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ON TWISTED SUMS OF BANACH SPACES

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Abstract

Using non-linear duality, we present several relations between the existence of non-trivial twisted sums of two Banach spaces and the existence of non-trivial twisted sums of their complemented subspaces and their duals, and we give some concrete examples. Also, we construct a quasi-linear map between two sequence spaces using an existed bounded linear map, and we prove that $Ext(Y, Z) = 0$ is a three space property for Z.

1. Introduction

A diagram $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ of quasi Banach spaces and bounded linear operators is called an *exact sequence* if the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem implies that X contains $i(Y)$ and the quotient $X/i(Y)$ is isomorphic to Z. In this case, we shall say that X is a *twisted sum* of Y and Z.

Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are said to be *equivalent* if there is a bounded linear operator T making

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the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, "T"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & X_2 \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

commutative. The three-lemma and the open mapping theorem imply that T must be an isomorphism, Cabello and Castillo [3, p. 525]. An exact sequence $0 \to Y \to X \to Z \to 0$ is said to be *split* if it is equivalent to the trivial exact sequence $0 \to Y \to Y \oplus Z \to Z \to 0,$ in this case, we say that X is *trivial*. We denote by $Ext(Z, Y)$ the space of all equivalence classes of locally convex twisted sums of Y and Z. Thus $Ext(Z, Y) = 0$ means that all locally convex twisted sums of Y and Z are equivalent to the direct sum $Y \oplus Z.$ An operator $T : X \to Y$ of Banach spaces is an *isomorphism* if it is an invertible bounded linear map, T is an *isometry* if $\| Tx \| = \| x \|$ for every $x \in X,$ it is a λ-*isomorphism*, $\lambda > 1,$ if T is an isomorphism and $\| T \| < \lambda,$ $\|T^{-1}\| < \lambda,$ Heinrich [8, II.6]. The distance between two homogeneous maps T_1 and T_2 acting between the same spaces is given by

$$\text{dist}(T_1, T_2) = \sup\{\| T_1x - T_2x \| : \| x \| \leq 1\}.$$

We note that bounded maps are those maps at finite distance from the zero map, also it should be kept in mind that linear maps are not assumed to be bounded.

The reader is referred to Castillo and González [6] for a detailed account of exact sequences. The classical theory of Kalton and Peck [11] describes short exact sequences of quasi-Banach spaces in terms of the so-called quasi-linear maps. A homogeneous map $F : Z \to Y$ between two Banach spaces Z and Y is said to be *quasi-linear* if for some constant k and all $z, w \in Z$ it satisfies

$$\| F(z + w) - F(z) - F(w) \| \leq k(\| z \| + \| w \|).$$

The smallest constant satisfying the above inequality is called the *quasi-linearity constant* of the map F and is denoted by $Q(F)$ Cabello and

Castillo [4]. If $F : Z \to Y$ is a quasi-linear map, then it is possible to construct a twisted sum $Y \oplus_F Z$ by endowing the product space $Y \times Z$ with the quasi-norm $\|(y, z)\| = \|y - F(z)\| + \|z\|$. Clearly, the subspace $\{(y, 0) : y \in Y\}$ of $Y \oplus_F Z$ is isometric to Y and the corresponding quotient $(Y \oplus_F Z)/Y$ is isometric to Z. Conversely, given a short exact sequences $0 \to Y \to X \to Z \to 0$, a quasi-linear map $F : Z \to Y$ can be obtained such that X is equivalent to $Y \oplus_F Z$ [6, 1.5]. Two quasi-linear maps F and G of a Banach space Z into a Banach space Y are said to be *equivalent* if the corresponding exact sequences $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y \to Y \oplus_G Z \to Z \to 0$ are equivalent, in this case, we say that F is a *version* of G. It is shown that quasi-linear maps F and G are equivalent if and only if $d(F - G, L(Z, Y)) = \inf\{\mathrm{dist}(F - G, L) : L \in L(Z, Y)\} < \infty$ [11, Theorem 2.5], where $L(Z, Y)$ is the space of all linear maps $L : Z \to Y$. A quasi-linear map $F : Z \to Y$ is said to be *trivial* if the exact sequence $0 \to Y \to Y \oplus_F Z \to Z \to 0$ is equivalent to $0 \to Y \to Y \oplus Z \to Z \to 0$. Consequently, F is trivial if and only if F is at a finite distance from some linear map [1, Theorem 16.2]. In particular, F is trivial if and only if it can be written as the sum of a bounded and a linear map. There is a one to one correspondence between the classes of twisted sums $Y \oplus_F Z$ and the classes of quasi-linear maps $F : Z \to Y$, Benyamini and Lindenstrauss [1, 16.2]. A homogeneous map $F : Z \to Y$ acting between two Banach spaces is said to be *zero-linear* if there is some constant k such that whenever $z_1, z_2, \ldots, z_n$ are finitely many elements of Z, then

$$\left\| F\left(\sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{n} F(z_i) \right\| \le k\left(\sum_{i=1}^{n} \|z_i\|\right).$$

The smallest constant satisfying the above inequality, denoted by $Z(F)$, is called the *zero-linearity constant* of F. We note that a zero-linear map is a quasi-linear map, and that a twisted sum $Y \oplus_F Z$ of Banach spaces Y and Z is locally convex if and only if F is zero-linear [6, 1.6.e].

2. Nonlinear Duality

Let $F : Z \to Y$ be a zero-linear map that induces the exact sequence $0 \to Y \to X \to Z \to 0$. Then the dual sequence $0 \to Z^* \to X^* \to Y^* \to 0$ is well defined and exact [6, 2.2.d], and for each $y^* \in Y^*$, the composition $y^* \circ F : Z \to \mathbb{K}$ is a zero-linear map with $Z(y^* \circ F) \leq Z(F)\| y^* \|$, so that there is a linear map $H(y^*) : Z \to \mathbb{K}$ such that $\| H(y^*) - y^* \circ F \| \leq Z(F)\| y^* \|$ [3, Lemma 1], and the map $H : Y^* \to L(Z, \mathbb{K})$ need not to be linear. Take a Hamel basis (g_α) for Y^*, and define a map $L_H : Y^* \to L(Z, \mathbb{K})$ by $L_H(g_\alpha) = H(g_\alpha)$ and linearity. Then the map $F^* = L_H - H$ is a zero-linear map from Y^* to Z^*, and is called a *dual map of F*. Moreover, $Z(F^*) \leq Z(F)$ and the sequences $0 \to Z^* \to Z^* \oplus_{F^*} Y^* \to Y^* \to 0$ and $0 \to Z^* \to (Y \oplus_F Z)^* \to Y^* \to 0$ are equivalent [3, Theorem 3]. A zero-linear map $G : Y^* \to Z^*$ is called a *version of* F^* if $G = L' - H'$, where $L', H' : Y^* \to L(Z, \mathbb{K})$ such that H' is a homogeneous map satisfying $\| H'(y^*) - y^* \circ F \| \leq M\| y^* \|$ for some constant M, and L' is linear that coincide with H' on any Hamel basis of Y^* [3, Remark 1]. There is a version G of the zero-linear map $F^{**} : Z^{**} \to Y^{**}$ such that the restriction of G to Z coincides with F [3, Lemma 2]. An exact sequence $0 \to A \to B \to C \to 0$ is said to be a *dual sequence* if there is an exact sequence $0 \to Y \to X \to Z \to 0$ such that Y, X and Z are preduals of A, B and C, respectively.

Theorem 2.1. *Let Z and Y be Banach spaces such that Y is complemented in its bidual. If* $Ext(Z^{**}, Y) = 0$, *then* $Ext(Z, Y) = 0$.

Proof. Suppose that $Ext(Z^{**}, Y) = 0$, then all locally convex twisted sums of Y and Z^{**} are equivalent to $Y \oplus Z^{**}$. Let $F : Z \to Y$ be a zero-linear map, and consider a version G of the dual zero-linear map $F^{**} : Z^{**} \to Y^{**}$ that coincides with F on Z, then the composition map

$Z^{**} \xrightarrow{G} Y^{**} \xrightarrow{\pi} Y$ is a trivial zero-linear map, where $\pi : Y^{**} \to Y$ is a projection. Hence, there is a linear map $L : Z^{**} \to Y$ and a constant c such that $\| (\pi \circ G)(z) - L(z) \| \leq c\| z \|$ for all $z \in Z^{**}$ [3, Lemma 1], which implies that $\| F(z) - L(z) \| \leq c\| z \|$ for all $z \in Z$, since $\pi \circ G|_Z = F$. That is, $\operatorname{dist}(F, L|_Z) \leq c$, proving that F is trivial, by [1, Theorem 16.2]. □

Duals of a zero-linear map play an important role in establishing a relation between the existence of a non-trivial twisted sum of Banach spaces and the existence of a non-trivial twisted sum of their duals as we see in the following:

Theorem 2.2. *Let Y and Z be Banach spaces. Then* $Ext(Y, Z^*) = 0$ *if and only if* $Ext(Z, Y^*) = 0$.

Proof. Suppose that $Ext(Y, Z^*) = 0$, and let $F : Z \to Y^*$ be a zero-linear map. Let $F^* : Y^{**} \to Z^*$ and $F^{**} : Z^{**} \to Y^{***}$ be dual zero-linear maps of F and F^*, respectively, such that $F^{**}|_Z = F$. Then F^{**} can be written as $L_H - H$, where $L_H : Z^{**} \to L(Y^{**}, \mathbb{K})$ is a linear map, and $H : Z^{**} \to L(Y^{**}, \mathbb{K})$ is a homogeneous map satisfying

$$\| H(z^{**})(y^{**}) - z^{**} \circ F^*(y^{**}) \| \leq cZ(F) \| z^{**} \| \| y^{**} \|,$$

for all $z^{**} \in Z^{**}$ and $z^{**} \in Y^{**}$ [3, Theorem 3]. Let $G = F^*|_Y : Y \to Z^*$, and define $\varphi : Z^{**} \to Y^*$ by $\varphi(z^{**}) = L_H(z^{**})|_Y - H(z^{**})|_Y$. It is clear that φ is a version of the dual zero-linear map $G^* : Z^{**} \to Y^*$. Since $Ext(Y, Z^*) = 0$, G is trivial, and so is φ [3, Theorem 3]. Hence, there is a linear map $L : Z^{**} \to Y^*$ such that $\operatorname{dist}(\varphi, L) < \infty$. But

$$(\varphi(z))(y) = (L_H(z))(y) - (H(z))(y) = (F^{**}(z))(y) = (F(z))(y),$$

for all $y \in Y$, $z \in Z$, and so, $\varphi|_Z = F$. Therefore, F is trivial, since $\operatorname{dist}(F, L|_Z) = \operatorname{dist}(\varphi|_Z, L|_Z) < \infty$. Proving that $Ext(Z, Y^*) = 0$. The converse follows by symmetry. □

Proposition 2.3. *Let* $Y_1, \; Y_2$ *and* Z *be Banach spaces. Then*

(i) $Ext(Z, Y_1 \oplus Y_2) \neq 0$ *if and only if* $Ext(Z, Y_i) \neq 0$, *for some* $i = 1, 2.$

(ii) $Ext(Y_1 \oplus Y_2, Z) \neq 0$ *if and only if* $Ext(Y_i, Z) \neq 0$, *for some* $i = 1, 2.$

Proof. (i) This is Lemma 4 of [3].

(ii) Suppose that $Ext(Y_1 \oplus Y_2, Z) \neq 0$, and $Ext(Y_i, Z) = 0$ for $i = 1, 2$, and let $F : Y_1 \oplus Y_2 \to Z$ be a non-trivial zero-linear map. By [11, Theorem 2.5], there is a linear map $L_i : Y_i \to Z$, and a constant t_i such that $\| F(y_i) - L_i(y_i) \| \leq t_i \| y_i \|$ for all $y_i \in Y_i$. Define a linear map $L : Y_1 \oplus Y_2 \to Z$ by $L(y) = L_1(y_1) + L_2(y_2)$, for $y = y_1 + y_2 \in Y_1 \oplus Y_2$, then

$$\begin{aligned} \| F(y) - L(y) \| &\leq Z(F)(\| y_1 \| + \| y_2 \|) + t_1 \| y_1 \| + t_2 \| y_2 \| \\ &\leq (Z(F) + t_1 + t_2)(\| y_1 \| + \| y_2 \|) \\ &= (Z(F) + t_1 + t_2) \| y \| \end{aligned}$$

for all $y = y_1 + y_2 \in Y_1 \oplus Y_2$, which implies that F is trivial, a contradiction.

Conversely, suppose that $Ext(Y_1, Z) \neq 0$, and assume on the contrary that $Ext(Y_1 \oplus Y_2, Z) = 0$. Let $\pi : Y_1 \oplus Y_2 \to Y_1$ be the canonical projection, $i : Y_1 \to Y_1 \oplus Y_2$ be the natural injection, and let $F : Y_1 \to Z$ be a non-trivial zero-linear map, then the composition map $Y_1 \oplus Y_2 \xrightarrow{\pi} Y_1 \xrightarrow{F} Z$ is a trivial zero-linear map, since $Ext(Y_1 \oplus Y_2, Z) = 0$. Hence, there is a linear map $L : Y_1 \oplus Y_2 \to Z$ such that $\| F \circ \pi(y) - L(y) \| < c \| y \|$, for all $y = y_1 + y_2 \in Y_1 \oplus Y_2$ [3, Lemma 1]. Therefore

$$\| F(y) - (L \circ i)(y) \| = \| F(\pi \circ i(y) - L(i(y))) \| < c \| i(y) \| = c \| y \|,$$

for all $y_1 \in Y_1$, which implies that F is trivial, by [1, Theorem 16.2], and hence, $Ext(Y_1, Z) = 0$, a contradiction. $\square$

Remark. It is important to note that if $Ext(Y, Z) \neq 0$ does not imply that $Ext(A, Z) \neq 0$ for every complemented subspace A of Y. Indeed, consider the projective presentation $0 \to K \to \ell_1 \to L_1(0,1) \to 0$ of $L_1(0,1)$. It is easy to see that this sequence does not split, for otherwise $L_1(0, 1)$ is a complemented subspace of ℓ_1 which is impossible, since the space $L_1(0, 1)$ contains ℓ_2 [14, Remarks p. 72]. Therefore, $Ext(L_1(0,1), K) \neq 0$ which implies that $Ext(\ell_1^{**}, K) \neq 0$, by Proposition 2.3(ii), since $L_1(0, 1)$ is complemented in $\ell_\infty^* = \ell_1^{**}$ [1, Proposition F9], while $Ext(\ell_1, K) = 0$ by projectivity of ℓ_1 [6, p. 9].

Corollary 2.4. *Let Y and Z be Banach spaces such that Z is complemented in its bidual. If $Ext(Z^*, Y^*) = 0$, then $Ext(Y, Z) = 0$.*

Proof. It follows from Theorem 2.2 and Proposition 2.3 (i). □

Corollary 2.5. *Let Y and Z be two Banach spaces such that $Ext(Z^{**}, (Y^{**}/Y)^*) = 0$. If $Ext(Y^{**}, Z^*) \neq 0$, then $Ext(Z, Y^*) \neq 0$.*

Proof. Suppose that $Ext(Z^{**}, (Y^{**}/Y)^*) = 0$, then $Ext(Z, (Y^{**}/Y)^*) = 0$, by Theorem 2.1. The result is now immediate by Proposition 2.3, since $Y^{(3)} = Y^* \oplus (Y^{**}/Y)^*$. □

The converse of Corollary 2.4 is valid for certain Banach spaces as given in the following theorem. Recall that a Banach space X is said to satisfy *Grothendieck's theorem* (or is a *GT space*) if whenever $T : X \to \ell_2$ is a bounded linear operator, and (x_n) is an infinite sequence in X such that $\sum |f(x_n)| < \infty \; \forall f \in X^*$, then $\sum \| T(x_n) \| < \infty$ [15, Chapter 6].

Theorem 2.6. *Let Y and Z be Banach spaces such that Y is complemented in its bidual by an $\mathcal{L}_2$ space, and there is a linear surjective map q of an $\mathcal{L}_1$ space P onto Z such that $\ker q$ is a GT-space. Then, any exact sequence $0 \to Z^* \to W \to Y^* \to 0$ is a dual sequence. In particular, if $Ext(Z, Y) = 0$, then $Ext(Y^*, Z^*) = 0$.*

Proof. Let $Y^{**} = Y \oplus X$, where X is an $\mathcal{L}_2$ space, and let π_Y, π_X be the natural projections of Y^{**} onto Y, X, respectively. Since $q^{**} : P^{**} \to Z^{**}$ is a linear surjection of the $\mathcal{L}_1$ space P^{**} onto Z^{**}, and $\ker q^{**} = (\ker q)^{**}$ is a GT-space [15, Proposition 6.2], $Ext(Z^{**}, \ell_2) = 0$, by [11, Theorem 3.1]. Hence $Ext(Z^{**}, X) = 0$ by [4, Theorem 2]. If $0 \to Z^* \to W \to Y^* \to 0$ is a given exact sequence, let $F : Y^* \to Z^*$ be a zero-linear map that induces the sequence, then it is clear that the dual zero-linear map $F^* : Z^{**} \to Y^{**}$ can be written as $\pi_Y F^* + \pi_X F^*$. Since $Ext(Z^{**}, X) = 0$, $\pi_X F^* : Z^{**} \to X$ is a trivial zero-linear map, that is, $\pi_X F^*$ is a sum of a bounded and a linear map. Hence $\pi_Y F^* : Z^{**} \to Y$ is a version of F^* with its range contained in Y, which implies that the given sequence is a dual sequence [3, Theorem 4]. Therefore, there is a predual ${}_*W$ of W such that $0 \to Z^* \to W \to Y^* \to 0$ is the dual of the exact sequence $0 \to Y \to {}_* W \to Z \to 0$. Thus, if $Ext(Z, Y) = 0$, then ${}_*W \simeq Y \oplus Z$, which implies that $W \simeq Y^* \oplus Z^*$, and so $Ext(Y^*, Z^*) = 0$. $\square$

The *Johnson-Lindenstrauss space* JL is defined to be the completion of the linear span of $c_0 \cup \{\chi_i : i \in I\}$ in ℓ_∞ with respect to the norm:

$$\left\| y = x + \sum_{j=1}^{k} a_{i(j)}\chi_{i(j)} \right\|$$
$$= \max\{\| y \|_\infty, \|(a_i)_{i\in I}\|_{\ell_2(I)}\}, \quad x \in c_0, \quad a_{i(j)} \text{ are scalars,}$$

where χ_i is the characteristic function of A_i, $\{A_i\}_{i\in I}$ is an almost disjoint uncountable family of infinite subsets of $\mathbb{N}$. It is shown that JL gives a negative solution for the three space problem of "being weakly compactly generated", WCG for abbreviation, although c_0 and $\ell_2(I)$ are WCG spaces, while JL is not, there is an exact sequence $0 \to c_0 \to JL \to \ell_2(I) \to 0$ (see [6, Theorem 4.10.a]).

A Banach space X is called an $\mathcal{L}_p$ *space* if there exists a constant $\lambda > 1$, such that every finite dimensional subspace A of X is contained in a finite dimensional subspace B of X such that $d_{BM}(B, \ell_p^n) < \lambda$, where

$d_{BM}(B, E) = \inf\{\| T \| \| T^{-1} \|; T : X \to Y$ is an isomorphism of X onto $Y\}$ is the multiplicative Banach-Mazur distance and $n = \dim B$ (see [13, II.5.2]). It is known that $\mathcal{L}_p$ spaces generalizes the $L_p(\mu)$ spaces, $1 \leq p \leq \infty$, where $L_p(\mu)$ is the Banach space of equivalence classes of measurable functions on $(\Omega, \mathcal{B}, \mu)$ [1, Theorem F.2 (i)], and every infinite dimensional $\mathcal{L}_p$ space has a complemented subspace isomorphic to ℓ_p.

Example 2.7. Since ℓ_1 is projective [6, p. 8], the dual sequence $0 \to \ell_2(I) \to JL^* \to \ell_1 \to 0$ of the exact sequence $0 \to c_0 \to JL \to \ell_2(I) \to 0$ is trivial. Hence $JL^* = \ell_1 \oplus \ell_2(I)$, and so, $JL^{**} = c_0 \oplus \ell_2(I)$.

(i) Since $Ext(\ell_2, \ell_1) \neq 0$ [4, 4.1], and $\ell_2(I)$ is an $\mathcal{L}_p$ space [13, p. 326], we have $Ext(\ell_2(I), \ell_1) \neq 0$, by Proposition 2.3 (ii), which implies that $Ext(JL^*, \ell_1) \neq 0$, by Proposition 2.3 (ii), and so, $Ext(c_0, JL^{**}) \neq 0$, by Theorem 2.2. Also, we have $Ext(\ell_2, JL^*) \neq 0$, by Proposition 2.3 (i), which implies that $Ext(JL, \ell_2) \neq 0$, by Theorem 2.2.

(ii) Since $Ext(c_0, \ell_1) \neq 0$, [4, 4.3], we have $Ext(c_0, JL^*) \neq 0$, by Proposition 2.3 (i), which implies that $Ext(JL, \ell_1) \neq 0$, by Theorem 2.2.

(iii) Since $Ext(\ell_2, \ell_2) \neq 0$ [11, 4.7, 4.8], we have $Ext(\ell_2(I), \ell_1) \neq 0$, and so $Ext(JL^*, \ell_2) \neq 0$, by Proposition 2.3 (ii), which implies that $Ext(\ell_2, JL^{**}) \neq 0$, by Theorem 2.2.

The James Tree space $(JT, \| \cdot \|)$ is defined to be the completion of the space of finite sequences over the dyadic tree Δ with respect to the norm

$$\| x \| = \sup_{n \in N} \sup_{S_1, \dots, S_n} \left[\sum_{i=1}^{n} \left(\sum_{\alpha \in S_i} x_\alpha \right)^2 \right]^{1/2},$$

where the supremum is taken over all finite sets of pairwise disjoint segments of Δ. The space JT is an example of a separable dual space that does not contain ℓ_1 although it has a non separable dual JT^* [6, 4.14.e].

If B is the predual of the space JT, and Γ is the uncountable set of branches of Δ, then there is a non-trivial exact sequence $0 \to B \to JT^* \to \ell_2(\Gamma) \to 0$ such that its dual sequence $0 \to \ell_2(\Gamma) \to JT^{**} \to B^* \to 0$ is trivial, that is, $JT^{**} = \ell_2(\Gamma) \oplus B^* = \ell_2(\Gamma) \oplus JT$ [6, 4.14].

Example 2.8. (i) Since $Ext(\ell_2, \ell_2) \neq 0$ [11, 4.7, 4.8], we have $Ext(\ell_2, \ell_2(\Gamma)) \neq 0$, and so $Ext(\ell_2, JT^{**}) \neq 0$, by Proposition 2.3 (i), which implies that $Ext(JT^*, \ell_2) \neq 0$ by Theorem 2.2. Also, we have $Ext(\ell_2(\Gamma), \ell_2) \neq 0$, and so $Ext(JT^{**}, \ell_2) \neq 0$, by Proposition 2.3 (ii), which implies that $Ext(JT^*, \ell_2) \neq 0$ by Theorem 2.1.

(ii) Since $Ext(c_0, \ell_2) \neq 0$ [4, 4.3, Corollary 1], we have $Ext(c_0, \ell_2(\Gamma)) \neq 0$, and so $Ext(c_0, JT^{**}) \neq 0$, by Proposition 2.3 (i), which implies that $Ext(JT^*, \ell_1) \neq 0$ by Theorem 2.2.

(iii) Since $\ell_2(\Gamma)$ is a cotype 2 space [14, Corollary 3.6], we have $Ext(\ell_\infty, \ell_2(\Gamma)) \neq 0$ [4, Corollary 1], and hence $Ext(\ell_\infty, JT^{**}) \neq 0$, by Proposition 2.3 (i), which implies that $Ext(JT^*, \ell_\infty^*) \neq 0$.

Example 2.9. (i) If a Banach space Z satisfies $Ext(Z^{**}, \ell_2) = 0$, then any exact sequence $0 \to Z^* \to W \to JT^* \to 0$ is a dual sequence. In particular, if $Ext(Z, JT) = 0$, then $Ext(JT^*, Z^*) = 0$, by applying the argument of the proof of Theorem 2.6, and the fact that $JT^{**} = \ell_2(\Gamma) \oplus JT$.

(ii) Let A be an uncomplemented subspace A of ℓ_1 isomorphic to ℓ_1 [2] and consider the natural quotient map $q : \ell_1 \to \ell_1/A$. Then $\ker q = A$ is a GT-space, since it is an $\mathcal{L}_1$ space [15, Chapter 6]. Since $JT^{**} = JT \oplus \ell_2(\Gamma)$, and $\ell_2(\Gamma)$ is an $\mathcal{L}_2$ space, any exact sequence $0 \to (\ell_1/A)^* \to W \to JT^* \to 0$ is a dual sequence.

Example 2.10. Since $Ext(L_1(0,1), \ell_2) = 0,$ we have $Ext(\ell_2, (L_1(0,1))^*)$ $= 0,$ by Theorem 2.1. Since $L_1(0, 1)$ is separable, there is a separable Banach space Y with a quotient map ϕ from Y^* onto $L_1(0, 1)$ and such that $Y^{**} \cong Y \oplus (L_1(0,1))^*$ [12]. Therefore, any exact sequence $0 \to \ell_2 \to W$ $\to Y^* \to 0$ is a dual sequence, by Theorem 2.2.

3. Twisted Sums of Sequence Spaces

A map $f : X \to Y$ between normed spaces X and Y, is said to be *quasi-additive* if it satisfies the following properties:

(i) $\| f(x + z) - f(x) - f(z) \| \leq K(\| x \| + \| z \|),\ x,\ z \in X,$

(ii) $\lim_{t \to 0} f(tx) = 0,\ x \in X,$

(iii) $f(-x) = -f(x),\ x \in X.$

Quasi-additive maps defined on dense subspaces of sequence spaces give rise to quasi-linear maps on the sequence spaces, Kalton and Peck [11].

Let $\mathfrak{L}$ denote the class of Lipschitz functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = 0$ for $t \leq 0,$ and X be a *solid quasi-normed FK* – space, that is, X is a Frechet sequence space with continuous coordinates and satisfies the following properties:

(1) The space X_0 of finite sequences is dense in X.

(2) $\| e_n \| = 1,$ where e_n is the nth basis vector $(e_n(k) = \delta_{nk})$.

(3) If $s \in \ell_\infty$ and $x \in X,$ then $\| sx \|_X \leq \| s \|_{\ell_\infty} \| x \|_X.$

(4) $\| x \|_{\ell_\infty} \leq \| x \|_X$ for all $x \in X.$

Theorem 3.1. *Let X and Y be two sequence spaces with the above properties* (1) *to* (4) *and unit vector bases* $\{e_n\}$ *and* $\{y_n\}$, *respectively. If* $T : X \to Y$ *is an injective bounded linear map such that* $Te_i = y_j$, *and* $\phi \in \mathfrak{L}$, *then there is a quasi-linear map* $F_\phi : X \to Y$ *such that*

$$F_{\phi}(x) = \begin{cases} \| x \| f\left(\frac{x}{\| x \|}\right), & \textit{if } x \neq 0 \\ 0, & \textit{otherwise} \end{cases}$$

for all $x \in X_0$, *where* $f : X_0 \to Y_0$ *is a quasi-additive map defined by*

$$f(x)(k) = \begin{cases} T(x)(k)\phi(-\log | T(x)(k) |), & \textit{if } T(x)(k) \neq 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let L_{ϕ} be the Lipschitz constant of ϕ. Then

$$\begin{aligned} & | (t_1 + t_2)\phi(-\log| t_1 + t_2 |) - t_1\phi(-\log| t_1 |) - t_2\phi(-\log| t_2 |) | \\ & \leq L_{\phi}(\log 2) | t_1 + t_2 | \end{aligned}$$

for all $t_1, t_2 \in \mathbb{R}$ [11, Theorem 3.7 (i)]. Define $f : X_0 \to Y_0$ by

$$f(x)(k) = \begin{cases} T(x)(k)\phi(-\log | T(x)(k) |), & \text{if } T(x)(k) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then for all $x, z \in X_0$

$$\begin{aligned} & | f(x + z)(k) - f(x)(k) - f(z)(k) | \\ & = | T(x + z)(k)\phi(-\log | T(x)(k) + T(z)(k) |) - T(x)(k)\phi(-\log | T(x)(k) |) \\ & \quad -T(z)(k)\phi(\log | T(z)(k) |)| \\ & \leq L_{\phi}(\log 2) | T(x)(k) + T(z)(k) |, \end{aligned}$$

so that

$$\begin{aligned} \| f(x + z) - f(x) - f(z) \|_Y & \leq L_{\phi}(\log 2)(\| T(x) \| + \| T(z) \|) \\ & \leq L_{\phi} \| T \| (\log 2)(\| x \| + \| z \|). \end{aligned}$$

It is easy to see that $\lim_{t \to 0} f(tx)(k) = 0$ for every $k \in \mathbb{N}$ since

$$| f(tx)(k) | \leq L_{\phi} \frac{\left| \log \frac{1}{tT(x)(k)} \right|}{\left| \frac{1}{tT(x)(k)} \right|}$$

which implies that $\lim_{t \to 0} f(tx) = 0$.

Since $f(-x) = -f(x)$ for all $x \in X_0$, that is, f is quasi additive. Now put

$$F_\phi(x) = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By [11, Theorem 3.5], F_ϕ is a quasi-linear map on X_0, which extends to a quasi-linear map $F_\phi : X \to Y$ [11, Theorem 3.1], proving the theorem.

The following theorem is proved in [11, Theorem 4.2] when $X = Y$. However, given the foregoing, inspection of the proof shows it to be valid more generally:

Theorem 3.2. *Let* X *and* Y *be as in Theorem* 2.1 *and let* $T : X \to Y$ *be an injective bounded linear map satisfying*:

(1) $Te_i = y_j$, *where* $j \geq i$.

(2) $\|Tx\| \geq \alpha \|x\|$, *for some* $\alpha > 0$ *and for all*

$$x \in \left\{ \sum_{i=n_0}^{n} \theta_i e_i : \theta_i = \pm 1, 0, \text{ for all } n \geq n_0 \right\}$$

for some $n_0 \in N$.

Suppose that no subsequence of the canonical basis $\{e_n\}$ *in* X *is equivalent to the canonical basis of* c_0. *Then*:

(i) *for any* $\phi, \psi \in \mathfrak{L}$, *the two twisted sums* $Y \oplus_{F_\phi} X$ *and* $Y \oplus_{G_\psi} X$ *are equivalent if and only if*

$$\sup_{0 < t < \infty} |\phi(t) - \psi(t)| < \infty,$$

(ii) *for any* $\phi \in \mathfrak{L}$, $Y \oplus_{F_\phi} X$ *is trivial if and only if* ϕ *is bounded.*

The Schreier Space S_p, $1 \leq p < \infty$, is the completion of the space of finite sequences with respect to the following norm:

$$\|x\|_{S_p} = \sup_A \left(\sum_{j \in A} |x_j|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all "admissible" subsets $A = \{n_1, n_2, \ldots, n_k\}$ of $\mathbb{N}$ such that $n_1 < n_2 < \cdots < n_k$ and $k \leq n_1$. Note that ℓ_p is algebraically contained in S_p, since $\| x \|_{S_p} \leq \| x \|_{\ell_p}$.

Example 3.3. Let $1 < p < \infty$, and let $T : \ell_p \to S_p$ be the identity map. Then T is a bounded injective linear map. For any finitely many $e_{i_1}, e_{i_2}, \ldots, e_{i_N}$, we have $\left\| \sum_{j=1}^{N} e_{i_j} \right\|_{S_p} \geq N/2$, and hence

$$\left(\left\| \sum_{j=1}^{N} e_{i_j} \right\|_{S_p} \right)^p = \left\| \sum_{j=1}^{N} e_{i_j} \right\|_{S_p} \geq N/2,$$

which implies that

$$\left\| \sum_{j=1}^{N} e_{i_j} \right\|_{S_p} \geq (N/2)^{1/p} = \left(\frac{1}{2} \right)^{1/p} \left\| \sum_{j=1}^{N} e_{i_j} \right\|_{\ell_p}.$$

Therefore

$$\| Tx \|_{S_p} \geq \left(\frac{1}{2} \right)^{1/p} \| x \|_{\ell_p},$$

for all $x \in \left\{ \sum_{i=1}^{n} \theta_i e_i : \theta_i = 0, \pm 1, n \in N \right\}$.

Since $(n^{-\frac{1}{p}}) \in S_p$ and $(n^{-\frac{1}{p}}) \notin \ell_p$, T is not an isomorphism onto S_p. So using any unbounded Lipschitz function, e.g., $\phi(t) = t$ for $t > 0$ and 0 otherwise, T induces a non-trivial twisted sum $S_p \oplus_\phi \ell_p$, where the non-trivial quasi-linear map $F : \{e_n\}_{\ell_p} \to \{e_n\}_{S_p}$ is given by

$$F(x) = \begin{cases} \| x \| f\left(\dfrac{x}{\| x \|} \right), & \text{if } x \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$f(x)(k) = \begin{cases} -x(k)\log|x(k)|, & \text{if } x(k) \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $x \in \{e_n\}_{\ell_p}$.

Recall that a Banach space property is said to be a *three space property* if whenever it is satisfied by a closed subspace Y of a Banach space X and the corresponding quotient X/Y, then it is satisfied by X. It has been proved in [5] that $Ext(Y, Z) = 0$ is a three space property for Y. The following theorem shows that $Ext(Y, Z) = 0$ is a three space property for Z.

Theorem 3.4. *Let Z and Y be Banach spaces and let E be a closed subspace of Z such that $Ext(Y, E) = 0$ and $Ext(Y, Z/E) = 0$. Then $Ext(Y, Z) = 0$.*

Proof. Let $0 \to Z \to \ell_\infty(I) \xrightarrow{q_Z} \ell_\infty(I)/Z \to 0$ be an injective presentation of Z, and let $T : Y \to \ell_\infty(I)/Z$ be a bounded linear operator. Consider the natural isomorphism $\eta : (\ell_\infty(I)/E)/(Z/E) \to \ell_\infty(I)/Z$, then $\eta^{-1}T : Y \to (\ell_\infty(I)/E)/(Z/E)$ is a bounded linear map, and so, there is a bounded linear operator $\gamma : Y \to \ell_\infty(I)/E$, by [10, Theorem 3.1], since $Ext(Y, Z/E) = 0$. Consequently, there is a bounded linear operator $\tilde{\gamma} : Y \to \ell_\infty(I)$, since $Ext(Y, E) = 0$ [10, Theorem 3.1]. Hence, we have the following commutative diagram

$$\begin{tikzcd}
(\ell_\infty(I)/E)/(Z/E) \arrow[r, "\eta"] & \ell_\infty(I)/Z \\
\ell_\infty(I)/E \arrow[u, "q"] & Y \arrow[l, "\gamma"'] \arrow[u, "T"'] \arrow[dl, "\tilde{\gamma}"] \\
\ell_\infty(I) \arrow[u, "p"] &
\end{tikzcd},$$

where q and p are the natural quotient maps. Clearly $\eta qp : \ell_\infty(I) \to \ell_\infty(I)/Z$ is the natural quotient map q_Z, and $T = q_Z\widetilde{\gamma}$. Therefore $Ext(Y, Z) = 0$ by [10, Theorem 3.1]. □

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