جامعة الملك سعود
King Saud University

Department of Mathematics
Midterm Exam Semester II (3/30/2021)

Time: 2 hours | Math 203 | Marks: 30

Q.1) Determine whether the sequence $\{2n\ln(n+1) - n\ln(n^2)\}_{n\geq 1}$ converges or diverges and if it converges, find its limit.

[5 Marks]

Q.2) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$ is absolutely convergent, conditionally convergent or divergent.

[5 Marks]

Q.3) Find the interval of convergence and radius of convergence of the power series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right) x^n$$

[5 Marks]

Q.4) Find the Maclaurin series of $f(x) = \frac{e^{x^2}-1}{x^2}$ and use its first three non-zero terms to approximate $\int_0^1 f(x)dx$.

[5 Marks]

Q.5) Evaluate the double integral $\int_1^{\sqrt{e}} \int_0^{\ln x} \frac{1}{e^y - \sqrt{e}\, y}\, dydx$.

[5 Marks]

Q.6) Find the surface area of the surface $z = \sqrt{x^2+y^2}$ that lies above the region R in xy-plane bounded by the circle $x^2 + y^2 = 9$.

[5 Marks]

End of exam

M-203

Term Exam. (II Semester 1441/1442)

Time: 2 Hours Solutions Max. Marks: 30

Q#1) Determine whether the sequence $\{2n\ln(n+1) - n\ln n^2\}$ converges or diverges and if it converges find its limit. [Marks: 5]

Sol. $\ln(n+1)^{2n} - \ln(n^{2n}) = \ln\left(\frac{n+1}{n}\right)^{2n} = \ln\left(1+\frac{1}{n}\right)^{2n}$

$$= 2n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{2n}} \quad (2)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/2n} \quad \frac{0}{0} \text{ form.} \quad 1$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\left(-\frac{1}{\cancel{n^2}}\right)}{+\frac{1}{2}\left(-\frac{1}{\cancel{n^2}}\right)} \quad (1) \quad = \lim_{n\to\infty} 2\,\frac{1}{1+\frac{1}{n}} = 2 \quad (1)$$

Conv.

Q#2) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$ is absolutely convergent, conditionally convergent or diverg.[Marks: 5]

Sol. $\lim \sum_{n=1}^{\infty} |u_n| = \lim \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad (1)$

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{(n!)}}{(n+1)^n\cdot\cancel{(n+1)}}\,\frac{n^n}{\cancel{n!}} \quad (2)$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \quad (2)$$

Hence, it is absolutely convergent.

Q#3) Find the Interval of convergence and radius of Convergence of the power Series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)x^n$. (2)

[Marks: 5]

Soln. Choose $u_n = \sin\left(\frac{1}{\sqrt{n}}\right)x^n$.

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{\sin\left(\frac{1}{\sqrt{n+1}}\right)}{\sin\left(\frac{1}{\sqrt{n}}\right)}|x|$$

$$= \frac{\sin\left(\frac{1}{\sqrt{n+1}}\right)}{\frac{1}{\sqrt{n+1}}} \frac{\frac{1}{\sqrt{n}}}{\sin\left(\frac{1}{\sqrt{n}}\right)} \frac{\sqrt{n}}{\sqrt{n+1}} |x|$$

as $n \to \infty \longrightarrow |x|$ $\therefore -1 < x < 1$

If $x=1$, we have $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$ (2)

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} = 1 \text{ and } \sum \sin\frac{1}{\sqrt{n}} \text{ is divg.}$$ (1)

If $x=-1$, $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{\sqrt{n}}$ is convergent as $\sin\left(\frac{1}{\sqrt{n}}\right)$ (1) is decreasing to zero. Hence Interval of Conv: $[-1, 1)$ and radius of convergence $r = 1$. (1)

Q#4) Find the Maclaurin Series of $f(x) = \frac{e^{x^2}-1}{x^2}$ and use its first three non-zero terms to approximate $\int_0^1 f(x)\,dx$.

[Marks: 5]

Soln. We have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Hence $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ (1)

$\therefore e^{x^2} = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$ $\therefore f(x) = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)!}$

$\therefore f(x) = \frac{e^{x^2}-1}{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)!} = 1 + \frac{x^2}{2} + \frac{x^4}{6} + \cdots$ (2)

$$\therefore \int_0^1 f(x)\,dx = \int_0^1 \left(1+\frac{x^2}{2}+\frac{x^4}{6}+\cdots\right)dx$$

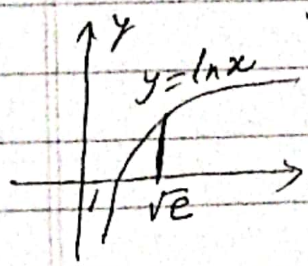
$$= \left[x+\frac{x^3}{3(2)}+\frac{x^5}{5(6)}+\cdots\right]_0^1$$

$$= 1+\frac{1}{6}+\frac{1}{30} = \frac{30+5+1}{30} = \frac{36}{30} = \frac{6}{5} = 1.2 \quad (2)$$

Q#5) Evaluate the double integral $\int_1^{\sqrt{e}}\int_0^{\ln x} \frac{1}{e^y-\sqrt{e}\,y}\,dy\,dx$ [Marks: 5]

Soln: Reversing the order of integration, we get

$$I = \int_1^{\sqrt{e}}\int_0^{\ln x}\frac{1}{e^y-\sqrt{e}\,y}\,dy\,dx = \int_0^{1/2}\int_{e^y}^{\sqrt{e}}\frac{1}{e^y-\sqrt{e}\,y}\,dx\,dy \quad (3)$$



$$= \int_0^{\frac{1}{2}}\frac{\sqrt{e}-e^y}{e^y-\sqrt{e}\,y}\,dy = -\ln\left(e^y-\sqrt{e}\,y\right)\Big]_0^{\frac{1}{2}} = -\ln\frac{\sqrt{e}}{2}$$

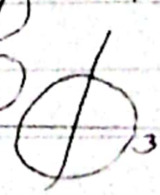
(2)

Q#6) Find the surface area of the surface $z=\sqrt{x^2+y^2}$ that lies above the region R in xy-plane bounded by the circle $x^2+y^2=9$. [Marks: 5]

Soln: We have $z=\sqrt{x^2+y^2}=g(x,y)$ (1)

$$\therefore g_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } g_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} \quad (1)$$

$$\therefore \sqrt{1+g_x^2+g_y^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2} \quad (1)$$



$$\therefore S.A. = \iint_R \sqrt{1+g_x^2+g_y^2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

$$= \sqrt{2}\left[\frac{r^2}{2}\right]_0^3 2\pi = \frac{1}{\sqrt{2}}\,9\pi\left(\frac{\sqrt{2}}{2}\right)$$

$$= 9\sqrt{2}\,\pi$$

(2)