# Chapter 2

# Solution of Nonlinear Equations

**2.** Use the bisection method to find solutions accurate to within $10^{-4}$ on the interval $[-5, 5]$ of the following functions:

(a) $f(x) = x^5 - 10x^3 - 4$

**4.** Estimate the number of iterations needed to achieve an approximation with accuracy $10^{-4}$ to the solution of $f(x) = x^3 + 4x^2 + 4x - 4$ lying in the interval $[0, 1]$ using the bisection method.

**5.** Use the bisection method for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to find:

(b) Find an error estimate $|\alpha - x_8|$.

**9.** Use the false position method to find solution accurate to within $10^{-4}$ on the interval $[3, 4]$ of the equation $e^x - 3x^2 = 0$.

**11.** Consider the nonlinear equation $g(x) = \frac{1}{2}e^{0.5x}$ defined on the interval $[0, 1]$. Then

(a) Show that there exists a unique fixed-point for $g$ in $[0, 1]$.
(b) Use the fixed-point iterative method to compute $x_3$, set $x_0 = 0$.
(c) Compute an error bound for your approximation in part (b).

**13.** Find value of $k$ such that the iterative scheme $x_{n+1} = \dfrac{x_n^2 - 4kx_n + 7}{4}$, $n \geq 0$ converges to 1. Also, find the rate of convergence of the iterative scheme.

**14.** Write the equation $x^2 - 6x + 5 = 0$ in the form $x = g(x)$, where $x \in [0, 2]$, so that the iteration $x_{n+1} = g(x_n)$ will converge to the root of the given equation for any initial approximation $x_0 \in [0, 2]$.

**15.** Which of the following iterations

(a) $x_{n+1} = \dfrac{1}{4}\left(x_n^2 + \dfrac{6}{x_n}\right)$

(b) $x_{n+1} = \left(4 - \dfrac{6}{x_n^2}\right)$

is suitable to find a root of the equation $x^3 = 4x^2 - 6$ in the interval $[3, 4]$ ? Estimate the number of iterations required to achieve $10^{-3}$ accuracy, starting from $x_0 = 3$.

**19.** Use the Newton's formula for the reciprocal of square root of a number 15 and then find the 3rd approximation of number, with $x_0 = 0.05$.

**21.** Find the Newton's formula for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to calculate $x_3$, if $x_0 = 1.5$. Also, find the rate of convergence of the method.

**23.** Given the iterative scheme $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n \geq 0$ with $f(\alpha) = f'(\alpha) = 0$ and $f''(\alpha) \neq 0$. Find the rate of convergence for this scheme.

**27.** Solve the equation $e^{-x} - x = 0$ by using the secant method, starting with $x_0 = 0$ and $x_1 = 1$, accurate to $10^{-4}$.

**29.** Find the root of multiplicity of the function $f(x) = (x-1)^2 \ln(x)$ at $\alpha = 1$.

**32.** If $f(x)$, $f'(x)$ and $f''(x)$ are continuous and bounded on a certain interval containing $x = \alpha$ and if both $f(\alpha) = 0$ and $f'(\alpha) = 0$ but $f''(\alpha) \neq 0$, show that

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

**33.** Show that iterative scheme $x_{n+1} = 1 + x_n - \frac{x_n^2}{2}$, $n \geq 0$ converges to $\sqrt{2}$. Find the rate of convergence of the sequence.

34. Let $\alpha$ be the exact solution of the function $f(x) = 0$ such that $f'(\alpha) \neq 0$, $f''(\alpha) \neq 0$, then find the conditions of the constant $K$ under which the rate of convergence of the sequence $x_{n+1} = x_n^2 - Kf(x_n)$, $n = 0, 1, 2, \ldots$ is quadratic.

**39.** Solve the following system using the Newton's method:

$$\begin{aligned} 4x^3 + y &= 6 \\ x^2 y &= 1 \end{aligned}$$

Start with initial approximation $x_0 = y_0 = 1$. Stop when successive iterates differ by less than $10^{-7}$.

# Chapter 3

# Systems of Linear Algebraic Equations

**14.** Use the simple Gaussian elimination method to show that the following system does not have a solution

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & & & = & 1.5 \\ 2x_1 & - & x_2 & - & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & x_3 & = & 0 \end{array}$$

**15.** Solve the following systems using the simple Gaussian elimination method

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \end{array}$$

**23.** Solve the following systems using the Gauss-Jordan method

(a)

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 9 \\ 3x_1 & + & 5x_2 & - & 2x_3 & = & 11 \end{array}$$

**21.** Solve the following linear systems using the Gaussian elimination with partial pivoting and without pivoting

(c)

$$\begin{array}{rcrcr} 6.122x_1 & + & 1500.5x_2 & = & 1506.622 \\ 2000x_1 & + & 3x_2 & = & 2003 \end{array}$$

**27.** Find the LU decomposition of each matrix $A$ using the Doolittle's method, and then solve the systems.

(c)

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 3 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}.$$

**28.** Solve the Problem 27 by the LU decomposition using the Crout's method.

**35.** Solve the following linear systems using the Jacobi method, start with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and iterate until $\|\mathbf{x}^{(\mathbf{k}+1)} - \mathbf{x}^{(\mathbf{k})}\|_\infty \le 10^{-5}$ for each system.

(a)

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & x_3 & = & 7 \\ 4x_1 & - & 8x_2 & + & x_3 & = & -21 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 15 \end{array}$$

**36.** Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & 7x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & 20x_3 & = & 7 \end{array}$$

(a) Show that the Jacobi method converges by using $\|T_J\|_\infty < 1$.
(b) Compute 2nd approximation $\mathbf{x}^{(2)}$, starting with $\mathbf{x}^{(0)} = [0, 0, 0]^T$.
(c) Compute an error estimate $\|\mathbf{x} - \mathbf{x}^{(2)}\|_\infty$ for your approximation.

**38.** Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 11 \\ -x_1 & + & 2x_2 & & & = & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & = & 16 \end{array}$$

(a) Show that the Gauss-Seidel method converges by using $\|T_G\|_\infty < 1$.
(b) Compute the second approximation $\mathbf{x}^{(2)}$, starting with $\mathbf{x}^{(0)} = [1, 1, 1]^T$.
(c) Compute an error estimate $\|\mathbf{x} - \mathbf{x}^{(2)}\|_\infty$ for your approximation.

**51.** Discuss the ill-conditioning (stability) of the linear system

$$\begin{array}{rcrcr} 1.01x_1 & + & 0.99x_2 & = & 2 \\ 0.99x_1 & + & 1.01x_2 & = & 2 \end{array}$$

If $\mathbf{x}^* = [2, 0]^T$ be an approximate solution of the system, then find the residual vector $\mathbf{r}$ and estimate the relative error.

# Chapter 4

# Polynomial Interpolation and Approximation

**7.** Let $f(x) = (x+2)\ln(x+2)$. Use the quadratic Lagrange interpolation formula based on the points $x_0 = 0, x_1 = 1, x_2 = 2$, and $x_3 = 3$ to approximate $f(0.5)$ and $f(2.8)$. Also, compute the error bounds for your approximations.

**9.** Let $f(x) = x^4 - 2x + 1$. Use cubic Lagrange interpolation formula based on the points $x_0 = -1, x_1 = 0, x_2 = 2$, and $x_3 = 3$ to find the polynomial $p_3(x)$ to approximate the function $f(x)$ at $x = 1.1$. Also, compute an error bound for your approximation.

**10.** Construct the Lagrange interpolation polynomials for the following functions and compute the error bounds for the approximations:

(a) $f(x) = x + 2^{x+1}$, $\quad x_0 = 0, x_1 = 1, x_2 = 2.5, x_3 = 3$.

(b) $f(x) = 3x^3 + 2x^2 + 1$, $\quad x_0 = 1, x_1 = 2, x_2 = 3$.

**13.** Consider the following table of the $f(x) = \sqrt{x}$:

| $x$ | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|
| $f(x)$ | 2.0000 | 2.2361 | 2.4495 | 2.6458 | 2.8284 |

(a) Construct the divided difference table for the tabulated function.

(b) Find the Newton interpolating polynomials $p_3(x)$ and $p_4(x)$ at $x = 5.9$.

(c) Compute error bounds for your approximations in part (b).

**17.** Let $f(x) = x^2 + e^x$ and $x_0 = 0, x_1 = 1$. Use the divided differences to find the value of the second divided difference $f[x_0, x_1, x_0]$.

21. Consider the following table for function $f(x) = \sin\theta$

| $x$ | $45^o$ | $50^o$ | $55^o$ | $60^o$ |
|---|---|---|---|---|
| $f(x)$ | 0.7071 | 0.7660 | 0.8192 | 0.8660 |

Use Newton's forward interpolation formula to find the value of $\sin 52^o$

# Chapter 5

# Numerical Differentiation and Integration

**1.** Let $f(x) = (x-1)e^x$ and take $h = 0.01$.
(**a**) Calculate approximation to $f'(2.3)$ using the two-point forward-difference formula. Also, compute the actual error and an error bound for you approximation.
(**b**) Solve part *(a)* using the two-point backward-difference formula.

**5.** Use the three-point central-difference formula to compute the approximate value for $f'(5)$ with $f(x) = (x^2+1)\ln x$, and $h = 0.05$. Compute the actual error and the error bound for you approximation.

**20.** Let $f(x) = x + \ln(x+2)$, with $h = 0.1$. Use the three-point formula to approximate $f''(2)$. Find error bound for your approximation and compare the actual error to the bound.

**28.** Use a suitable composite integration formula for the approximation of the integral $\int_1^2 \frac{dx}{3-x}$, with $n = 5$. Compute an upper bound for your approximation.

**29.** Use the composite Trapezoidal rule for the approximation of the integral $\int_1^3 \frac{dx}{7-2x}$ with $h = 0.5$. Also, compute an error term.

**30.** Find the step size $h$ so that the absolute value of the error for the composite Trapezoidal rule is less than $5 \times 10^{-4}$ when it is used to approximate the integral $\int_2^7 \frac{dx}{x}$.

**35.** Evaluate $\int_0^1 e^{x^2} dx$ by the Simpson's rule choosing $h$ small enough to guarantee five decimal accuracy. How large can $h$ be ?

# Chapter 6

# Numerical Solution of Ordinary Differential Equations

**3.** Solve the following initial-value problems using the Euler's method.

(a) $y' = y + x^2, \quad x = 0(0.2)1, \quad y(0) = 1.$

**5.** Solve the following initial-value problems using the Taylor's method of order two.

(a) $y' = 2x^2 - y, \quad x = 0(0.2)1, \quad y(0) = -1.$

**7.** Solve the following initial-value problems using the Modified Euler's method.

(a) $y' = y^2 x^2, \quad x = 1(0.2)2, \quad y(1) = -1.$

**11.** Solve the following initial-value problems using the fourth-order Runge-Kutta formula using $h = 0.2$

(a) $y' = 1 + \frac{y}{x}, \quad 1 \leq x \leq 2 \quad y((1) = 1.$