(1) By contradiction, suppose that $\alpha \neq \inf A$.

Let $\varepsilon = \inf A - \alpha > 0$

By assumption there exists $a \in A$ such that 2

$$a < \alpha + \varepsilon = \alpha + \inf A - \alpha = \inf A.$$

On the other hand α is a lower bound then

$$\alpha \leq \inf A.$$

This is a contradiction This proves that $\alpha = \inf A$.

(2) Let $\varepsilon = \frac{|x|}{2}$ Since $x_n \to x$, there exists $N \in \mathbb{N}$

~~such that $|x_n - x| < 1$ for all $n \geq N$.~~

~~$-1 < x_n - x < \frac{|x|}{2}$~~

~~$0 < x - \frac{|x|}{2} < x_n < \frac{|x|}{2} + x$~~

~~Put $M = x - \frac{|x|}{2}$~~ 2

then: $\big||x_n| - |x|\big| < |x_n - x| < \frac{|x|}{2}$ $\Longleftrightarrow$

for every $n \geq N$.

It follows

$$\frac{|x|}{2} = |x| - \frac{|x|}{2} < |x_n| \quad \text{for all } n \geq N$$

Put $M = \frac{|x|}{2}$, then

for all $n \geq N$. $|x_n| \geq M$.

Question 2

(1) let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{2}$; $\boxed{f(0) = 0}$

$$|f(x) - f(0)| \leq 2|x - 0| < 2\delta = \varepsilon$$

3

(2) let $h(x) = f(x) - g(x)$

- h is continuous on $[a, b]$
- $h(a) < 0$ and $h(b) > 0$

By I.V.T there exists $c \in (a, b)$ such that

$$h(c) = 0$$

then $f(c) - g(c) = 0$, 3

(3) (a) $x_n = n + \frac{1}{n}$, $y_n = n$

3

- $|x_n - y_n| = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$
- $x_n^2 - y_n^2 = n^2 + 2 + \frac{1}{n^2} - \frac{1}{n^2} = n^2 + 2 \to +\infty$

then $f(x) = x^n$ is not uniformly continuous on $\mathbb{R}$

(b) ~~$x \to$ cos(x)~~

unknown

(4) Since for all $x, y \in \mathbb{R}$

$$|\cos(x) - \cos y| \leq |x - y|.$$ 3

This shows that $x \to \cos(x)$ is uniformly continuous on $\mathbb{R}$

Question 3

(1) By Cauchy criterion: for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ / for all $n \geq m \geq N$ we have

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon.$$ 2

Take $m = n$, then

for all $n \geq N$, $|a_n| < \varepsilon$.

This shows that $a_n \to 0$ as $n \to \infty$. 2

(2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely 3

Converge.

(3) $\sum_{k=1}^{\infty} \frac{\sin k}{(1.001)^k}$ 2

$\left|\frac{\sin k}{(1.001)^k}\right| < \frac{1}{(1.001)^k}$

$\sum_{k=1}^{\infty} \frac{1}{(1.001)^k}$ converges.

By Comparison test $\sum \frac{\sin(k)}{(1-\alpha)^k}$ converges (2)

ii) $\sum \frac{k^2+3}{4k^2+5}$; diverges since $\frac{k^2+3}{4k^2+5} \to \frac{1}{4} \neq 0$ (3)

iii) $\sum_{k=1}^{\infty} \frac{e^{-k}}{k!}$ converges (3)

$0 \leq \frac{e^{-k}}{k!} \leq e^{-k} = \left(\frac{1}{e}\right)^k$ (1) (1)

$\sum \left(\frac{1}{e}\right)^k$ converges

By Comparison test $\sum_{k=1}^{\infty} \frac{e^{-k}}{k!}$ converges

Question 4

(1) $f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$ (1)

$= \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$ (1)

Since f is differentiable at c (1)

necessarily $f'(c) = 0$

(2) $f(x) \leq |x^2| \Rightarrow f(0) = 0$ (1)

$\left|\frac{f(x)-f(0)}{x}\right| \leq |x|$ (2)

then $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$.

So f is differentiable at $x=0$ and $f'(0)=0$

Question 5

(1) $\overline{\int_{-1}^{1}} f(x)\,dx = \sup_{P} \{ U(f,P) ; \ P \text{ partition of } [-1,1] \}$ 2

for every partition of $[-1,1]$. 2

$U(f,P) = \sum_{i=1} f(x_i)\, \Delta_i = \sum$

$= 1-(-1) = 2.$

$\overline{\int_{-1}^{1}} f(x)\,dx = 2.$

and similarly $\underline{\int_{-1}^{1}} f(x)\,dx = -2$

(ii) Since $\overline{\int_{-1}^{1}} f(x)\,dx \neq \underline{\int_{-1}^{1}} f(x)\,dx$ 2

f is not Riemann-integrable on $[-1,1]$

(iii) $|f(x)| = 1$ is constant function so its Riemann integrable on $[-1,1]$. 2