Ph.D Qualifying Examination [1]
Analysis (General Paper)

2003

Exercise 1 :

1. Let Ω be a bounded domain in the complex plane. Suppose that f is continuous on $\bar{\Omega}$ and analytic on Ω. Let $\alpha \geq 0$ be a constant such that $|f(z)| = \alpha$ for all z on the boundary of Ω. Show that f is a constant function or f has a zero on Ω.

2. Show that the most general linear transformation from the upper half-plane to the unit disc is of the form

$$W = e^{i\alpha}\frac{z-\beta}{z-\bar{\beta}}.$$

where α is real and $\operatorname{Im}\beta > 0$.

3. Let Ω be a domain in the complex plane. Let $(f_n)_n$ be a sequence of analytic functions on Ω, is without zeros and converging uniformly to f on compact sets in Ω. Show that f is analytic on Ω and $f \equiv 0$ or f is without zeros in Ω.

Exercise 2 :

1. Show that

$$\int_0^{2\pi} \frac{\cos^2\theta}{5+3\sin\theta}d\theta = \frac{2\pi}{9}.$$

2. Let f be an analytic function defined on the annulus $r < |z-a| < R$. Show that there exists two uniquely determined

[1] This document is free. Prepared by:
Professor Mongi BLEL, King Saud University, Department of Mathematics .
mblel@ksu.edu.sa ,

analytic functions f_1 on $|z-a|<R$ and f_2 on $|z-a|>r$ such that $\lim_{|z|\to+\infty} f_2(z)=0$ and $f=f_1+f_2$ on the annulus $r<|z-a|<R$.

Answer of Ph.D Qualifying Examination Analysis (General Paper)

March 2003

Solution of the Exercise 1:

1. If f is without zeros on Ω, the function $\dfrac{1}{f}$ is analytic on Ω and $\dfrac{1}{|f(z)|}=\dfrac{1}{\alpha}$ for all z on the boundary of Ω. Then by the maximum principle $|f|\leq\alpha$ and $\dfrac{1}{|f|}\leq\dfrac{1}{\alpha}$ on Ω. Then $|f|=\alpha$ on Ω, which proves that f is constant.

2. Show that the most general linear transformation from the upper half-plane to the unit disc is of the form $\operatorname{Im}\beta>0$ and $f(\beta)=0$. Moreover by symmetry, $f(\overline{\beta})=\infty$, then $f(z)=\lambda\frac{z-\beta}{z-\overline{\beta}}$. The function f transforms the real axis to the unit circle, then for all $x\in\mathbb{R}$, $\left|\lambda\dfrac{x-\beta}{x-\overline{\beta}}\right|=|\lambda|=1$, then

$$f(z)=e^{\mathrm{i}\alpha}\frac{z-\beta}{z-\overline{\beta}},$$

where $\alpha\in\mathbb{R}$ and $\operatorname{Im}\beta>0$.

3. Since the sequence $(f_n)_n$ is uniformly convergent on any compact subset of Ω, then f is holomorphic. We assume that f is not identically zero and there exists $a\in\Omega$ a zero of multiplicity $k\geq 1$ of f. Let $r>0$ such that $f(z)\neq 0$ for any $z\in\overline{D(a,r)}\setminus\{a\}$ and let γ be the closed curve defined by the

circle of radius r and centered at a traversed in the clockwise direction. Then $\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = k$. Since f never vanishing on γ, the sequence $\left(\frac{f_n'}{f_n}\right)_n$ converges uniformly on γ to $\frac{f'}{f}$, thus

$$k = \frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \lim_{n\to+\infty}\frac{1}{2i\pi}\int_\gamma \frac{f_n'(z)}{f_n(z)}\,dz = 0,$$

which is absurd.

Solution of the Exercise 2:

1.
$$\begin{aligned}\int_0^{2\pi}\frac{\cos^2\theta}{5+3\sin\theta}d\theta &= \int_{|z|=1}\frac{(z^2+1)^2}{4z^2(5+3\frac{z^2-1}{2\mathrm{i}z})}\frac{dz}{\mathrm{i}z}\\ &= \int_{|z|=1}\frac{(z^2+1)^2}{2z^2(3z^2+10\mathrm{i}z-3)}dz\\ &= 2\mathrm{i}\pi\left(\mathrm{Res}(f,0)+\mathrm{Res}(f,-\frac{\mathrm{i}}{3})\right) = -2\mathrm{i}\pi(\frac{\mathrm{i}}{9}) = \frac{2\pi}{9},\end{aligned}$$

where $f(z) = \frac{(z^2+1)^2}{2z^2(3z^2+10\mathrm{i}z-3)}$.

$\mathrm{Res}(f,0) = -\frac{5\mathrm{i}}{9}$ and $\mathrm{Res}(f,-\frac{\mathrm{i}}{3}) = \frac{4\mathrm{i}}{9}$.

2. For all $r<|z-a|<R$,

$$f(z) = \sum_{-\infty}^{+\infty}a_n(z-a)^n = \sum_{-\infty}^{-1}a_n(z-a)^n + \sum_{n=0}^{+\infty}a_n(z-a)^n.$$

Define $f_1(z) = \sum_{n=0}^{+\infty}a_n(z-a)^n$ and $f_2(z) = \sum_{-\infty}^{-1}a_n(z-a)^n$. f_1 is analytic on $\{z\in\mathbb{C}:\ |z-a|<R\}$ and f_2 analytic

$\{z \in \mathbb{C} : \ |z-a| > r\}$, $f = f_1 + f_2$ on the annulus $\{z \in \mathbb{C} : r < |z-a| < R\}$ and $\lim_{|z| \to +\infty} f_2(z) = 0$.

Ph.D Qualifying Examination
Analysis (General Paper)

October 2004

Exercise 1 :

For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists a number R, $0 \leq R \leq \infty$, called the radius of convergence. Prove that

1. The series converges absolutely for every $|z| < R$, if $\rho < R$ the convergence is uniform on $\{z \in \mathbb{C} : \ |z| \leq \rho\}$.

2. If $|z| > R$ the terms of the series are unbounded, and the series is consequently divergent.

3. The sum of the series is an analytic function on $\{z \in \mathbb{C} : |z| < R\}$, the derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.

Exercise 2 :

1. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$$

2. State the definition of a conformal mapping.

3. Find a function $w = f(z)$ that maps the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ conformally onto the upper plane $\{w \in \mathbb{C} : \ \operatorname{Im} w > 0\}$.

Answer of Ph.D Qualifying Examination
Analysis (General Paper)

October 2004

Solution of the Exercise 1:

Let $\sum_{n\geq 0} a_n z^n$ be a power series. Define $R = \sup\{r > 0;\ \sum_{n=1}^{+\infty} |a_n| r^n < +\infty\}$. $R \in [0, +\infty]$.

1. If $|x| < R$, the series $\sum_{n\geq 1} a_n x^n$ is absolutely convergent by definition of R.

   Consider $\rho < R$ and the domain $D_\rho = \{z \in \mathbb{C};\ |z| \leq \rho\}$. Let $\rho < S < R$ and $z \in D_\rho$. Since the series $\sum_{n\geq 1} |a_n| S^n$ is convergent, there is $M > 0$ such that $|a_n| S^n \leq M$ for all $n \in \mathbb{N}$. Then $|a_n z^n| \leq \dfrac{M\rho^n}{S^n}$ and the series is uniformly convergent on D_ρ.

2. If $|x| > R$ and the sequence $(a_n x^n)_n$ is bounded, then the series $\sum_{n\geq 1} |a_n| r^n$ converges for every $R < r < |x|$, which is impossible.

3. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}$.

We denote R' the radius of convergence of the power series $\sum_{n\geq 1} n a_n z^{n-1}$. It is obvious that $R' \leq R$. Let $r > 0$ such that $|z| + r < R$. We have $|n a_n z^{n-1}| \leq \dfrac{1}{r}\big(2|a_n|(|z| + r)^n + |a_n||z|^n\big)$ and thus $\sum_{n\geq 1} n a_n z^{n-1}$ converges absolutely on $D(0, R)$. Thus the radius of convergence of the series defining g is greater than R. Thus $R = R'$.

Moreover $|\dfrac{f(z+h) - f(z)}{h} - g(z)| \leq \dfrac{|h|}{r} \sum_{n=1}^{+\infty} |a_n|(|z| + r)^n$, this proves that when h tends to 0, $f'(z) = g(z)$, for any $z \in D(0, R)$.

Solution of the Exercise 2:

1.

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \int_{\{|z|=1\}} \frac{2dz}{(1-2\mathrm{i})z^2+6\mathrm{i}z-(1+2\mathrm{i})}$$
$$= 2\mathrm{i}\pi \mathrm{Res}(f, -\frac{\mathrm{i}}{1-2\mathrm{i}}) = \pi.$$

with $f(z) = \dfrac{2}{(1-2\mathrm{i})z^2+6\mathrm{i}z-(1+2\mathrm{i})}$.

2. A function $f\colon \Omega \longrightarrow \mathbb{C}$ is conformal if it is holomorphic and its derivative is without zeros in Ω.

3. The function $f(z) = \mathrm{i}\dfrac{1+z}{1-z}$ maps the unit disc $|z| < 1$ conformally onto the upper half plane $\{z \in \mathbb{C};\ \mathrm{Im}\, z > 0\}$.

Ph.D Qualifying Examination
Analysis (General Paper)
December 2014

Exercise 1 : [Note that parts 1) and 2) are independent]

1. Compute the following integrals $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^3}$ and $\displaystyle\int_0^{+\infty} \frac{\ln(x)}{1+x^3}dx$.

2. Consider the function defined by the power series

$$f(z) = \sum_{n=1}^{+\infty} z^{n!}.$$

   (a) Prove that f is holomorphic on the unit disc $D = \{z \in \mathbb{C};\ |z| < 1\}$.

   (b) Let $\alpha \in \mathbb{C}$ such that $\alpha^m = 1$, for some $m \in \mathbb{N}$. (α is called a root of unity).
   Prove that $\displaystyle\lim_{r \to 1, r<1} |f(r\alpha)| = +\infty$.

   (c) Deduce that f can not be extended to a holomorpic function on an open set U such that $D \subset U$ and $D \neq U$.

Exercise 2 : [Note that parts 1) and 2) are independent]

1. Let f be a holomorphic function on $D \setminus \{0\}$ and $|f(z)| \leq \ln\left(\frac{1}{|z|}\right)$, for all $z \in D \setminus \{0\}$, where D is the unit disc.

   (a) Prove that 0 is a removable singularity of f. (Hint: you can consider the function $zf(z)$ and calculate its limit at 0).

   (b) Deduce that $f = 0$.

2. Let (f_n) be a sequence of holomorphic functions on the unit disc D such that $f_n(D) \subset D$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} f_n(0) = 1$.

   (a) Prove that there is a subsequence $(f_{n_j})_j$ which converges uniformly on any compact to a holomorphic function g on the unit disc D and $g(0) = 1$.

   (b) We assume that g is not constant.

      i. Prove that there exists $R > 0$ such that $g - 1$ is without zeros in $D(0, R) \setminus \{0\}$.

      ii. Prove that for j sufficiently large and $|z| = r < R$, we have

$$\left|(f_{n_j}(z) - 1) - (g(z) - 1)\right| < \inf\{|g(z)-1|;\ |z| = r\}.$$

      iii. Deduce that $f_{n_j}(z)-1$ has the same number of zeros as $g - 1$ in $D(0, r)$.

      iv. Prove that $f_{n_j}(z) - 1$ is without zero on $D(0, r)$.

      v. Deduce that $g(z) = 1$ for all $z \in D$.

   (c) Prove that $(f_n)_n$ converges uniformly to 1 on any compact.

Solution of Ph.D Qualifying Examination Analysis (General Paper) December 2014

Solution of the Exercise 1:

1. Let $f(z) = \dfrac{\log^2(z)}{1+z^3}$, $I = \displaystyle\int_0^{+\infty} \frac{dx}{1+x^3}$ and $J = \displaystyle\int_0^{+\infty} \frac{\ln(x)}{1+x^3}dx$.
   By Residue Theorem

$$-4i\pi J + 4\pi^2 I = 2i\pi\left(\text{Res}(f,-1) + \text{Res}(f, e^{i\pi}) + \text{Res}(f, e^{\frac{5i\pi}{3}})\right).$$

$\text{Res}(f,-1) = -\dfrac{\pi^2}{3}$.

$\text{Res}(f,e^{i\pi}) = \dfrac{\pi^2}{27}(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2})$.

$\text{Res}(f,e^{\frac{5i\pi}{3}}) = \dfrac{25\pi^2}{27}(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2})$.

Then $-4i\pi J + 4\pi^2 I = \dfrac{8i\pi^3}{27} + \dfrac{8\pi^3\sqrt{3}}{9}$, $I = \dfrac{2\pi\sqrt{3}}{9}$ and $J = -\dfrac{2\pi^2}{27}$.

2. Let f be a holomorphic function on $D\setminus\{0\}$ and $|f(z)| \leq \ln\left(\dfrac{1}{|z|}\right)$, for all $z \in D\setminus\{0\}$, where D is the unit disc.

   (a) $\lim\limits_{z\to 0} zf(z) = 0$, then 0 is a removable singularity of f.

   (b) For all $z \in D(0,1-r)$, $|f(z)| \leq \sup\limits_{|w|=1-r} |f(w)| \ln\left(\dfrac{1}{1-r}\right)$.

   (c) It results that for all $z \in D$, $|f(z)| \leq \lim\limits_{r\to 0} \ln\left(\dfrac{1}{1-r}\right) = 0$, then $f = 0$.

Solution of the Exercise 2:

1. (a) For all $z \in D$, $|z^{n!}| \leq |z|^n$ and the series $\sum\limits_{n\geq 1} |z|^n$ is convergent. Then f is holomorphic on the unit disc $D = \{z \in \mathbb{C};\ |z| < 1\}$.

   (b) $f(r\alpha) = \sum\limits_{n=1}^{m-1} r^{n!}\alpha^{n!} + \sum\limits_{n=m}^{+\infty} r^{n!}$. Then $\lim\limits_{r\to 1, r<1} |f(r\alpha)| = +\infty$. ($\sum\limits_{n=m}^{+\infty} r^{n!} \geq \sum\limits_{n=m}^{p} r^{n!}$ for all $p > m$.)

(c) Let U be an open set such that $D \subset U$ and $D \neq U$. There is α a root of unity in U. But $\lim\limits_{r \to 1} |f(r\alpha)| = \infty$, which is absurd. The function f can not be extended to a holomorpic function on an open set U such that $D \subset U$ and $D \neq U$.

2. Let (f_n) be a sequence of holomorphic functions on the unit disc D such that $f_n(D) \subset D$ for all $n \in \mathbb{N}$ and $\lim\limits_{n \to +\infty} f_n(0) = 1$.

   (a) The sequence (f_n) is bounded, then by Montel Theorem, there is a subsequence $(f_{n_j})_j$ which converges uniformly on any compact to a holomorphic function g on the unit disc D. Since $\lim\limits_{n \to +\infty} f_n(0) = 1$, then $g(0) = 1$.

   (b) We assume that g is not constant.

      i. By Theorem of isolated zero of non constant holomorphic function, there exists $R > 0$ such that $g(z) - 1 \neq 0$ for all $z \in D(0, R) \setminus \{0\}$.

      ii. The convergence of the sequence $(f_{n_j})_j$ is uniform on the compact $\{z \in \mathbb{C};\ |z| = r < R\}$. Then for j large enough

$$\begin{aligned} \left| f_{n_j}(z) - g(z) \right| &= \left| (f_{n_j}(z) - 1) - (g(z) - 1) \right| \\ &< \inf\{|g(z) - 1|;\ |z| = r\}. \end{aligned}$$

      iii. By Theorem $f_{n_j} - 1$ and $g - 1$ have the same number of zeros on $D(0, r)$.

      iv. $f_{n_j}(z) - 1 \neq 0$ for all $z \in D(0, r)$ since $f_n(D) \subset D$, which is absurd since $g(0) = 1$.

      v. We deduce that g is constant, then $g(z) = 1$ for all $z \in D$.

   (c) Since the sequence $(f_n)_n$ is bounded and any subsequence converges to 0 in the space of holomorphic functions, then $(f_n)_n$ converges uniformly to 1 on any compact.

Ph.D Qualifying Examination
Analysis (General Paper)

1424-1425

Exercise 1 :

1. Let f be analytic on a domain Ω and suppose that for $z_0 \in \Omega$, $f^{(n)}(z_0) = 0$, $\forall n \in \mathbb{N}$.

   Show that f is constant.

2. Let f be an analytic function on the unit disc and continuous on $|z| \leq 1$. If $|f(z)| \leq 1 - |z|^2$ for $|z| < 1$. Show that $f \equiv 0$.

Exercise 2 :

1. Let E b the ellipse $x^2 + 4y^2 = 4$. Use the residue theorem to find the value of $\displaystyle\int_E \frac{dz}{(z-3)(2z-1)^3}$.

2. Define a conformal mapping.

   Show that the most general linear transformation from the unit disc to the unit disc can be represented as

$$w = e^{\mathrm{i}\lambda} \frac{z - \alpha}{\bar{\alpha} z - 1}, \ |\alpha| < 1 \text{ and } \lambda \text{ real.}$$

Answer of Ph.D Qualifying Examination
Analysis (General Paper)

1424-1425

Solution of the Exercise 1:

1. f is analytic, then there is $r > 0$ such that
   $$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = f(z_0), \ \forall z \in D(z_0, r).$$

Let $A = \{z \in \Omega;\ f \equiv f(z_0) \text{ on a neighborhood of z}\}$. $z_0 \in A$ and A is an open subset. Let $(z_n)_n$ be a convergent sequence of A and $a \in \Omega$ its limit. Since $z_n \in A$, then $f^{(k)}(z_n) = 0$ for any $k \in \mathbb{N}$ and by continuity $f^{(k)}(a) = 0$. f is analytic, this yields that f is constant on a neighborhood of a. This proves that A closed and open, then $A = \Omega$ and f is constant.

2. $\lim\limits_{|z|\longrightarrow 1} f(z) = 0$, then by Maximum principle, $f = 0$.

Solution of the Exercise 2:

1. $\displaystyle\int_E \frac{dz}{(z-3)(2z-1)^3} = 2i\pi \text{Res}(f, \frac{1}{2}) = \frac{-2i\pi}{125}$, with $f(z) = \dfrac{1}{(z-3)(2z-1)^3}$.

2. A function $f\colon \Omega \longrightarrow \mathbb{C}$ is conformal if and only if it is holomorphic and its derivative is without zeros on Ω.

   Let f be a linear transformation from the unit disc $\mathbb{D}$ to the unit disc $\mathbb{D}$. There is $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. By symmetry $f(\dfrac{1}{\bar{\alpha}}) = \infty$. Then there is $\lambda \in \mathbb{C}$ such that $f(z) = \lambda \dfrac{z-\alpha}{\bar{\alpha}z - 1}$. Since $\left|\dfrac{z-\alpha}{\bar{\alpha}z-1}\right| = 1$ for $|z| = 1$, then $|\lambda| = 1$ and

$$f(z) = e^{i\theta}\frac{z-\alpha}{\bar{\alpha}z-1},\ |\alpha| < 1 \text{ and } \theta \text{ real.}$$

Ph.D Comprehensive Examination Analysis

1425-1426

Exercise 1 :

1. Show that $w = \mathrm{i}\frac{1-z}{1+z}$ maps the domain $\{z \in \mathbb{C};\ |z| > 1\}$ conformally onto the lower half plane $\{w \in \mathbb{C};\ \operatorname{Im} z < 0\}$.

2. Find the number of zeros of $f(z) = z^8 - 5z^5 - 2z + 1$ in the region $\{z \in \mathbb{C};\ 1 < |z| < 2\}$.

Exercise 2 :

1. Evaluate the integral $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x} dx$, α real.

2. Find the Laurent series of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in the region $\{z \in \mathbb{C};\ 1 < |z| < 2\}$.

Answer Ph.D Comprehensive Examination Analysis

1425-1426

Solution of the Exercise 1:

1. The Möbius transformation $f(z) = \mathrm{i}\dfrac{1-z}{1+z}$ transforms the unit circle onto the real axis. ($f(1) = 0$, $f(-1) = \infty$ and $f(\mathrm{i}) = 1$). Since $f(0) = \mathrm{i}$, then f transforms the unit disc onto the upper halph plane and transforms the domain $\{z \in \mathbb{C};\ |z| > 1\}$ onto the lower half plane $\{w \in \mathbb{C};\ \operatorname{Im} z < 0\}$.

2. Let $g(z) = -5z^5$. For $|z| = 1$, $|f(z) - g(z)| = |z^8 - 2z + 1| \leq 4 < |g(z)| = 5$, the by Rouché's Theorem, f has exactly 5 roots in the unit disc.

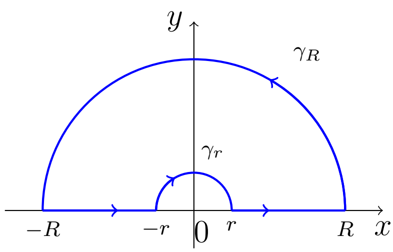
We consider the function $h(z) = z^8$. For $|z| = 2$, $|h(z) - f(z)| = |-5z^5 - 2z + 1| \leq 165 < |h(z)| = 256$, then by Rouché's Theorem, f has 8 roots in the disc of center 0 and radius 2. Then f has 3 roots in the annulus $\{z \in \mathbb{C};\ 1 < |z| < 2\}$.

Solution of the Exercise 2:

1. Let $I(\alpha) = \displaystyle\int_0^{\infty} \frac{\sin \alpha x}{x} dx$.

   The mapping $\alpha \longmapsto I(\alpha)$ is odd and $I(0) = 0$. We compute $I(\alpha)$ for $\alpha > 0$. For a change of variable $t = \alpha x$, $I(\alpha) = I(1)$ for $\alpha > 0$.

   $I(1) = \displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x}$. We set $f(z) = \dfrac{\mathrm{e}^{\mathrm{i}z}}{z}$. We integrate the function f on the following closed path



   By residue theorem, we have:

$$\int_{-R}^{-r} f(x)\ dx - \int_{\gamma_r} f(z)\ dz + \int_r^R f(x)\ dx + \int_{\gamma_R} f(z)\ dz = 0.$$

$$\left|\int_{\gamma_R} f(z)\ dz\right| = \left|\int_0^{\pi} \mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}} \mathrm{i}\ d\theta\right| \leq \int_0^{\pi} \mathrm{e}^{-R\sin\theta}\ d\theta \underset{R\to+\infty}{\longrightarrow} 0.$$

$$\int_{\gamma_r} f(z)\ dz \underset{r\to 0}{\longrightarrow} \mathrm{i}\pi, \text{ thus } I = \pi.$$

2. $f(z) = \dfrac{1}{z(z-1)(z-2)} = \dfrac{1}{2z} - \dfrac{1}{z-1} + \dfrac{1}{2(z-2)}.$

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{+\infty}\frac{1}{z^n} = -\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}}, \quad \forall |z| > 1.$$

$$\frac{1}{2(z-2)} = -\frac{1}{4}\frac{1}{1-\frac{z}{2}} = -\frac{1}{4}\sum_{n=0}^{+\infty}\frac{z^n}{2^n}, \quad \forall |z| < 2.$$

Then

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} - \frac{1}{4}\sum_{n=0}^{+\infty}\frac{z^n}{2^n}$$

in the region $\{z \in \mathbb{C};\ 1 < |z| < 2\}$.

Ph.D Comprehensive Examination
Analysis

1425-1426- Second semester

Exercise 1 :

1. (a) Let f be a analytic function on $\mathbb{C}$. Prove that for any $a, b \in \mathbb{C}$, $a \neq b$ we have for $R > \sup(|a|, |b|)$

$$\frac{1}{2i\pi} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz = \frac{f(a) - f(b)}{a - b}.$$

   (b) Prove that if in addition, f is bounded on $\mathbb{C}$, then

$$\frac{1}{2i\pi} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz \longrightarrow 0, \quad \text{when } R \longrightarrow +\infty.$$

   deduce that any bounded analytic function on $\mathbb{C}$ is constant.

2. Prove that the function $f(z) = \dfrac{z-1}{z+1}$ is a conformal mapping from the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$ into the unit disc $\{z \in \mathbb{C};\ |z| < 1\}$.

Exercise 2 :

For $R > 1$, let γ_R be the half-circle defined by $\gamma_R(t) = Re^{it}$, $t \in [0, \pi]$. We consider the function $f(z) = \dfrac{ze^{3iz}}{(z^2+1)^2}$.

1. Prove that the integral $\displaystyle\int_{\gamma_R} f(z) dz \longrightarrow 0, \quad \text{when } R \longrightarrow +\infty.$

2. Use the residue theorem to find the value of the integral $\displaystyle\int_0^{+\infty} \frac{x \sin(3z)}{(x^2+1)^2} dx.$

Answer of Ph.D Comprehensive Examination Analysis

1425-1426- Second semester

Solution of the Exercise 1:

1. (a) For R large enough ($\max(|a|,|b|)<R$),

$$\begin{aligned}\frac{1}{2\mathrm{i}\pi}\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz &= \frac{1}{2\mathrm{i}\pi}\int_{|z|=R}\frac{1}{a-b}\Big[\frac{f(z)}{(z-a)}-\frac{f(z)}{(z-b)}\Big]dz\\ &= \frac{1}{a-b}(f(a)-f(b)).\end{aligned}$$

   (b) If $|f(z)|\leq M$, for any $z\in\mathbb{C}$, then $\left|\frac{f(a)-f(b)}{a-b}\right|\leq \frac{MR}{(R-|a|)(R-|b|)}$. Since $\lim\limits_{R\to+\infty}\frac{MR}{(R-|a|)(R-|b|)}=0$, then $f(a)=f(b)$.

   Then if f is a bounded analytic function on $\mathbb{C}$, $f(a)=f(b)$ for all $a,b\in\mathbb{C}$ and f is constant.

2. f is a Möbius transformation, $|f(\mathrm{i}t)|=\left|\frac{\mathrm{i}t-1}{\mathrm{i}t+1}\right|=1$ and $f(1)=0$, f is a conformal mapping from the half-plane $\{z\in\mathbb{C};\ \mathrm{Re}\, z>0\}$ into the unit disc $\{z\in\mathbb{C};\ |z|<1\}$.

Solution of the Exercise 2:

For $R>1$, let γ_R be the half-circle defined by $\gamma_R(t)=Re^{\mathrm{i}t}$, $t\in[0,\pi]$. We consider the function $f(z)=\frac{ze^{3\mathrm{i}z}}{(z^2+1)^2}$.

1.

$$\begin{aligned}\left|\int_{\gamma_R}f(z)dz\right| &= \left|\int_0^\pi\frac{\mathrm{i}R^2e^{2\mathrm{i}\theta}e^{3\mathrm{i}R\cos\theta}e^{-3R\sin\theta}}{(R^2e^{2\mathrm{i}\theta}+1)^2}d\theta\right|\\ &\leq \int_0^\pi\frac{R^2}{(R^2-1)^2}d\theta=\frac{\pi R^2}{(R^2-1)^2}\underset{R\to+\infty}{\longrightarrow}0.\end{aligned}$$

2.

$$\begin{aligned}\int_0^{+\infty} \frac{x\sin(3z)}{(x^2+1)^2}dx &= \lim_{R\to+\infty}\left(-\mathrm{i}\int_{-R}^{R} f(x)dx + \int_{\gamma_R} f(z)dz\right)\\ &= 2\pi\mathrm{Res}(f,\mathrm{i}) = \frac{3\pi}{2e^3}.\end{aligned}$$

Ph.D Comprehensive Examination
Analysis (General Paper)

First semester 1426-1427

Exercise 1 :

1. The aim of this question is to prove Liouville's theorem.

   Let f be a holomorphic function on $\mathbb{C}$. Use the Cauchy's theorem to prove that

   $$\frac{1}{2i\pi}\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz=\frac{f(a)-f(b)}{a-b}\qquad \text{for } R \text{ large.}$$

   Prove that if f is bounded $\lim_{R\to+\infty}\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz=0$ and hence f is constant.

2. We consider the polynomial $P(z)=z^7+5z^4+z^3-z+1$.

   Prove that P has exactly 4 roots in the unit disc and 3 roots in the annulus $\{z\in\mathbb{C};\ 1<|z|<2\}$.

Exercise 2 :

Let f be a holomorphic function on a domain $\Omega\subset\mathbb{C}$. We suppose that $\Omega\supset\overline{D(0,1)}$ and

$$(P)\qquad |f(e^{i\theta})|=1\quad \forall\ \theta\in\mathbb{R}.$$

1. Let $a\in D(0,1)$. Prove that the function $h_a(z)=\dfrac{a-z}{1-\bar{a}z}$ verifies the property (P).

2. Prove that if f is without zeros in the unit disc $D(0,1)$, then f is constant in Ω.

3. (a) Prove that the set of zeros of f in $D(0,1)$ is finite.
   (b) Deduce that if f is not constant, there exist $z_1, \ldots, z_n$ in $D(0,1)$ and $p_1, \ldots, p_n \in \mathbb{N}$ such that

$$f(z) = \lambda \prod_{j=1}^{n} (\frac{z - z_j}{1 - \overline{z_j} z})^{p_j} \quad \text{with } |\lambda| = 1$$

Answer of Ph.D Comprehensive Examination Analysis (General Paper)

First semester 1426-1427

Solution of the Exercise 1:

1. For R large enough ($\max(|a|, |b|) < R$),

$$\begin{aligned} \frac{1}{2i\pi} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz &= \frac{1}{2i\pi} \int_{|z|=R} \frac{1}{a-b} \Big[\frac{f(z)}{(z-a)} - \frac{f(z)}{(z-b)} \Big] dz \\ &= \frac{1}{a-b}(f(a) - f(b)). \end{aligned}$$

   If $|f(z)| \le M$, for all $z \in \mathbb{C}$, then $\left| \frac{f(a) - f(b)}{a-b} \right| \le \frac{MR}{(R - |a|)(R - |b|)}$.
   Since $\lim_{R \to +\infty} \frac{MR}{(R-|a|)(R-|b|)} = 0$, then $f(a) = f(b)$ and f is constant.

2. Let $f(z) = 5z^4$. For $|z| = 1$, $|f(z) - P(z)| = |z^7 + z^3 - z + 1| \le 4 < |f(z)|$, the by Rouché's Theorem, P has exactly 4 roots in the unit disc.

   We consider the function $g(z) = z^7$. For $|z| = 2$, $|g(z) - P(z)| = |5z^4 + z^3 - z + 1| \le 91 < |g(z)| = 128$, then by Rouché's Theorem, P has 7 roots in the disc of center 0 and radius 2. Then P has 3 roots in the annulus $\{z \in \mathbb{C};\ 1 < |z| < 2\}$.

Solution of the Exercise 2:

1. $h_a(a) = 0$, $h_a(0) = a$ and $|h_a(e^{i\theta})| = \left|\frac{a - e^{i\theta}}{1 - \bar{a}e^{i\theta}}\right| = \left|\frac{a - e^{i\theta}}{e^{-i\theta} - \bar{a}}\right| = 1$. Then h_a fulfills the property (P).

2. If f is without zeros in the unit disc D, then $\frac{1}{f}$ is holomorphic on a neighborhood of $\overline{D}$. Since $|f(e^{i\theta})| = 1$ and $\left|\frac{1}{f(e^{i\theta})}\right| = 1$ for any $\theta \in \mathbb{R}$, then from the Maximum Principle applied to f and $\frac{1}{f}$, we have $|f| \leq 1$ and $\frac{1}{|f|} \leq 1$ on D, thus f is constant on Ω.

3. (a) $\overline{D}$ is compact, thus the number of zeros of f in D is finite.

   (b) If $z_1, \ldots, z_n$ are the zeros of f in D and $p_1, \ldots, p_n \in \mathbb{N}$ their multiplicities respective, then $g(z) = \frac{f(z)}{\prod_{j=1}^{n}(\frac{z - z_j}{1 - \overline{z_j}z})^{p_j}}$ is holomorphic on a neighborhood of $\overline{D}$, without zeros in D and $|g(e^{i\theta})| = 1$ for any $\theta \in \mathbb{R}$. From the above question g is constant, which yields that

$$f(z) = \lambda \prod_{j=1}^{n}(\frac{z - z_j}{1 - \overline{z_j}z})^{p_j}, \quad \text{with } |\lambda| = 1.$$

Ph.D Comprehensive Examination
Analysis (Spacial Paper)

First semester 1429-1430 H

Exercise 1 :

1. Justify the non existence of a conformal transformation from $\mathbb{C}$ into the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

2. (a) State the maximum principle.

   (b) Let Ω be the square region $\{z \in \mathbb{C};\ |\operatorname{Re} z| < 1,\ |\operatorname{Im} z| < 1\}$. Suppose that f is continuous on $\overline{\Omega}$, holomorphic on Ω and $f(z) = 0$ whenever $\operatorname{Re} z = 1$. Prove that f is identiquely 0 on $\overline{\Omega}$. (Hint: consider the function $g(z) = f(z)f(-z)f(\mathrm{i}z)f(-\mathrm{i}z)$).

3. Let $a, b, c \in D(0,1)$. Prove that the function

$$f(z) = z\left(\frac{z-a}{1-\bar{a}z}\right)^n\left(\frac{z-b}{1-\bar{b}z}\right)^p - c, \quad n, p \in \mathbb{N},$$

   has exactly $n+p+1$ roots in $D(0,1)$.

Exercise 2 :

1. Let g be an entire function and assume that there exists a constant $M > 0$ such that

$$|g(z)| \leq M|z^2 e^z|, \quad \forall z \in \mathbb{C}.$$

   Prove that there exists a constant $K \in \mathbb{C}$ such that $g(z) = Kz^2e^z$, $\forall z \in \mathbb{C}$ with $|K| \leq M$.

2. (a) Prove that the integral $\displaystyle\int_0^{+\infty} \frac{\ln x}{1+x^2}dx$ is convergent and use Cauchy's residue theorem to prove that $\displaystyle\int_0^{+\infty} \frac{\ln x}{1+x^2}dx = 0$.

(b) Using a suitable variable change, show that $\int_0^{+\infty} \frac{\ln x}{1+x^2} dx = 0$.

3. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic automorphism.

   (a) Prove that $\lim_{|z|\to+\infty} |f(z)| = +\infty$ and that 0 is a pole for the function $f(\frac{1}{z})$.
   Deduce that f is a polynomial function.

   (b) b) Deduce that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$.

Answer of Ph.D Comprehensive Examination Analysis (Spacial Paper)

First semester 1429-1430 H

Solution of the Exercise 1:

1. By Riemann Theorem there is a conformal transformation from the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$ into the unit disc. Then if there is a conformal transformation from $\mathbb{C}$ into the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$, there is a conformal transformation from $\mathbb{C}$ into the unit disc, which is impossible by Liouville Theorem.

   Second proof: If $f = U + \mathrm{i}V$ is a conformal transformation from $\mathbb{C}$ into the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$, then $V \geq 0$ and V is harmonic on $\mathbb{R}^2$, then V is constant and f is constant.

2. (a) The maximum principle: Let Ω be a bounded domain and f a continuous function on $\overline{\Omega}$ and holomorphic on Ω. If $M = \sup_{z\in\overline{\Omega}\setminus\Omega} |f(z)|$, then $|f(z)| \leq M$ for every $z \in \Omega$, and if there exists $a \in \Omega$ such that $|f(a)| = M$, then f is constant on Ω. (Furthermore, $|f|$ does not attains a maximum at an interior point unless f is constant.)

(b) the function $g(z) = f(z)f(-z)f(\mathrm{i}z)f(-\mathrm{i}z)$ is continuous on $\overline{\Omega}$, holomorphic on Ω. Moreover $g = 0$ on $\partial\Omega$, then $g = 0$ and $f = 0$.

(c) Let g be the function defined by $g(z) = z\left(\frac{z-a}{1-\bar{a}z}\right)^n \left(\frac{z-b}{1-\bar{b}z}\right)^p$.
For $|z| = 1$, $|g(z)| = 1$ and $|f(z) - g(z)| = |c| < 1 = |g(z)|$. Then f and g have the same number of zeros on the unit disc. Then f has exactly $n + p + 1$ roots in $D(0, 1)$.

Solution of the Exercise 2:

1. Consider the function $f(z) = g(z)e^{-z}$. For all $z \in \mathbb{C}$, $|f(z)| \leq M|z^2|$, then f is a polynomial of degree ≤ 2. But $f(0) = 0$ and $f'(0) = 0$. Then there exists $K \in \mathbb{C}$ such that $f(z) = Kz^2$, $\forall z \in \mathbb{C}$ and $|K| \leq M$.

2. (a) In a neighborhood of 0, $\frac{\ln x}{1+x^2} \approx \ln x$ which is integrable.
For $x > 1$, $\frac{\ln x}{1+x^2} \leq \frac{\ln x}{x^2}$ which is integrable on $[1, +\infty[$.
By residue theorem

$$2\mathrm{i}\pi\left(\mathrm{Res}(\frac{\log^2 z}{1+z^2}, \mathrm{i}) + \mathrm{Res}(\frac{\log^2 z}{1+z^2}, -\mathrm{i})\right) = 4\pi^2 \int_0^{+\infty} \frac{dx}{1+x^2} - 4\mathrm{i}\pi \int_0^{+\infty}$$

$\mathrm{Res}(\frac{\log^2 z}{1+z^2}, \mathrm{i}) = -\frac{\pi^2}{8\mathrm{i}}$, $\mathrm{Res}(\frac{\log^2 z}{1+z^2}, -\mathrm{i}) = \frac{9\pi^2}{8\mathrm{i}}$. Then $\int_0^{+\infty} \frac{\ln x}{1+x^2}dx = 0$ and $\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.

(b) $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx \overset{t=\frac{1}{x}}{=} -\int_0^1 \frac{\ln x}{1+t^2}dt$. Then $\int_0^{+\infty} \frac{\ln x}{1+x^2}dx = 0$.

3. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic automorphism.

(a) f^{-1} is continuous, then for all $R > 0$, $f^{-1}(\overline{D(0,R)})$ is a compact. There is $R' > 0$ such that $f^{-1}(\overline{D(0,R)}) \subset D(0,R')$. This is equivalent to

$$\forall R > 0, \exists R' > 0 \text{ such that if} |z| \geq R', \ |f(z)| \geq R.$$

Then $\lim\limits_{|z| \to +\infty} |f(z)| = +\infty$.

Since $\lim\limits_{z \to 0} |f(\dfrac{1}{z})| = +\infty$, then 0 is a pole for the function $f(\frac{1}{z})$.

(b) Since f is injective, 0 is a simple pole of the function $f(\frac{1}{z})$ and f is a polynomial function of degree 1.

Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1429-1430 H

Exercise 3 :

1. Let $\Omega = \{z = x + \mathrm{i}y \in \mathbb{C}, a < x < b,\ y > 0\}$ and $g\colon \overline{\Omega} \longrightarrow \mathbb{C}$ be a continuous function and holomorphic on Ω. Assume that $g(x) \in \mathbb{R}$, for all $a < x < b$.

   (a) Prove that the function $\tilde{g}$ defined on the strip $\{z = x + \mathrm{i}y \in \mathbb{C}, a < x < b\}$ by

   $$\tilde{g}(z) = \begin{cases} g(z) & \text{if } z \in \overline{\Omega} \\ g(z) = \overline{g(\overline{z})} & \text{if } \overline{z} \in \Omega \end{cases}$$

   is holomorphic.

   (b) Deduce that if $g(x) = 0$ for all $a < x < b$, then $g \equiv 0$ on Ω.

2. Let h be the holomorphic function defined on a neighborhood of the closed unit disc $\overline{D}$ by: $h(z) = \mathrm{i}\dfrac{1-z}{1+z}$.

   (a) Prove that h is a conformal mapping from the unit disc D onto the upper half-plane $\mathcal{H} = \{x + \mathrm{i}y \in \mathbb{C};\ y > 0\}$.

   (b) Find the image of $\{e^{\mathrm{i}t};\ 0 < t < \frac{\pi}{2}\}$ by h.

3. Let f be a holomorphic function on the unit disc D and continuous on $\overline{D}$. Assume that $f(e^{\mathrm{i}t}) = 0$, for all $t \in [0, \frac{\pi}{2}]$. Prove that $f \circ h^{-1} \equiv 0$ and that $f \equiv 0$.

4. We can prove the same result otherwise. Define the function F by: $F(z) = f(z)f(\mathrm{i}z)f(-z)f(-\mathrm{i}z)$.

   Prove that $F \equiv 0$, and deduce that $f \equiv 0$.

Exercise 4 :

Let P be a polynomial of degree $n \geq 1$ and let $R > 0$.

1. Let h be an entire function (i.e. holomorphic on $\mathbb{C}$). Assume that $|h(z)| \leq |P(z)|$, for all $|z| \geq R$.

   Prove that h is a polynomial of degree at least n.

2. Prove that $\lim\limits_{|z| \longrightarrow +\infty} |P(z)| = +\infty$.

3. Let $(z_n)_n$ be a sequence of complex numbers such that the sequence $(P(z_n))_n$ is convergent.

   Prove that the sequence $(z_n)_n$ is bounded.

4. Prove that $P(\mathbb{C})$ is an open and closed subset of $\mathbb{C}$ and deduce D'Alembert's theorem, namely: Every non constant polynomial has at least one zero in $\mathbb{C}$.

5. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function such that $\lim\limits_{|z| \to +\infty} |f(z)| = +\infty$.

   (a) Prove that f has a finite number of zeros in $\mathbb{C}$.

   (b) Prove that there exists a polynomial P such that the function $h = \dfrac{P}{f}$ is holomorphic in $\mathbb{C}$ and $h(z) \neq 0$, for all $z \in \mathbb{C}$.

   (c) Prove that there exists an $R > 0$ such that $|h(z)| \leq |P(z)|$, for all $|z| \geq R$.

   (d) Deduce that there exists a constant C such that $f = CP$.

6. Now let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be a conformal mapping.

   (a) Prove that $\lim\limits_{|z| \to +\infty} |g(z)| = +\infty$.

   (b) Deduce that $g(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$.

Answer Ph.D Comprehensive Examination Analysis (General Paper)

Second semester 1429-1430 H

Solution of the Exercise 3:

1. (a) $\tilde{g}$ is holomorphic on Ω and on $\Omega^- = \{z = x+iy \in \mathbb{C}, a < x < b,\ y < 0\}$ and $\tilde{g}$ is continuous on $\{z = x+iy \in \mathbb{C}; a < x < b\}$.

If $g(z) = U(x,y)+iV(x,y)$ on Ω, then $\tilde{g}(z) = U(x,-y)-iV(x,-y) = U_1(x,y)+iV_1(x,y)$ on Ω^-.

$$\begin{cases} \dfrac{\partial U_1}{\partial x}(x,y) = \dfrac{\partial U}{\partial x}(x,-y) = \dfrac{\partial V}{\partial y}(x,-y) = \dfrac{\partial V_1}{\partial y}(x,y) \\ \dfrac{\partial U_1}{\partial y}(x,y) = -\dfrac{\partial U}{\partial y}(x,-y) = \dfrac{\partial V}{\partial x}(x,-y) = -\dfrac{\partial V_1}{\partial x}(x,y) \end{cases}$$

Then $\tilde{g}$ is holomorphic on Ω^-. Moreover $\tilde{g}$ is continuous on $\{z = x+iy \in \mathbb{C}, a < x < b\}$. To show that $\tilde{g}$ is holomorphic on $\{z = x+iy \in \mathbb{C}, a < x < b\}$, we use Morera's theorem and we prove that for all triangle $\Delta \subset \{z = x+iy \in \mathbb{C}, a < x < b\}$, $\displaystyle\int_{\Delta} \tilde{g}(z)dz = 0$.

Let $\Delta = (A,B,C)$ be a triangle in $\{z = x+iy \in \mathbb{C}, a < x < b\}$.

If $\Delta \subset \Omega$ or $\Delta \subset \Omega^-$, then $\displaystyle\int_{\partial\Delta} \tilde{g}(z)dz = 0$.

If Δ meets the real axis, then we can suppose that $\Delta\cap\Omega$ is a triangle $\Delta_1 = (A,\alpha,\beta)$ and $\Delta\cap\Omega^-$ is a polygon (α,B,C,β), (cf figure 1).

Since the triangle $\Delta_1 = (A,A_1,A_2)$ is in Ω and the quadrilateral $R_1 = (B,C,B_2,B_1)$ is in Ω^-, then $\displaystyle\int_{\partial\Delta_1} \tilde{g}(z)dz =$

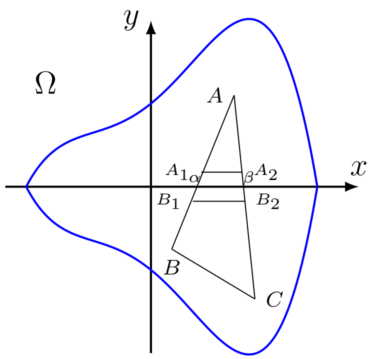


Figure 1:

$\displaystyle\int_{\partial R_1} f(z)dz = 0$, thus $\displaystyle\int_{\partial\Delta} \tilde{g}(z)dz = \int_{\partial R_2} \tilde{g}(z)dz = 0$, with R_2 the quadrilateral (A_1, B_1, B_2, A_2).

If the points A_1 and B_1 tend to α, then the integral $\displaystyle\int_{[A_1,B_1]} \tilde{g}(z)dz$ tends to 0. The same result for the integral $\displaystyle\int_{[B_2,A_2]} \tilde{g}(z)dz$ tends to 0 when the points A_2 and B_2 tend to β.

It follows from Morera's Theorem that $\tilde{g}$ is holomorphic on Ω.

(b) If $g(x) = 0$ for all $a < x < b$, $\tilde{g}(x) = 0$ for all $a < x < b$, then $\tilde{g} \equiv 0$ on $\{z = x + \mathrm{i}y \in \mathbb{C}, a < x < b\}$ and then $g \equiv 0$ on Ω.

2. (a) $h'(z) = \dfrac{-2\mathrm{i}}{(1+z)^2}$, then h is a conformal mapping. $h(z) = \dfrac{2y + \mathrm{i}(1-|z|^2)}{|1+z|^2} \in \mathcal{H}$ with $z = x + \mathrm{i}y \in D$. Moreover $h^{-1}(z) = \dfrac{1+\mathrm{i}z}{1-\mathrm{i}z} = \dfrac{1-|z|^2+2\mathrm{i}x}{|1-\mathrm{i}z|^2}$, then $|h^{-1}(z)|^2 = \dfrac{1+x^2-y^2}{(1+y)^2+x^2} \in D$ if $y > 0$.

Otherwise, we can see that h is a möbius transform and the image of the unit circle is the real axis and $h(0) = \mathrm{i}$, then h is a conformal mapping from the unit disc D onto the upper half-plane $\mathcal{H} = \{x + \mathrm{i}y \in \mathbb{C};\ y > 0\}$.

(b) The image of $\{e^{\mathrm{i}t};\ 0 < t < \frac{\pi}{2}\}$ by h is the interval $]0, 1[$.

3. From the first question $f \circ h^{-1}$ is holomorphic on the open set $\{z = x + \mathrm{i}y \in \mathbb{C}, 0 < x < 1,\ y > 0\}$ and $f \circ h^{-1}(x) = 0$ on the interval $]0, 1[$. Then $f \circ h^{-1} \equiv 0$ and $f \equiv 0$.

4. $F \equiv 0$ on the unit circle and from the maximum principle, $F \equiv 0$ on D. Then $f \equiv 0$.

Solution of the Exercise 4:

If $P(z) = a_n z^n + \ldots + a_0$ with $a_n \neq 0$. So

$$\lim_{|z|\longrightarrow+\infty} \frac{|P(z)|}{|a_n||z|^n} = \lim_{|z|\longrightarrow+\infty} \left|1 + \frac{a_{n-1}}{a_n z} + \ldots + \frac{a_0}{a_n z^n}\right| = 1, \qquad (0.1)$$

then there exists $R_1 > 0$ such that $|P(z)| \leq 2|a_n||z|^n$ for $|z| \geq R_1$.

1. If $h(z) = \displaystyle\sum_{k=0}^{+\infty} b_k z^k$ and $|h(z)| \leq |P(z)|$, for all $|z| \geq R$. The Cauchy's inequalities gives that for all $m \geq 1$ and $|z| \geq \max(R, R_1)$, $|b_m| \leq 2|a_n||z|^{n-m}$; which gives that $b_m = 0$ if $m \geq n + 1$. Then h is a polynomial of degree at least n.

2. The relation (0.1) proves that $\displaystyle\lim_{|z|\longrightarrow+\infty} |P(z)| = +\infty$.

3. Let $(z_n)_n$ be a sequence of complex numbers such that the sequence $(P(z_n))_n$ is convergent. If the sequence $(z_n)_n$ is not bounded, there exists a subsequence $(z_{\varphi(n)})_n$ such that $\displaystyle\lim_{n\to+\infty} |z_{\varphi(n)}| = +\infty$. Then $\displaystyle\lim_{n\to+\infty} |P(z_{\varphi(n)})| = +\infty$ which is impossible.

4. By the open mapping theorem $P(\mathbb{C})$ is an open subset which deduced from the open mapping theorem.

   If there exists a sequence $(z_n)_n$ such that the sequence $(P(z_n))_n$ is convergent, there exists a convergent subsequence $(z_{\varphi(n)})_n$. Let $a = \lim\limits_{n\to+\infty} z_{\varphi(n)}$ and $\alpha = \lim\limits_{n\to+\infty} P(z_n)$. Then $\alpha = P(a)$, and then $P(\mathbb{C})$ is closed. $P(\mathbb{C})$ is connected, then $P(\mathbb{C}) = \mathbb{C}$, which proves the D'Alembert's theorem.

5. (a) There exists $R > 0$ such that for $|z| > R$, $|f(z)| \geq 1$, then the set of zeros of f is in the compact $\overline{D(0,R)}$, then f has a finite number of zeros in $\mathbb{C}$.

   (b) It suffices to take $P(z) = \prod\limits_{j=1}^{n}(z - z_j)$, with $z_1, \ldots, z_n$ the zeros of f cited with their order of multiplicity.

   (c) It suffices to take R the same as in the question a).

   (d) We deduce from the first question that $\dfrac{P}{f}$ is a polynomial without zeros, then it is a constant. Then $f = CP$.

6. (a) As g is a conformal mapping, then g^{-1} is continuous, then for all $R > 0$, $g^{-1}(\overline{D(0,R)}$ is a compact subset, then is bounded. It follows that there exists $R' > 0$ such that $g^{-1}(\overline{D(0,R)} \subset D(0,R')$. Then for all $R > 0$, there exists $R' > 0$ such that for $|z| \geq R'$, $|g(z)| \geq R$, which proves that $\lim\limits_{|z|\to+\infty} |g(z)| = +\infty$.

   (b) From the above question g is a polynomial, but it has only one zero, then $g(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$.

Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1996

Exercise 1 :

1. (a) Let f be analytic in $\mathbb{C}$, $z = x + iy$. If $\operatorname{Re} f(z) = e^x(x\cos y - y\sin y)$ when $|z| < 1$, find the general form of $f(z) \in \mathbb{C}$.

   (b) Map the region between $|z-1| = 1$ and $|z-2| = 2$ conformally onto $\operatorname{Re} z > 0$.

2. (a) Calculate $\displaystyle\int_{|z-1|=3} (z^2 - z + 1)d\bar{z}$.

   (b) Let $f(z)$ be analytic in $|z| < 1$. Show that $|f^n(0)| \leq n!n^n$ for some integer n.

   (c) By the method of contour integration show that $\displaystyle\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin \pi\alpha}$, $0 < \alpha < l$.

3. (a) Find the number of zeros of $2z^2 - e^{\frac{z}{2}}$ in $|z| < 1$.

   (b) Expand $f(z) = \dfrac{1}{z(z-1)(z-2)}$ as a Laurent series in the annulus $1 < |z| < 2$.

Answer of Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1996

Solution of the Exercise 1:

1. If $f = U + \mathrm{i}V$, $U(x,y) = e^x(x\cos y - y\sin y)$. $\dfrac{\partial U}{\partial x} = e^x(x\cos y - y\sin y + \cos y) = \dfrac{\partial V}{\partial y}$. Then $V(x,y) = e^x(x\sin y + y\cos y) + h(y)$. Moreover $\dfrac{\partial U}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = -\dfrac{\partial V}{\partial x} = e^x(-x\sin y - y\cos y - \sin y) - h'(y)$. Then $h = C$ and

$$f(z) = e^x(x\cos y - y\sin y) + \mathrm{i}e^x(x\sin y + y\cos y) + \mathrm{i}C = ze^z + \mathrm{i}C.$$

2. We denote Ω the region between $|z-1| = 1$ and $|z-2| = 2$. The function $z \longmapsto f_1(z) = \frac{1}{z}$ maps conformally Ω onto the strip $\Omega_1 = \{z \in \mathbb{C}; \frac{1}{4} < \operatorname{Re} z < \frac{1}{2}\}$. The function $z \longmapsto f_2(z) = 4\mathrm{i}\pi z - \frac{3\mathrm{i}\pi}{2}$ maps conformally Ω_1 onto the strip $\Omega_2 = \{z \in \mathbb{C}; -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$. The function $z \longmapsto f_3(z) = e^z$ maps conformally Ω_2 onto the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$. Then the function $z \longmapsto f(z) = f_3 \circ f_2 \circ f_1(z)$ maps conformally Ω onto the half plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

3. (a) $\displaystyle\int_{|z-1|=3} (z^2 - z + 1)d\bar{z} = -\mathrm{i}\int_0^{2\pi} ((1+3e^{\mathrm{i}\theta})^2 - 1 - 3e^{\mathrm{i}\theta} - 1)e^{-\mathrm{i}\theta}d\theta = -6\mathrm{i}\pi.$

   (b) The power series $\displaystyle\sum_{n\geq 1} n!n^n z^n$ has 0 as radius of convergence. Then if $|f^n(0)| \geq n!n^n$ for all integers n, the function f can not be analytic on the unit disc.

   (c) Let $f(z) = \dfrac{z^{\alpha-1}}{1+z}$, with $z^{\alpha-1} = \mathrm{e}^{(\alpha-1)\log z}$, $\log z$ is the branch of $\log z$ such that $\log z = \ln|z| + \mathrm{i}\theta$, $0 < \theta < 2\pi$. We take the closed curve defined by the figure (2).

   $\operatorname{Res}(f,-1) = -\mathrm{e}^{\mathrm{i}\pi\alpha}$. Then by the residue theorem $\displaystyle\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin\pi\alpha}$, $0 < \alpha < l$.

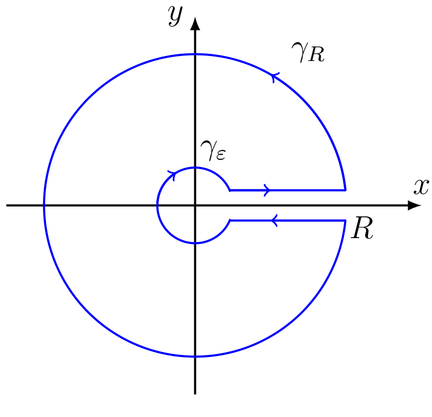


Figure 2:

4. (a) Let $f(z) = 2z^2 - e^{\frac{z}{2}}$ and $g(z) = 2z^2$. For $|z| = 1$, $|f(z) - g(z)| = |e^{\frac{z}{2}}| \leq \sqrt{e} < 1 < |g(z)|$. Then the number of zeros of $2z^2 - e^{\frac{z}{2}}$ in $|z| < 1$ is 2.

   (b)

$$\begin{aligned} f(z) &= \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)} = \frac{1}{2z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{4}\frac{1}{1-\frac{z}{2}} \\ &= \frac{1}{2z} - \sum_{n=0}^{+\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{+\infty} \frac{z^n}{2^{n+2}}. \end{aligned}$$

Ph.D Comprehensive Examination
Analysis (General Paper)

Autumn 1997

Exercise 1 :

1. (a) Let f be an analytic function in a domain Ω. If the $\arg f$ is constant, show that f is a constant.

   (b) Map the region between $\{z \in \mathbb{C};\ |z| = 1\}$ and $\{z \in \mathbb{C};\ |2z-1| = 1\}$ conformally onto the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

2. (a) Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{5+3\cos\theta} d\theta$.

   (b) State Rouche's theorem and use it to prove the fundamental theorem of algebra about the zeros of a polynomial.

3. (a) Let $(f_n)_n$ be a sequence of analytic functions in a domain D. Suppose $f_n(z) \neq 0$ for any n and any $z \in D$. Suppose $(f_n)_n$ converges to f uniformly on every compact subset of D. Show that if $f(z_0) = 0$ for some $z \in D$, then $f(z) = 0$ for all $z \in D$.

   (b) Let $f(z) = \dfrac{\cos z}{z^2 \log(1+z)}$. Use the Laurent series to find the residue of f at $z = 0$.

Answer of Ph.D Comprehensive Examination
Analysis (General Paper)

Autumn 1997

Solution of the Exercise 1:

1. (a) Let f be an analytic function in a domain Ω. If the $\arg f$ is constant, show that f is a constant.

(b) Map the region between $\{z \in \mathbb{C};\ |z| = 1\}$ and $\{z \in \mathbb{C};\ |2z-1| = 1\}$ conformally onto the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

2. (a)

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2\theta}{5+3\sin\theta}d\theta &= \int_{|z|=1} \frac{(z^2+1)^2}{4z^2(5+3\frac{z^2-1}{2\mathrm{i}z})}\frac{dz}{\mathrm{i}z} \\ &= \int_{|z|=1} \frac{(z^2+1)^2}{2z^2(3z^2+10\mathrm{i}z-3)}dz \\ &= 2\mathrm{i}\pi\left(\operatorname{Res}(f,0)+\operatorname{Res}(f,-\frac{\mathrm{i}}{3})\right) = -2\mathrm{i}\pi(\frac{\mathrm{i}}{9}) = \frac{2\pi}{9}\end{aligned}$$

where $f(z) = \dfrac{(z^2+1)^2}{2z^2(3z^2+10\mathrm{i}z-3)}$.

$\operatorname{Res}(f,0) = -\dfrac{5\mathrm{i}}{9}$ and $\operatorname{Res}(f,-\dfrac{\mathrm{i}}{3}) = \dfrac{4\mathrm{i}}{9}$.

(b) The Rouché's Theorem: Let f and g be two holomorphic functions on a neighborhood of the disc $\{z \in \mathbb{C};\ |z-a| \leq r\}$ and $|f(z)-g(z)| < |f(z)|;\ \forall\ z \in \mathscr{C}(a,r) = \{z \in \mathbb{C};\ |z-a| = r\}$, then f and g have the same number of zeros inside the disc $D(a,r)$. (The zeros are counted according to their order or multiplicity.)

The Fundamental Theorem of Algebra: If $P(z) = a_n z^n + \ldots + a_0$, then for $|z|$ large enough, $|P(z) - a_n z^n| < |a_n||z^n|$, because $\lim\limits_{|z|\to+\infty}\left|\dfrac{P(z)-a_n z^n}{a_n z^n}\right| = 0$. It results that P has the same number of zeros that the polynomial $Q(z) = a_n z^n$.

3. (a) Since the sequence $(f_n)_n$ is uniformly convergent on any compact subset of Ω, then f is holomorphic. We assume that $f(z_0) = 0$ and z_0 is a zero of multiplicity $k \geq 1$ of f and f is not identically 0. Let $r > 0$ such that $f(z) \neq 0$ for any $z \in \overline{D(z_0,r)} \setminus \{z_0\}$ and let γ be the

closed curve defined by the circle of radius r and centered at z_0 traversed in the clockwise direction. Then $\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = k$. Since f never vanishing on γ, the sequence $\left(\frac{f_n'}{f_n}\right)_n$ converges uniformly on γ to $\frac{f'}{f}$, thus

$$k = \frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \lim_{n\to+\infty}\frac{1}{2i\pi}\int_\gamma \frac{f_n'(z)}{f_n(z)}\,dz = 0,$$

which is absurd.

(b) Let $f(z) = \frac{\cos z}{z^2 \log(1+z)}$. Use the Laurent series to find the residue of f at $z = 0$.

Ph.D Qualifying Examination
Analysis (General Paper)

Dhu Al-Hijjah 1425, October 2014

Exercise 1 : [Note that parts 1) and 2) are independent]

1. Compute the following integrals $\int_0^{+\infty} \frac{dx}{1+x^3}$ and $\int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$.

2. Let f be a holomorphic function on $D \setminus \{0\}$ and $|f(z)| \leq \ln\left(\frac{1}{|z|}\right)$, for all $z \in D \setminus \{0\}$, where D is the unit disc.

   (a) Prove that 0 is a removable singularity of f. (Hint: you can consider the function $zf(z)$ and calculate its limit at 0).

   (b) Prove that for all $0 < r < 1$, $|f(z)| \leq \ln\left(\frac{1}{1-r}\right)$, for all $z \in D(0, 1-r)$.

   (c) Deduce that $f = 0$.

Exercise 2 : [Note that parts 1) and 2) are independent]

1. Consider the function defined by the power series

$$f(z) = \sum_{n=1}^{+\infty} z^{n!}.$$

   (a) Prove that f is holomorphic on the unit disc $D = \{z \in \mathbb{C};\ |z| < 1\}$.

   (b) Let $\alpha \in \mathbb{C}$ such that $\alpha^m = 1$, for some $m \in \mathbb{N}$. (α is called a root of unity).

   Prove that $\lim_{r \to 1, r<1} |f(r\alpha)| = +\infty$.

(c) Deduce that f can not be extended to a holomorpic function on an open set U such that $D \subset U$ and $D \neq U$.

2. Let (f_n) be a sequence of holomorphic functions on the unit disc D such that $f_n(D) \subset D$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} f_n(0) = 1$.

   (a) Prove that there is a subsequence $(f_{n_j})_j$ which converges uniformly on any compact to a holomorphic function g on the unit disc D and $g(0) = 1$.

   (b) We assume that g is not constant.

      i. Prove that there exists $R > 0$ such that $g - 1$ is without zeros in $D(0, R) \setminus \{0\}$.

      ii. Prove that for j sufficiently large and $|z| = r < R$, we have

$$\left|(f_{n_j}(z) - 1) - (g(z) - 1)\right| < \inf\{|g(z)-1|;\ |z| = r\}.$$

      iii. Deduce that $f_{n_j}(z)-1$ has the same number of zeros as $g - 1$ in $D(0, r)$.

      iv. Prove that $f_{n_j}(z) - 1$ is without zero on $D(0, r)$.

      v. Deduce that $g(z) = 1$ for all $z \in D$.

   (c) Prove that $(f_n)_n$ converges uniformly to 1 on any compact.

Answer of Ph.D Qualifying Examination Analysis (General Paper)

Dhu Al-Hijjah 1425, October 2014

Solution of the Exercise 1:

1. Let $f(z) = \dfrac{\log^2(z)}{1+z^3}$, $I = \displaystyle\int_0^{+\infty} \frac{dx}{1+x^3}$ and $J = \displaystyle\int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$.
   By Residue Theorem

   $$-4\mathrm{i}\pi J + 4\pi^2 I = 2\mathrm{i}\pi \left(\mathrm{Res}(f,-1) + \mathrm{Res}(f, e^{\frac{\mathrm{i}\pi}{3}}) + \mathrm{Res}(f, e^{\frac{5\mathrm{i}\pi}{3}})\right).$$

   $\mathrm{Res}(f,-1) = -\dfrac{\pi^2}{3}$.

   $\mathrm{Res}(f, e^{\frac{\mathrm{i}\pi}{3}}) = \dfrac{\pi^2}{27}(\dfrac{1}{2} + \mathrm{i}\dfrac{\sqrt{3}}{2})$.

   $\mathrm{Res}(f, e^{\frac{5\mathrm{i}\pi}{3}}) = \dfrac{25\pi^2}{27}(\dfrac{1}{2} - \mathrm{i}\dfrac{\sqrt{3}}{2})$.

   Then $-4\mathrm{i}\pi J + 4\pi^2 I = \dfrac{8\mathrm{i}\pi^3}{27} + \dfrac{8\pi^3\sqrt{3}}{9}$, $I = \dfrac{2\pi\sqrt{3}}{9}$ and $J = -\dfrac{2\pi^2}{27}$.

2. (a) For all $z \in D$, $|z^{n!}| \leq |z|^n$ and the series $\displaystyle\sum_{n\geq 1} |z|^n$ is convergent. Then f is holomorphic on the unit disc $D = \{z \in \mathbb{C};\ |z| < 1\}$.

   (b) $f(r\alpha) = \displaystyle\sum_{n=1}^{m-1} r^{n!}\alpha^{n!} + \sum_{n=m}^{+\infty} r^{n!}$. Then $\displaystyle\lim_{r\to 1, r<1} |f(r\alpha)| = +\infty$. ($\displaystyle\sum_{n=m}^{+\infty} r^{n!} \geq \sum_{n=m}^{p} r^{n!}$ for all $p > m$.)

   (c) Let U be an open set such that $D \subset U$ and $D \neq U$. There is α a root of unity in U. But $\displaystyle\lim_{r\to 1} |f(r\alpha)| = +\infty$, which is absurd. The function f can not be extended to a holomorpic function on an open set U such that $D \subset U$ and $D \neq U$.

Solution of the Exercise 2:

1. (a) $\lim_{z \to 0} z f(z) = 0$, then 0 is a removable singularity of f.

   (b) For all $z \in D(0, 1-r)$, $|f(z)| \leq \sup_{|w|=1-r} |f(w)| \ln\left(\frac{1}{1-r}\right)$.

   (c) It results that for all $z \in D$, $|f(z)| \leq \lim_{r \to 0} \ln\left(\frac{1}{1-r}\right) = 0$, then $f = 0$.

2. (a) The sequence (f_n) is bounded, then by Montel Theorem, there is a subsequence $(f_{n_j})_j$ which converges uniformly on any compact to a holomorphic function g on the unit disc D. Since $\lim_{n \to +\infty} f_n(0) = 1$, then $g(0) = 1$.

   (b) We assume that g is not constant.

      i. By Theorem of isolated zero of non constant holomorphic function, there exists $R > 0$ such that $g(z) - 1 \neq 0$ for all $z \in D(0, R) \setminus \{0\}$.

      ii. The convergence of the sequence $(f_{n_j})_j$ is uniform on the compact $\{z \in \mathbb{C};\ |z| = r < R\}$. Then for j large enough

$$\left|f_{n_j}(z) - g(z)\right| = \left|(f_{n_j}(z) - 1) - (g(z) - 1)\right| < \inf\{|g(z) - 1|;\ |z| =$$

      iii. By Theorem $f_{n_j} - 1$ and $g - 1$ have the same number of zeros on $D(0, r)$.

      iv. $f_{n_j}(z) - 1 \neq 0$ for all $z \in D(0, r)$ since $f_n(D) \subset D$, which is absurd since $g(0) = 1$.

      v. We deduce that g is constant, then $g(z) = 1$ for all $z \in D$.

   (c) Since the sequence $(f_n)_n$ is bounded and any subsequence converges to 0 in the space of holomorphic functions, then $(f_n)_n$ converges uniformly to 1 on any compact.

Ph.D Comprehensive Examination
Analysis (Special Paper)

Second semester 27-28

Exercise 1 :

1. Precise the image of the line $\{z \in \mathbb{C};\ \operatorname{Re} z = 0\}$ by the möbius transformation $f(z) = \dfrac{1}{1-z}$. Deduce the image of the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$ by the function f.

2. If Ω is a simply connected domain in $\mathbb{C}$ different from $\mathbb{C}$, justify the non existence of a conformal transformation from $\mathbb{C}$ to Ω.

3. Let $(a_n)_n$ be a sequence of complex numbers such that $\sum_{n=1}^{+\infty} \frac{1}{|a_n|} < +\infty$. Construct an entire function such that its set of zeros is equal to $\{a_n;\ n \geq 1\}$.

Question 2

For $a \in \mathbb{C}$ and $s > 0$, we consider the set $\mathcal{F}$ of family of analytic functions on a domain $\Omega \subset \mathbb{C}$ satisfying to $|f(z) - a| > s$ for all $z \in \Omega$ and all $f \in \mathcal{F}$. We consider the family

$$\mathcal{G} = \{g;\ g(z) = \frac{1}{f(z) - a},\ f \in \mathcal{F}\}.$$

1. State the definition of a normal family and prove that $\mathcal{G}$ is normal.

2. Deduce that for any sequence $(g_n)_n$ of $\mathcal{G}$, we can extract a sub-sequence that converges to a function g which is either identically equal to zero or without zero on Ω.

3. $\mathcal{F}$ is it a normal family?

Answer of Ph.D Comprehensive Examination Analysis (Special Paper)

Second semester 27-28

Solution of the Exercise 1:

1. $f(0)=1$, $f(\mathrm{i})=\dfrac{1+\mathrm{i}}{2}$ and $f(\infty)=0$, then the image of the line $\{z\in\mathbb{C};\ \operatorname{Re} z=0\}$ by f is the circle of center $\frac{1}{2}$ and radius $\frac{1}{2}$. Since $f(1)=\infty$ then the image of the half-plane $\{z\in\mathbb{C};\ \operatorname{Re} z>0\}$ by the function f is the complement of the disc of center $\frac{1}{2}$ and radius $\frac{1}{2}$.

2. Ω is a simply connected domain in $\mathbb{C}$ different from $\mathbb{C}$, then there is a conformal transformation from Ω into the unit disc. If there is a conformal transformation from $\mathbb{C}$ to Ω, we find a conformal transformation from $\mathbb{C}$ into the unit disc, which is impossible by Liouville theorem.

3. The function $f(z)=\displaystyle\prod_{n=1}^{+\infty}(1-\frac{z}{a_n})$ is an entire function and its set of zeros is equal to $\{a_n;\ n\geq 1\}$.

Solution of the Exercise 2:

1. A family $\mathcal{F}\subset\mathcal{O}(\Omega)$ is called a normal family if from any sequence $(f_n)_n\in\mathcal{F}$, we can extract a convergent sub-sequence.

   By Montel's theorem, $\mathcal{G}$ is normal since $\mathcal{G}$ is bounded.

2. Let $(g_n)_n$ be a sequence of $\mathcal{G}$, we can extract a sub-sequence that converges to a function g. Since the functions g_n are without zeros, then g is either identically equal to zero or without zero on Ω.

3. $\mathcal{F}$ is not a normal family. We can take the sequence $(f_n = n+a+s)_n$.

Ph.D Comprehensive Examination
Analysis (Special Paper)

Second semester 28-29

Exercise 1 :

1. (a) Prove that the principal determination (branch) of the argument is a continuous function on $\mathbb{C} \setminus \mathbb{R}^-$. Verify that it can not be extended continuously at any point of $\mathbb{R}^-$.

   (b) We denote by Log the principal determination (branch) of the logarithmic function and by log the determination of the logarithm defined on $\mathbb{C} \setminus i\mathbb{R}^+$, $(\theta \in]\frac{\pi}{2}, \frac{5\pi}{2}[)$. On which domain of $\mathbb{C}$, log = Log.

2. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function such that $\lim\limits_{|z| \to +\infty} |f(z)| = +\infty$.

   (a) Prove that the set of zeros of f is non empty and is a finite set.

   (b) We denote by $z_1, \ldots, z_p$ the zeros of f counted with order of multiplicity. Let $P(z) = \prod\limits_{j=1}^{p}(z - z_j)$ and $g(z) = \dfrac{P(z)}{f(z)}$.
   Prove that g extends analytically on $\mathbb{C}$ and $g(z) \neq 0$ for all $z \in \mathbb{C}$.

   (c) Use the Cauchy inequalities to prove that g is a polynomial function.

   (d) Deduce that f is a polynomial function.

3. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a proper analytic function on $\mathbb{C}$. Deduce from 2) that f is a polynomial function. (Proper means the pre-image of any compact is a compact).

Exercise 2 :

1. Let f be a holomorphic function on $D(a,r)\setminus\{a\}$. Assume that $\alpha>0$ such that $f(D(a,r)\setminus\{a\})\cap D(0,\alpha)=\emptyset$. Prove that either a is a removable singularity or a pole.
2. Determine the singularities of the function $\dfrac{z}{\sin\pi z}$ and find its corresponding residues.
3. Determine all möbius mappings transforming the half-plane $\{z\in\mathbb{C};\ \operatorname{Im} z>0\}$ onto the unit disc.
4. Evaluate the following integral $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{\mathrm{i}ax}}{x-\mathrm{i}}dx$, with $a\neq 0$.

Answer of Ph.D Comprehensive Examination Analysis (Special Paper)

Second semester 28-29

Solution of the Exercise 1:

1. (a) $z=r(\cos\theta+\mathrm{i}\sin\theta)=x+\mathrm{i}y$, with $\theta\in]-\pi,\pi[$.
   $x=r\cos\theta=2r\cos^2(\frac{\theta}{2})-r$, $y=r\sin\theta=2r\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})$.
   $x+r=2r\cos^2(\frac{\theta}{2})$ and $y=2r\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})$. Then
   $\dfrac{y}{x+\sqrt{x^2+y^2}}=\tan(\frac{\theta}{2})\Rightarrow\theta=2\tan^{-1}(\dfrac{y}{x+\sqrt{x^2+y^2}})$,
   which is a continuous function.
   For $a<0$, $\displaystyle\lim_{(x,y)\to(a,0^+)}\theta(x,y)=\pi$ and $\displaystyle\lim_{(x,y)\to(a,0^-)}\theta(x,y)=-\pi$. Then θ can not be extended continuously at any point $(a,0)$.

   (b) $\log(z)=\operatorname{Log}(z)$ for all $z\in\mathbb{C}$ such that $\operatorname{Im} z>0$ and $\operatorname{Re} z<0$.

2. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function such that $\lim\limits_{|z|\to+\infty} |f(z)| = +\infty$.

   (a) If $f(z) \neq 0$ for all $z \in \mathbb{C}$, the function $\dfrac{1}{f}$ is an entire function and $\lim\limits_{|z|\to+\infty} \dfrac{1}{|f(z)|} = 0$. Then $\dfrac{1}{f}$ is the null function, which is absurd. Moreover there is $R > 0$ such that $|f(z)| \geq 1$ for all $|z| \geq R$. Then the set of zeros of f is in the compact $\overline{D(0, R)}$, then it is finite.

   (b) The function g is analytic on $\mathbb{C} \setminus \{z_1, \ldots, z_p\}$ and each point z_j is a removable singularity of g, then g can be extended analytically on $\mathbb{C}$. Moreover by definition of the points z_j, $g(z) \neq 0$ for all $z \in \mathbb{C}$.

   (c) For $|z| \geq R$, $|g(z)| \leq |P(z)|$. Since $\lim\limits_{|z|\to+\infty} \dfrac{|P(z)|}{|z|^p} = C < +\infty$, there exists a constant $C' > 0$ such that $|g(z)| \leq C(1 + |z|)^p$. From the Cauchy's inequalities, g is a polynomial of degree less or equal then p.

   (d) Since g is zero free, thus $\deg g = 0$, this which yields that f is a polynomial.

3. Since f is proper, then $\lim\limits_{|z|\to+\infty} |f(z)| = +\infty$. (For all $R > 0$, $f^{-1}(D(0, R))$ is bounded. Then there is $R' > 0$ such that $f^{-1}(D(0, R)) \subset D(0, R')$. This is equivalent to: for all $R > 0$ there is $R' > 0$ such that for all $|z| \geq R'$, $|f(z)| \geq R$.) From 2) f is a polynomial function.

Solution of the Exercise 2:

1. Since $f(D(a, r) \setminus \{a\})$ is not dense, then a is not an essential singularity.

2. The singularities of the function $f(z) = \dfrac{z}{\sin \pi z}$ are $n \in \mathbb{Z}$. 0 is a removable singularity. $\operatorname{Res}(f, n) = \dfrac{n(-1)^n}{\pi}$, for $n \neq 0$.

3. Let f be such Möbius transformation and $\alpha \in \mathcal{H}^+ = \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$ such that $f(\alpha) = 0$, thus $f(\bar{\alpha}) = \infty$ and $f(z) = e^{\mathrm{i}\theta}\dfrac{z-\alpha}{z-\bar{\alpha}}$, with $\theta \in \mathbb{R}$.

4. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}ax}}{x-\mathrm{i}}dx = 2\mathrm{i}\pi(\mathrm{Res} f, \mathrm{i}) = 2\mathrm{i}\pi e^{-a}$, with $f(z) = \dfrac{e^{\mathrm{i}az}}{z-\mathrm{i}}$.

Ph.D Comprehensive Examination
Analysis (Special Paper)

First semester 29-30

$\mathcal{O}(D)$ denotes the space of holomorphic functions on unit disc $D = D(0,1)$ and $D^* = D \setminus \{0\}$.

Exercise 1 :

1. Let h be a holomorphic function on D.

   (a) Assume that h is injective on D. Justify that if $h(a) = 0$ for some $a \in D$, then a is a simple zero of h.

   (b) Now, assume that h is injective on D^*. Prove that $h'(0) \neq 0$ and h is necessary injective on the disc D.

2. Let f be an injective holomorphic function on D^*, a be a point in D^* and $r > 0$ be a positive real number such that $D(a,r) \subset D^*$.

   (a) Prove that there exists $\alpha > 0$ such that for all $z \in D^* \setminus D(a,r)$

   $$|f(z) - f(a)| \geq \alpha.$$

   (b) Deduce that either f extends as a holomorphic function, injective on the disc $D(0,1)$, or 0 is a simple pole of f. Give an example of a such function.

3. Let $(a_n)_n$ be a sequence of complex numbers such that the series $\sum_{n\geq 1} \frac{1}{|a_n|}$ is convergent. Prove that $f(z) = \prod_{n\geq 1}(1 - \frac{z}{a_n})$ is holomorphic on $\mathbb{C}$.

Exercise 2 :

1. Let D be a bounded domain in $\mathbb{C}$ containing the origin and f be a holomorphic function on D with $f(D) \subset D$ and $f(0) = 0$.

   (a) Prove that the sequence $(f_n'(0))_n$ is bounded, where $f_n = f \circ \dots f$ denotes the n-th iteration of f. Deduce that $|f_n'(0)| \leq 1$. (Hint: express $f_n'(0)$ as a function of $f'(0)$).

   (b) Prove that if $f'(0) = 1$, then $f = \text{id}$. (Cartan's theorem).

2. Assume that $f_k'(0) = 1$ for some $k \geq 1$. Prove that f is an automorphism of D.

3. Let $\mathcal{F} = \{f \in \mathcal{O}(D);\ f(D) \subset D \text{ and } f(0) = 0\}$.

   (a) Prove that $\mathcal{F}$ is closed in $\mathcal{O}(D)$.

   (b) Justify that $\mathcal{F}$ is a normal family of $\mathcal{O}(D)$.

Answer of Ph.D Comprehensive Examination Analysis (Special Paper)

First semester 29-30

Solution of the Exercise 1:

1. (a) If h is injective on D, then $h'(z) \neq 0$ for all $z \in D$. Then if $h(a) = 0$, a is a simple zero of h.

   (b) If $h'(0) = 0$, h can not be injective in any neighborhood of 0. If h is injective on D^*, then there is a sequence $z_n \neq 0$ and $z_n \neq z_m$ for $m \neq n$ such that $h(z_n) = h(0)$, which is absurd. Then $h'(0) \neq 0$ and h is injective in a neighborhood of 0. With the same arguments h is necessary injective on the disc D.

2. Let f be a holomorphic function on D^*, a be a point in D^* and $r > 0$ be a positive real number such that $D(a, r) \subset D^*$.

(a) f is a holomorphic function and injective on D^*, $f(D(a,r))$ is a neighborhood of $f(a)$, thus there exists $\alpha > 0$ such that $D(f(a), \alpha) \subset f(D(a,r))$. Since f is injective, then $\forall z \notin D(a,r)$, $f(z) \notin D(f(a), \alpha)$, i.e. $|f(z) - f(a)| \geq \alpha$, $\forall z \in D^* \setminus D(a,r)$.

(b) We deduce that 0 can not be an essential singularity of f. Then either f extends as a holomorphic function, injective on the disc $D(0,1)$, or 0 is a pole of f. In use the function $h = \frac{1}{f}$ and the previous question, we deduce that 0 is a simple pole of f.

As example, we take the function $f(z) = \frac{1}{z}$.

3. If the series $\displaystyle\sum_{n\geq 1} \frac{1}{|a_n|}$ is convergent, the series $\displaystyle\sum_{n\geq 1} \frac{z}{a_n}$ is uniformly convergent on any compact of $\mathbb{C}$, then $f(z) = \displaystyle\prod_{n\geq 1}(1 - \frac{z}{a_n})$ is holomorphic on $\mathbb{C}$.

Solution of the Exercise 2:

1. (a) If $D \subset D(0,R)$ and $D(0,r) \subset D$, with $r > 0$, we have:

$$f'(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} d\theta.$$

Thus $|f'(0)| \leq \frac{R}{r}$.

We prove by induction that $f_n'(0) = (f'(0))^n$. Since the sequence $(f_n'(0))_n$ is bounded, then $|f'(0)| \leq 1$.

(b) If $f'(0) = 1$, then $f_n'(0) = 1$ for all $n \in \mathbb{N}$. We assume that the expansion in power series of f is $f(z) = z + \displaystyle\sum_{n=m}^{+\infty} a_n z^n$ for $|z| < r$, with $m \geq 2$. We assume that the expansion in power series of $f^{[k]}$ is $f^{[k]}(z) = z + k a_m z^m +$

$\sum_{n=m+1}^{+\infty} a_{n,k} z^n$ and let proves the expansion in power series of $f^{[k+1]}$ is $f^{[k+1]}(z) = z + (k+1)a_m z^m + \sum_{n=m+1}^{+\infty} a_{n,k+1} z^n$.

$f \circ f^{[k]}(z) = f^{[k]}(z) + a_m (f^{[k]})^m(z) + z^{m+1} g(z)$, with g a holomorphic function on $D(0, r)$. The first term of the function $a_m (f^{[k]})^m(z)$ is $a_m z^m$ and the first term of $f^{[k]}(z) - z$ is $k a_m z^m$, thus the expansion in power series of $f^{[k+1]}$ is $f^{[k+1]}(z) = z + (k+1)a_m z^m + \sum_{n=m+1}^{+\infty} a_{n,k+1} z^n$.

$$k a_m z^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{[k]}(e^{i\theta} z) e^{-im\theta} d\theta, \quad \text{for } |z| < r.$$

Since $f^{[k]}$ is a holomorphic function from D in D, then $|f^{[k]}(e^{i\theta} z)| < R$. Thus for any $k \in \mathbb{N}$, $k|a_m| r^m < R$. Then it results that $a_m = 0$ and $f(z) = z$.

2. If $f_k'(0) = 1$, then $f_k = \text{id}$. If $k = 1$, f is an automorphism of D. If $k \geq 2$, $f \circ f_{k-1} = f_{k-1} \circ f = \text{id}$. Then f is an automorphism of D.

3. (a) If $(f_n)_n$ is a sequence in $\mathcal{F}$ and convergent to f. Since $f_n(D) \subset D$ and $f_n(0) = 0$, then $f(D) \subset D$ and $f(0) = 0$. Then $\mathcal{F}$ is closed in $\mathcal{O}(D)$.

   (b) Since $\mathcal{F}$ is bounded, then it is a normal family of $\mathcal{O}(D)$.