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Poisson Process

* pb 5.1.3 p. 228

$X \sim \text{Poisson}(\alpha), Y \sim \text{Poisson}(\beta)$

المطلوب

$$\Pr\{X=k | N=n\}$$

$$= \Pr\{X=k | X+Y=n\}$$

Ans: $X \sim \text{Poisson}(\alpha), Y \sim \text{Poisson}(\beta)$
 $\therefore X+Y \sim \text{Poisson}(\alpha+\beta)$

$$\Pr\{X=k | N=n\} = \Pr\{X=k | X+Y=n\}$$

$$= \frac{\Pr\{X=k\} \cap \Pr\{X+Y=n\}}{\Pr\{X+Y=n\}}$$

$$= \frac{\Pr\{X=k\} \cap \Pr\{Y=n-k\}}{\Pr\{X+Y=n\}}$$

$$= \frac{e^{-\alpha} \alpha^k / k! \cdot e^{-\beta} \beta^{n-k} / (n-k)!}{e^{-(\alpha+\beta)} (\alpha+\beta)^n / n!}$$

$$= \alpha^k \beta^{n-k} \left(\frac{1}{\alpha+\beta}\right)^n \frac{n!}{k! (n-k)!}$$

$$= \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \left(\frac{\beta}{\alpha+\beta}\right)^{n-k}$$

$\therefore \Pr\{X=k | N=n\} = \binom{n}{k} p^k (1-p)^{n-k}, p = \frac{\alpha}{\alpha+\beta}$
Note that $1-p = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$ which is Binomial dist'n

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Given $P_k = \Pr\{X=k\}$ for poisson (λ)

Verify

$$\begin{cases} P_0 = e^{-\lambda} \\ P_k = \left(\frac{\lambda}{k}\right) P_{k-1} \end{cases} \quad \begin{array}{l} \text{علاقة التكرار} \\ \text{recurrence relation} \end{array}$$

Ans: $\therefore P_k = \frac{e^{-\lambda} \lambda^k}{k!}, k = 0, 1, 2, \dots$

$$\therefore P_0 = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}, k = 0$$

$$\text{and } \therefore P_k = \frac{\lambda}{k} \cdot \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!}$$

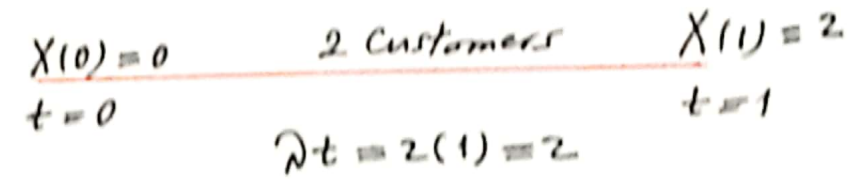
$$\therefore P_k = \left(\frac{\lambda}{k}\right) P_{k-1}$$

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$X(t)$ is the # of customers (arrivals), $\lambda = 2$

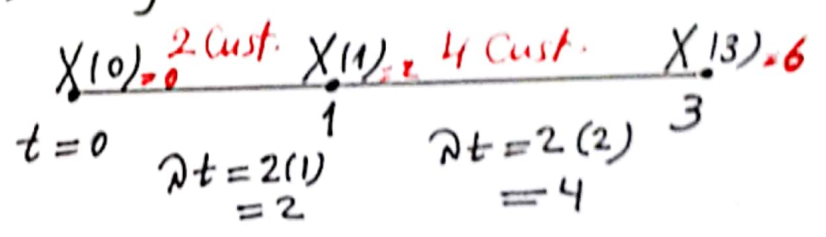


a) $\text{pr}\{X(1) = 2\}$

$= \text{pr}\{X(1) - X(0) = 2\}$

$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{2^2 e^{-2}}{2!} = 2 e^{-2}$

b) $\text{pr}\{X(1) = 2 \text{ and } X(3) = 6\}$



$= \text{pr}\{X(1) - X(0) = 2, X(3) - X(1) = 4\}$

indep. r.v.s

$= \frac{2^2 e^{-2}}{2!} \cdot \frac{4^4 e^{-4}}{4!}$

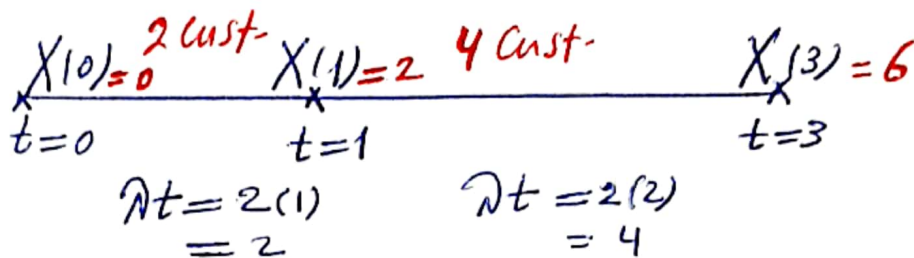
$= 2 e^{-2} \cdot \frac{64}{6} e^{-4} = \frac{64}{3} e^{-6}$

$\text{pr}(x, y)$
 $= \text{pr}(x) \text{pr}(y)$
 if X and Y are
 indep. r.v.s

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d) $pr \{ X(3) = 6 \mid X(1) = 2 \}$

$= pr \{ X(3) - X(1) = 4 \mid X(1) - X(0) = 2 \}$



$= pr \{ X(3) - X(1) = 4 \}$

$= \frac{4^4 e^{-4}}{4!}$

$= \frac{64}{6} e^{-4} = \frac{32}{3} e^{-4}$

$X(3) - X(1)$
and $X(1) - X(0)$ are
2 indep. r.v

Note that $pr(A|B) = pr(A)$
where A and B are
independent events.

c) $pr \{ X(1) = 2 \mid X(3) = 6 \}$

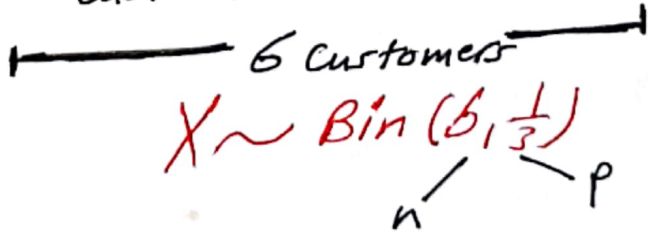
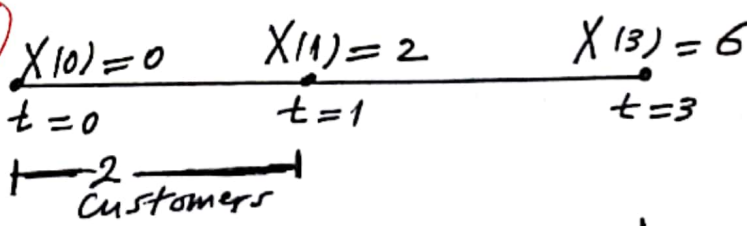
كلنا التجربة اجريت 6 مرات، كل مره واحد
وصول الباتر عند $t=1$ هو

$P = \frac{2}{6} = \frac{1}{3}$

وواصل عدم الوصول هو

$q = \frac{4}{6} = \frac{2}{3}$

$\Rightarrow X \sim Bin(6, \frac{1}{3})$



$X \sim Bin(6, \frac{1}{3})$

$\therefore pr \{ X(1) = 2 \mid X(3) = 6 \} = \binom{n}{x} p^x q^{n-x}$
 $= \binom{6}{2} (\frac{1}{3})^2 (\frac{2}{3})^4$
 $= \frac{80}{243} \approx 0.3292$

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* حل آخر للفقرة (c)

$$\begin{aligned} & \Pr \{ X_{(1)} = 2 \mid X_{(3)} = 6 \} \\ &= \frac{\Pr \{ X_{(1)} = 2 \text{ and } X_{(3)} = 6 \}}{\Pr \{ X_{(3)} = 6 \}} \\ &= \frac{(2e^{-2}/2!) \cdot (4^4 e^{-4}/4!)}{6^6 e^{-6}/6!} \\ &= \frac{64/3 \cancel{e^6}}{64 \cdot 8 \cancel{e^6}} \\ &\simeq 0.3292 \quad \# \end{aligned}$$

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pb 5.1.9 p. 229

For the poisson process $\{X(t); t \geq 0\}$

$$E[X(t)] = \lambda t \text{ and } \text{Var}[X(t)] = \lambda t$$

a) $E[X(1)] = 2(1) = 2, \lambda = 2$

$$E[X(2)] = 2(2) = 4 \quad \#$$

b) To get $E[\{X(1)\}^2]$

$$\text{Var}[X(t)] = E[\{X(t)\}^2] - [E\{X(t)\}]^2$$

$$\therefore \text{Var}[X(1)] = E[\{X(1)\}^2] - [E\{X(1)\}]^2$$

$$\therefore 2 = E[\{X(1)\}^2] - 2^2$$

$$\therefore E[\{X(1)\}^2] = 2 + 4 = 6 \quad \#$$

c) $E[X(1) X(2)]$

$$= E[X(1)] E[X(2)], \quad X(1) \text{ and } X(2) \text{ are indep. r.v.s}$$

$$= 2(4) = 8 \quad \#$$