 (ii) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

(i) By altrenating series test

Let $f(x) = \frac{\sqrt{x}}{x^2+1} > 0 \; x > 1$ (1)

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x^2+1) - 2x\sqrt{x}}{(x^2+1)^2} = \frac{\frac{1}{2}x\sqrt{x} + \frac{1}{2\sqrt{x}} - 2x\sqrt{x}}{(x^2+1)^2}$$

$$= \frac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2} = \frac{-3x^2+1}{2\sqrt{x}(x^2+1)} < 0 \quad \forall x \geq 1$$

(2) $\therefore$ f is decreasing.

(3) $\lim_{x\to\infty} f(x) = 0$

$\therefore$ Cgt.

(ii) By ration test $a_n = \frac{2^n n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n \cdot n!}$$

$$= \lim_{n\to\infty} \frac{2(n+1) \cdot n^n}{(n+1) \cdot (n+1)^n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} 2\frac{1}{e} < 1, \therefore \text{Cgt.}$$

Question 4 $4+3$

$2+2$ 1. Find the following limits, if they exist, and prove using the definition of the limit or sequence characterization:

a) $\lim_{x\to 0} \frac{x^2}{|x|}$ (b) $\lim_{x\to\infty} \frac{x^2}{e^x}$.

a) $\lim\limits_{x\to 0} \frac{x^2}{|x|} =$

$$f(x) = \begin{cases} \frac{x^2}{x} = x & x>0 \\ \frac{x^2}{-x} = -x & x<0 \end{cases}$$

$\therefore \lim\limits_{x\to 0^+} x = 0$, $\lim\limits_{x\to 0^-} -x = 0$

$\forall \varepsilon > 0, \exists \delta = \varepsilon > 0 \ni$:

if $0<x<\delta$, then $|f(x)-0| = |x| = x < \varepsilon$

Similarly $\forall \varepsilon > 0, \exists \delta = +\varepsilon > 0$

$|x-0|$

if $0<-x<\delta$, then $|f(x)-0| = |-x| = -x < \varepsilon$

$\therefore \lim\limits_{x\to \infty} \frac{x^2}{|x|} = 0$

(b) $\lim\limits_{x\to\infty} \frac{x^2}{e^x}$

$\because f(x) = x^2, g(x) = e^x$ are cont on $\mathbb{R}$ diff on $\mathbb{R}$

$g'(x) = e^x \neq 0$, $\lim\limits_{x\to\infty} \frac{x^2}{e^x}$ of the form $\frac{\infty}{\infty}$

$\therefore$ By L'Hop Rule $\lim\limits_{x\to\infty} \frac{x^2}{e^x} = \lim\limits_{x\to\infty} \frac{2x}{e^x} \underset{\text{L'Hop}}{\overset{\text{again}}{=}} \lim\limits_{x\to\infty} \frac{2}{e^x} = 0$

3 2. Let

$$f(x) = \begin{cases} x^2 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Prove that f is differentiable at $x = 0$, and evaluate $f'(0)$.

$x \to 0$

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad \cancel{0.5 + 0.5}$$

$\therefore$ if $x > 0$ $\underline{\text{or}}$

$$\underset{\to 0}{0} \leq \frac{f(x) - f(0)}{x - 0} \leq \underset{\to 0}{x} \leftarrow 0.5 + 0.5$$

if $x < 0$

$$\underset{\text{or}}{x} \leq \frac{f(x) - f(0)}{x - 0} \leq \underset{\to 0}{0}$$

Question 5 3+3+3

3 1. Determine a real interval of length $\frac{1}{2}$ where the equation

$$x^3 - 6x^2 + \frac{5}{2} = 0,$$

has a solution. Justify your answer.

3 2. Prove that if f is continuous on $[a, b]$ and has zero derivative on (a, b), then f is constant.

1) $\because f(\frac{1}{2}) = \frac{9}{8} > 0$

$f(1) = \frac{-5}{2} < 0$

$\therefore$ By the I. V. Th $\exists\, c \in (\frac{1}{2}, 1) \ni: f(c) = 0$

2). $\forall\, x_1, x_2 \in (a,b) \ni: x_1 < x_2$, apply

M.V. Th on $[x_1, x_2]$, $\therefore \exists\, c \in (x_1, x_2) \ni,$

$$\frac{f(x_2) - f(x_1)}{x_2 - x} = f'(c) = 0 \implies f(x_2) = f(x_1)$$

$\therefore f$ is constant.

3 3.Use Taylor's theorem with $n = 3$ and $x_0 = 0$ to obtain a suitable approximation of the function $f(x) = \sqrt{1-x}$ by a polynomial of degree 3.

$f(x) = (1-x)^{+1/2} \quad f(0) = 1$

$f'(x) = \frac{-1}{2}(1-x)^{-1/2} \quad = -\frac{1}{2}$

$f''(x) = \frac{-1}{4}(1-x)^{-3/2} \quad = -\frac{1}{4}$

$f''' = -\frac{3}{8}(1-x)^{-5/2} \quad = -\frac{3}{8}$

$f^{(4)} = \frac{-15}{16}(1-x)^{-7/2} \quad = -\frac{15}{16}$

$\therefore f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$+ \frac{f^{(4)}(c)}{4!}x^4$

$= 1 - \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2} - \frac{3}{8}\frac{x^3}{6} + \left(\frac{-15}{16}\frac{(1-c)^{-\frac{7}{2}}}{4!}\right)$

Question 6 8

Let

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \cap [-2,2] \\ -1 \text{ if } x \in \mathbb{Q}^c \cap [-2,2] \end{cases}$$

i) Find the upper and the lower integral of f over $[-2,2]$.

ii) Is f integrable on $[-2,2]$? justify your answer.

iii) Is $|f|$ integrable on $[-2,2]$? justify your answer.

Let P be a partion of $[-2,2]$, $P = \{x_0, x_1, \dots, x_n\}$

$\therefore M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\} = 1$ 0.5

$m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\} = -1$ 0.5

$U(f,P) = \sum_{i=0}^{n-1} M_i \Delta x_i = 1.(2-(-2)) = 4$ 0.5

$L(f,P) = \sum_{i=0}^{n-1} m_i \Delta x_i = (-1)(2-(-2)) = -4$ 0.5

$\therefore U(f) = 4$,

$L(f) = -4$

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(ii) $\because U(f) \neq L(f)$ $\therefore f$ is not integrable.

(iii) $|f| = 1$,

$\therefore f$ is constant on $[-2,2]$, $\therefore M_i = 1$ $\forall p \in P[a,b)$

$m_i = 1$

$U(f) = 1$

$L(f) = 1$ $\Rightarrow f$ is integrable.