1

poisson processes

* pb 5.1.3 p.228

$X \sim \text{poisson}(\alpha)$, $Y \sim \text{poisson}(\beta)$

المطلوب

$$pr\{X=k \mid N=n\}$$
$$= pr\{X=k \mid X+Y=n\}$$

Ans: $X \sim \text{poisson}(\alpha)$, $Y \sim \text{poisson}(\beta)$

$\therefore X+Y \sim \text{poisson}(\alpha+\beta)$

$$pr\{X=k \mid N=n\} = pr\{X=k \mid X+Y=n\}$$
$$= \frac{pr\{X=k\} \cap pr\{X+Y=n\}}{pr\{X+Y=n\}}$$
$$= \frac{pr\{X=k\} \cap pr\{Y=n-k\}}{pr\{X+Y=n\}}$$
$$= \frac{\cancel{e^{-\alpha}} \alpha^k / k! \cdot \cancel{e^{-\beta}} \beta^{n-k} / (n-k)!}{\cancel{e^{-(\alpha+\beta)}} (\alpha+\beta)^n / n!}$$
$$= \alpha^k \beta^{n-k} \left(\frac{1}{\alpha+\beta}\right)^n \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \left(\frac{\beta}{\alpha+\beta}\right)^{n-k}$$

$\therefore pr\{X=k \mid N=n\} = \binom{n}{k} p^k (1-p)^{n-k}$, $p = \dfrac{\alpha}{\alpha+\beta}$ which is Binomial distn

Note that $1-p = 1 - \dfrac{\alpha}{\alpha+\beta} = \dfrac{\beta}{\alpha+\beta}$

pb 5.1.2 p. 228

Given $P_k = Pr\{X = k\}$ for poisson (λ)

verify
$$\begin{cases} P_0 = e^{-\lambda} \\ P_k = \left(\frac{\lambda}{k}\right) P_{k-1} \end{cases}$$

علاقة تكرارية
recurrence relation

Ans: $\because P_k = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$

$\therefore P_0 = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$, $k = 0$

and $\because P_k = \frac{\lambda}{k} \cdot \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}$

$\therefore P_k = \left(\frac{\lambda}{k}\right) P_{k-1}$

#

* pb 5.1.7 p.229

$X(t)$ is the # of customers (arrivals), $\lambda = 2$

$X(0)=0$ — 2 customers — $X(1)=2$

$t=0$ — $t=1$

$\lambda t = 2(1) = 2$

a) $pr\{X(1)=2\}$

$= pr\{X(1)-X(0)=2\}$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{2^2 e^{-2}}{2!} = 2e^{-2}$$

b) $pr\{X(1)=2 \text{ and } X(3)=6\}$

$X(0)=0$ — 2 Cust. — $X(1)=2$ — 4 Cust. — $X(3)=6$

$t=0$ — 1 — 3

$\lambda t = 2(1) = 2$ $\quad$ $\lambda t = 2(2) = 4$

$$= pr\{\underbrace{X(1)-X(0)}=\underset{k}{2}, \underbrace{X(3)-X(1)}=\underset{k}{4}\}$$

indep. r.v.s

$$= \frac{2^2 e^{-2}}{2!} \cdot \frac{4^4 e^{-4}}{4!}$$

$$= 2e^{-2} \cdot \frac{64}{6} e^{-4} = \frac{64}{3} e^{-6}$$

$pr(x,y)$
$= pr(x)\, pr(y)$
iff X and Y are indep. r.v.s

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d) $pr\{X(3)=6 \mid X(1)=2\}$

$$= pr\{X(3)-X(1)=4 \mid X(1)-X(0)=2\}$$

$X(0)=0$ — 2 cust. — $X(1)=2$ — 4 cust. — $X(3)=6$

$t=0$ $\quad$ $t=1$ $\quad$ $t=3$

$\lambda t = 2(1) = 2$ $\qquad$ $\lambda t = 2(2) = 4$

$$= pr\{X(3)-X(1)=4\}$$

$$= \frac{4^4 e^{-4}}{4!}$$

$$= \frac{64}{6} e^{-4} = \frac{32}{3} e^{-4}$$

$X(3)-X(1)$ and $X(1)-X(0)$ are 2 indep. r.v.s

Note that $pr(A|B) = pr(A)$ where A and B are independent events.

c) $pr\{X(1)=2 \mid X(3)=6\}$

كأنما التجربة أجريت 6 مرات وكان احتمال وصول الزبائن عند $t=1$ هو

$$P = \frac{2}{6} = \frac{1}{3}$$

واحتمال عدم الوصول هو

$$q = \frac{4}{6} = \frac{2}{3}$$

$X(0)=0$ $\quad$ $X(1)=2$ $\quad$ $X(3)=6$

$t=0$ $\quad$ $t=1$ $\quad$ $t=3$

2 customers

6 customers

$X \sim Bin(6, \frac{1}{3})$ (n, p)

$$\Rightarrow X \sim Bin\left(6, \tfrac{1}{3}\right)$$

$$\therefore pr\{X(1)=2 \mid X(3)=6\} = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$$

$$= \frac{80}{243} \simeq 0.3292$$

* حل آخر للفقرة (c)

$$pr\{X(1)=2 \mid X(3)=6\}$$

$$= \frac{pr\{X(1)=2 \text{ and } X(3)=6\}}{pr\{X(3)=6\}}$$

$$= \frac{(2e^{-2}/2!)\cdot(4^4 e^{-4}/4!)}{6^6 e^{-6}/6!}$$

$$= \frac{64/3 \; \cancel{e^{-6}}}{64.8 \; \cancel{e^{-6}}}$$

$$\simeq 0.3292$$

#

6

pb 5.1.9 p. 229

For the poisson process $\{X(t);\ t \geq 0\}$

$E[X(t)] = \lambda t$ and $Var[X(t)] = \lambda t$

a) $E[X(1)] = 2(1) = 2$, $\lambda = 2$

$E[X(2) = 2(2) = 4$ #

b) To get $E[\{X(1)\}^2]$

$Var[X(t)] = E[\{X(t)\}^2] - [E\{X(t)\}]^2$

$\therefore Var[X(1)] = E[\{X(1)\}^2] - [E\{X(1)\}]^2$

$= E[\{X(1)\}^2] - 2^2$

$\therefore 2$

$\therefore E[\{X(1)\}^2] = 2 + 4 = 6$ #

c) $E[X(1)\ X(2)]$

$= E[X(1)]\ E[X(2)]$, $X(1)$ and $X(2)$ are indep. r.vs

$= 2(4) = 8$ #