Chapter 5: Some Discrete Probability Distributions:

5.2: Discrete Uniform Distribution:
If the discrete random variable $X$ assumes the values $x_1, x_2, \ldots, x_k$ with equal probabilities, then $X$ has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x; k) = \begin{cases} \frac{1}{k}; & x = x_1, x_2, \ldots, x_k \\ 0; & elsewhere \end{cases}$$

Note:
- $f(x)=f(x;k)=P(X=x)$
- $k$ is called the parameter of the distribution.

Example 5.2:
- Experiment: tossing a balanced die.
- Sample space: $S=\{1,2,3,4,5,6\}$
- Each sample point of $S$ occurs with the same probability $1/6$.
- Let $X=\text{the number observed when tossing a balanced die}$.
- The probability distribution of $X$ is:

$$f(x) = P(X = x) = f(x; 6) = \begin{cases} \frac{1}{6}; & x = 1,2,\ldots, 6 \\ 0; & elsewhere \end{cases}$$

Theorem 5.1:
If the discrete random variable $X$ has a discrete uniform distribution with parameter $k$, then the mean and the variance of $X$ are:

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k}$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Example 5.3:
Find $E(X)$ and $Var(X)$ in Example 5.2.

Solution:
E(X) = \mu = \frac{\sum x_i}{k} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5

Var(X) = \sigma^2 = \frac{\sum (x_i - \mu)^2}{k} = \frac{\sum (x_i - 3.5)^2}{6}
= \frac{(1 - 3.5)^2 + (2 - 3.5)^2 + \cdots + (6 - 3.5)^2}{6} = \frac{35}{12}

5.3 Binomial Distribution:

Bernoulli Trial:
- Bernoulli trial is an experiment with only two possible outcomes.
- The two possible outcomes are labeled: success (s) and failure (f)
- The probability of success is P(s) = p and the probability of failure is P(f) = q = 1 - p.
- Examples:
  1. Tossing a coin (success=H, failure=T, and p=P(H))
  2. Inspecting an item (success=defective, failure=non-defective, and p=P(defective))

Bernoulli Process:
Bernoulli process is an experiment that must satisfy the following properties:
1. The experiment consists of n repeated Bernoulli trials.
2. The probability of success, P(s) = p, remains constant from trial to trial.
3. The repeated trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial

Binomial Random Variable:
Consider the random variable:
X = The number of successes in the n trials in a Bernoulli process
The random variable $X$ has a binomial distribution with parameters $n$ (number of trials) and $p$ (probability of success), and we write:

$$X \sim \text{Binomial}(n,p) \text{ or } X \sim \text{b}(x; n, p)$$

The probability distribution of $X$ is given by:

$$f(x) = P(X = x) = b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & ; x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

We can write the probability distribution of $X$ as a table as follows.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = P(X = x) = b(x; n, p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$</td>
</tr>
<tr>
<td>1</td>
<td>$\binom{n}{1} p^1 (1-p)^{n-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$\binom{n}{2} p^2 (1-p)^{n-2}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$n-1$</td>
<td>$\binom{n}{n-1} p^{n-1} (1-p)^{1}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\binom{n}{n} p^n (1-p)^0 = p^n$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1.00</strong></td>
</tr>
</tbody>
</table>
Example:
Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, inspected, and classified as defective (D) or non-defective (N). Find the probability distribution of the number of defective items.

Solution:
- Experiment: selecting 3 items at random, inspected, and classified as (D) or (N).
- The sample space is
  \[ S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\} \]
- Let \( X \) be the number of defective items in the sample.
- We need to find the probability distribution of \( X \).

(1) First Solution:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NNN</td>
<td>[ \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64} ]</td>
<td>0</td>
</tr>
<tr>
<td>NND</td>
<td>[ \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{64} ]</td>
<td>1</td>
</tr>
<tr>
<td>NDN</td>
<td>[ \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{64} ]</td>
<td>1</td>
</tr>
<tr>
<td>NDD</td>
<td>[ \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{64} ]</td>
<td>2</td>
</tr>
<tr>
<td>DNN</td>
<td>[ \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{64} ]</td>
<td>1</td>
</tr>
<tr>
<td>DND</td>
<td>[ \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{64} ]</td>
<td>2</td>
</tr>
<tr>
<td>DDN</td>
<td>[ \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{64} ]</td>
<td>2</td>
</tr>
<tr>
<td>DDD</td>
<td>[ \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64} ]</td>
<td>3</td>
</tr>
</tbody>
</table>

The probability distribution of \( X \) is

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = P(X = x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[ \frac{27}{64} ]</td>
</tr>
<tr>
<td>1</td>
<td>[ \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64} ]</td>
</tr>
<tr>
<td>2</td>
<td>[ \frac{3}{64} + \frac{3}{64} + \frac{3}{64} = \frac{9}{64} ]</td>
</tr>
<tr>
<td>3</td>
<td>[ \frac{1}{64} ]</td>
</tr>
</tbody>
</table>

(2) Second Solution:

Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success \( P(s) = 25/100 = 1/4 = 0.25 \).

The experiments is a Bernoulli process with:
- number of trials: \( n = 3 \)
- Probability of success: \( p = 1/4 = 0.25 \)
- \( X \sim \text{Binomial}(n,p) = \text{Binomial}(3, 1/4) \)
The probability distribution of $X$ is given by:

$$f(x) = P(X = x) = b(x; 3, \frac{1}{4}) = \begin{cases} \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x}; & x = 0, 1, 2, 3 \\ 0; & \text{otherwise} \end{cases}$$

The probability distribution of $X$ is

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = P(X = x) = b(x; 3, \frac{1}{4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{27}{64}$</td>
</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>$\frac{9}{64}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{64}$</td>
</tr>
</tbody>
</table>

Theorem 5.2:
The mean and the variance of the binomial distribution $b(x; n, p)$ are:

$$\mu = np$$
$$\sigma^2 = np(1-p)$$

Example:
In the previous example, find the expected value (mean) and the variance of the number of defective items.

Solution:
• $X$ = number of defective items
• We need to find $E(X)=\mu$ and $Var(X)=\sigma^2$
• We found that $X \sim \text{Binomial}(n,p) = \text{Binomial}(3,1/4)$
• $n=3$ and $p=1/4$

The expected number of defective items is

\[
E(X)=\mu = n \cdot p = (3) \cdot (1/4) = 3/4 = 0.75
\]

The variance of the number of defective items is

\[
Var(X)=\sigma^2 = n \cdot p \cdot (1-p) = (3) \cdot (1/4) \cdot (3/4) = 9/16 = 0.5625
\]

**Example:**
In the previous example, find the following probabilities:
1. The probability of getting at least two defective items.
2. The probability of getting at most two defective items.

**Solution:**

$X \sim \text{Binomial}(3,1/4)$

\[
f(x) = P(X=x) = b(x;3,1/4) = \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x} \text{ for } x = 0, 1, 2, 3
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)=P(X=x)$=b(x;3,1/4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>27/64</td>
</tr>
<tr>
<td>1</td>
<td>27/64</td>
</tr>
<tr>
<td>2</td>
<td>9/64</td>
</tr>
<tr>
<td>3</td>
<td>1/64</td>
</tr>
</tbody>
</table>

(1) The probability of getting at least two defective items:

\[
P(X \geq 2) = P(X=2)+P(X=3) = f(2)+f(3) = \frac{9}{64} + \frac{1}{64} = \frac{10}{64} = \frac{5}{32}
\]

(2) The probability of getting at most two defective items:

\[
P(X \leq 2) = P(X=0)+P(X=1)+P(X=2)
\]

\[
= f(0)+f(1)+f(2) = \frac{27}{64} + \frac{27}{64} + \frac{9}{64} = \frac{63}{64}
\]

or

\[
P(X \leq 2) = 1-P(X>2) = 1-P(X=3) = 1- f(3) = 1- \frac{1}{64} = \frac{63}{64}
\]

**Example 5.4:** Reading assignment

**Example 5.5:** Reading assignment

**Example 5.6:** Reading assignment
5.4 Hypergeometric Distribution:

- Suppose there is a population with 2 types of elements:
  1-st Type = success
  2-nd Type = failure
- \( N \) = population size
- \( K \) = number of elements of the 1-st type
- \( N - K \) = number of elements of the 2-nd type
- We select a sample of \( n \) elements at random from the population
- Let \( X \) = number of elements of 1-st type (number of successes) in the sample
- We need to find the probability distribution of \( X \).

There are two methods of selection:
1. selection with replacement
2. selection without replacement

(1) If we select the elements of the sample at random and with replacement, then

\[ X \sim \text{Binomial}(n,p); \quad \text{where} \quad p = \frac{K}{N} \]

(2) Now, suppose we select the elements of the sample at random and without replacement. When the selection is made without replacement, the random variable \( X \) has a hypergeometric distribution with parameters \( N, n, \) and \( K \). and we write \( X \sim \text{h}(x;N,n,K) \).

The probability distribution of \( X \) is given by:
\[ f(x) = P(X = x) = h(x;N,n,K) \]
\[
= \begin{cases} 
\binom{K}{x} \times \binom{N-K}{n-x} \times \binom{N}{n}^{-1}; & x = 0, 1, 2, \ldots, n \\
0; & \text{otherwise}
\end{cases}
\]

Note that the values of \(X\) must satisfy:

\[0 \leq x \leq K\] and \[0 \leq n-x \leq N-K\]  
\[\iff \quad 0 \leq x \leq K\] and \[n-N+K \leq x \leq n\]

**Example 5.8:** Reading assignment  

**Example 5.9:**  
Lots of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the lot is to select 5 components at random (without replacement) and to reject the lot if a defective is found. What is the probability that exactly one defective is found in the sample if there are 3 defectives in the entire lot.  

**Solution:**  
- Let \(X=\) number of defectives in the sample  
- \(N=40, K=3,\) and \(n=5\)  
- \(X\) has a hypergeometric distribution with parameters \(N=40, n=5,\) and \(K=3.\)  
- \(X \sim h(x;N,n,K)=h(x;40,5,3).\)  
- The probability distribution of \(X\) is given by:
\[ f(x) = P(X = x) = h(x; 40, 5, 3) = \begin{cases} \binom{3}{x} \times \frac{37}{5-x} \cdot \frac{40}{5} ; & x = 0, 1, 2, \ldots, 5 \\ 0; & \text{otherwise} \end{cases} \]

But the values of X must satisfy:
\[ 0 \leq x \leq K \quad \text{and} \quad n-K \leq x \leq n \quad \Leftrightarrow \quad 0 \leq x \leq 3 \quad \text{and} \quad -32 \leq x \leq 5 \]

Therefore, the probability distribution of \( X \) is given by:
\[ f(x) = P(X = x) = h(x; 40, 5, 3) = \begin{cases} \binom{3}{x} \times \frac{37}{5-x} \cdot \frac{40}{5} ; & x = 0, 1, 2, 3 \\ 0; & \text{otherwise} \end{cases} \]

Now, the probability that exactly one defective is found in the sample is
\[ f(1) = P(X=1) = h(1; 40, 5, 3) = \frac{\binom{3}{1} \times \frac{37}{4} \times \frac{40}{5}}{\binom{40}{5}} = \frac{\binom{3}{1} \times \frac{37}{4} \times \frac{40}{5}}{\binom{40}{5}} = 0.3011 \]

**Theorem 5.3:**
The mean and the variance of the hypergeometric distribution \( h(x; N, n, K) \) are:
\[ \mu = n \frac{K}{N} \]
\[ \sigma^2 = n \frac{K}{N} \left( 1 - \frac{K}{n} \right) \frac{N-n}{N-1} \]

**Example 5.10:**
In Example 5.9, find the expected value (mean) and the variance of the number of defectives in the sample.

**Solution:**
- \( X = \) number of defectives in the sample
- We need to find \( E(X) = \mu \) and \( Var(X) = \sigma^2 \)
- We found that \( X \sim h(x; 40, 5, 3) \)
- \( N=40, \ n=5, \) and \( K=3 \)

The expected number of defective items is
\[ \text{E}(X) = \mu = n \frac{K}{N} = 5 \times \frac{3}{40} = 0.375 \]

The variance of the number of defective items is
\[ \text{Var}(X) = \sigma^2 = n \frac{K}{N} \left(1 - \frac{K}{N}\right) \frac{N - n}{N - 1} = 5 \times \frac{3}{40} \left(1 - \frac{3}{40}\right) \frac{40 - 5}{40 - 1} = 0.311298 \]

**Relationship to the binomial distribution:**

* Binomial distribution: \( b(x; n, p) = \binom{n}{x} p^x (1 - p)^{n-x}; x = 0, 1, \ldots, n \)

* Hypergeometric distribution: \( h(x; N, n, K) = \binom{N}{n} \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}} ; x = 0, 1, \ldots, n \)

If \( n \) is small compared to \( N \) and \( K \), then the hypergeometric distribution \( h(x; N, n, K) \) can be approximated by the binomial distribution \( b(x; n, p) \), where \( p = \frac{K}{N} \); i.e., for large \( N \) and \( K \) and small \( n \), we have:
\[ h(x; N, n, K) \approx b(x; n, \frac{K}{N}) \]
\[ \binom{K}{x} \frac{\binom{N-K}{n-x}}{\binom{N}{n}} \approx \left(\frac{n}{x}\right) \left(\frac{K}{N}\right)^x \left(1 - \frac{K}{N}\right)^{n-x} ; x = 0, 1, \ldots, n \]

Note:

If \( n \) is small compared to \( N \) and \( K \), then there will be almost no difference between selection without replacement and selection with replacement (\( \frac{K}{N} \approx \frac{K-1}{N-1} \approx \cdots \approx \frac{K-n+1}{N-n+1} \)).
Example 5.11:
N=5000  K=1000  n=10

X=Number of blemished tires in the Sample

X~h(x;5000,10,1000)

The exact probability is

\[ P(X=3) = \frac{\binom{1000}{3} \binom{4000}{7}}{\binom{5000}{10}} \]

\[ = 0.20147715 \]

\[ \approx 0.201 \]

Since \( n=10 \) is small relative to \( N=5000 \) and \( K=4000 \), we can approximate the hypergeometric probabilities using binomial probabilities as follows:

\[ n=10 \quad \text{(no. of trials)} \]
\[ p=K/N=1000/5000=0.2 \quad \text{(probability of success)} \]

\[ X \sim h(x;5000,10,1000) \approx b(x;10,0.2) \]

\[ P(X=3) \approx \binom{10}{3} (0.2)^3 (0.8)^7 = 0.201326592 \]

\[ \approx 0.201 \]
5.6 Poisson Distribution:

- Poisson experiment is an experiment yielding numerical values of a random variable that count the number of outcomes occurring in a given time interval or a specified region denoted by \( t \).

  \( X \) = The number of outcomes occurring in a given time interval or a specified region denoted by \( t \).

- Example:
  1. \( X = \) number of field mice per acre \((t = 1 \text{ acre})\)
  2. \( X = \) number of typing errors per page \((t = 1 \text{ page})\)
  3. \( X = \) number of telephone calls received every day \((t = 1 \text{ day})\)
  4. \( X = \) number of telephone calls received every 5 days \((t = 5 \text{ days})\)

- Let \( \lambda \) be the average (mean) number of outcomes per unit time or unit region \((t = 1)\).

- The average (mean) number of outcomes (mean of \( X \)) in the time interval or region \( t \) is:
  \[
  \mu = \lambda t
  \]

- The random variable \( X \) is called a Poisson random variable with parameter \( \mu \) \((\mu = \lambda t)\), and we write \( X \sim \text{Poisson}(\mu) \), if its probability distribution is given by:

  \[
  f(x) = P(X = x) = p(x; \mu) = \begin{cases} 
  \frac{e^{-\mu} \mu^x}{x!} ; & x = 0, 1, 2, 3, \ldots \\
  0 ; & \text{otherwise}
  \end{cases}
  \]

Theorem 5.5:
The mean and the variance of the Poisson distribution \( \text{Poisson}(x; \mu) \) are:

\[
\begin{align*}
\mu &= \lambda t \\
\sigma^2 &= \mu = \lambda t
\end{align*}
\]

Note:
- \( \lambda \) is the average (mean) of the distribution in the unit time \((t = 1)\).
- If \( X = \) The number of calls received in a month (unit time \( t = 1 \text{ month} \)) and \( X \sim \text{Poisson}(\lambda) \), then:
(i) Y = number of calls received in a year.
\[ Y \sim \text{Poisson} (\mu); \ \mu = 12\lambda \quad (t=12) \]
(ii) W = number of calls received in a day.
\[ W \sim \text{Poisson} (\mu); \ \mu = \lambda/30 \quad (t=1/30) \]

**Example 5.16:** Reading Assignment

**Example 5.17:** Reading Assignment

**Example:**
Suppose that the number of typing errors per page has a Poisson distribution with average 6 typing errors.

1. What is the probability that in a given page:
   (i) The number of typing errors will be 7?
   (ii) The number of typing errors will be at least 2?

2. What is the probability that in 2 pages there will be 10 typing errors?

3. What is the probability that in a half page there will be no typing errors?

**Solution:**

1. X = number of typing errors per page.
\[ X \sim \text{Poisson} (6) \quad (t=1, \ \lambda = 6, \ \mu = \lambda t = 6) \]
\[ f(x) = P(X = x) = p(x;6) = \frac{e^{-6}6^x}{x!}; \ x = 0, 1, 2, \ldots \]
   (i) \[ f(7) = P(X = 7) = p(7;6) = \frac{e^{-6}6^7}{7!} = 0.13768 \]
   (ii) \[ P(X \geq 2) = P(X=2)+P(X=3)+\ldots = \sum_{x=2}^{\infty} P(X = x) \]
   \[ = 1 - [P(X=0)+P(X=1)] = 1 - [f(0) + f(1)] = 1 - \left[ \frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} \right] \]
   \[ = 1 - [0.00248 + 0.01487] = 0.982650 \]

2. X = number of typing errors in 2 pages
\[ X \sim \text{Poisson}(12) \quad (t=2, \ \lambda = 6, \ \mu = \lambda t = 12) \]
\[ f(x) = P(X = x) = p(x;12) = \frac{e^{-12}12^x}{x!}; \ x = 0, 1, 2, \ldots \]
\[ f(10) = P(X = 10) = \frac{e^{-12}12^{10}}{10!} = 0.1048 \]
(3) $X = \text{number of typing errors in a half page.}$

$X \sim \text{Poisson (3)} \quad (t=1/2, \lambda=6, \mu=\lambda t=6/2=3)$

$f(x) = P(X = x) = p(x;3) = \frac{e^{-3} 3^x}{x!}: \quad x = 0, 1, 2…$

$P(X = 0) = \frac{e^{-3}(3)^0}{0!} = 0.0497871$

**Theorem 5.6: (Poisson approximation for binomial distribution):**

Let $X$ be a binomial random variable with probability distribution $b(x; n, p)$. If $n \to \infty$, $p \to 0$, and $\mu = np$ remains constant, then the binomial distribution $b(x; n, p)$ can approximated by Poisson distribution $p(x; \mu)$.

- For large $n$ and small $p$ we have:

\[
\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{e^{-\mu} \mu^x}{x!}; \quad x = 0, 1, \cdots, n; \quad (\mu = np)
\]

**Example 5.18:**

$X = \text{number of items producing bubbles in a random sample of 8000 items}$

$n=8000$ and $p=1/1000 = 0.001$

$X \sim b(x;8000, 0.001)$

The exact probability is:

$P(X<7)=P(X\leq 6)=\sum_{x=0}^{6} \left( \frac{8000}{x} \right) (0.001)^x (0.999)^{8000-x} = \ldots = 0.313252$

The approximated probability using Poisson approximation:

- $n=8000$ (n is large, i.e., $n \to \infty$)
- $p=0.001$ (p is small, i.e. $p \to 0$)
- $\mu= np = 8000(0.001)=8$

$X \approx \text{Poisson (8)}$

$f(x) = P(X = x) = p(x;8) = \frac{e^{-8} 8^x}{x!}: \quad x = 0, 1, 2…$

$P(X<7)=P(X\leq 6)=\sum_{x=0}^{6} \frac{e^{-8} 8^x}{x!} = e^{-8} \sum_{x=0}^{6} \frac{8^x}{x!} = \ldots = 0.313374$