King Saud University

Department of Mathematics

Ph.D Qualifying Examinations with solutions

Measure Theory

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2023

Ph.D Qualifying Examination[1]
Analysis (General Paper)

2003

1. The first question.

   (a) Does there exist a Lebesgue measurable set on $(0,1)$ which is not Borel?

   (b) Let $(f_n)_n$ be a sequence of measurable functions on $(0,1)$ such that

   $$|f_n(x)| \leq \frac{1}{\sqrt{x}} \quad \text{and} \quad \lim_{n\to+\infty} f_n(x) = f(x) \ a.e.$$

   Show that

   $$\lim_{n\to+\infty} \int_0^1 e^{-x} f_n(x)dx = \int_0^1 e^{-x} f(x)dx.$$

   (c) If f is integrable on $[a,b]$, show that the function $F(x) = \displaystyle\int_a^x f(t)dt$ is absolutely continuous on $[a,b]$.

2. The second question.

   (a) On a measure space $(X, \mathscr{S}, \mu)$, consider a sequence $(E_n)_n$ of measurable sets. Show that

   $$\mu(\liminf_{n\to+\infty} E_n) \leq \liminf_{n\to+\infty} \mu(E_n).$$

   (b) On a measurable space $(X, \mathscr{S})$, let μ and ν be two signed measures such that for every $E \in \mathscr{S}$

   $$\nu(E) = \int_E f(x)d\mu(x)$$

   and

   $$|\nu|(E) = \int_E g(x)d|\mu(x)|.$$

   Show that $g = |f|$ (μ a.e.).

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(c) Let $f\colon [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x,y) = \begin{cases} y^{-2} & \text{if } 0 < x < y < 1 \\ -x^{-2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the iterated and the double integrals. Explain why the Fubini's theorem is not applicable in this example.

Ph.D Qualifying Examination
Answer Analysis (General Paper)

March 2003

1. The first question.

   (a) The Borel $\sigma-$algebra is not complete, then there is nulls subsets which are not Borel subsets.

   (b) For all $x \in (0,1)$, $|e^{-x} f_n(x)| \leq \dfrac{1}{\sqrt{x}}$ and the function $g(x) = \dfrac{1}{\sqrt{x}}$ is integrable on $(0,1)$. Then by Dominate Convergence Theorem

   $$\lim_{n \to +\infty} \int_0^1 e^{-x} f_n(x) dx = \int_0^1 e^{-x} f(x) dx.$$

   (c) Let (a_k, b_k), $k = 1, \ldots, m$ be a finite number of non overlapping intervals with $[a_k, b_k] \subset [a,b]$
   For $n \in \mathbb{N}$, define $f_n = \inf(|f|, n)$ and $A_n = \{x \in [a,b];\ |f(x)| \geq n\}$. The sequence $(f_n)_n$ increases to $|f|$, then by Monotone Convergence Theorem, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall n \geq N$,

   $$\int_{[a,b]} |f(x)| - f_n(x) dx \leq \frac{\varepsilon}{2}.$$

   Let $\delta = \frac{\varepsilon}{2N}$ and $A = \cup_{k=1}^{m} (a_k, b_k)$ a measurable subset such that $\displaystyle\sum_{k=1}^{m} b_k - a_k \leq \frac{\varepsilon}{2N}$.

   $$\begin{aligned} \int_A |f(x)| dx &= \int_A |f(x)| - f_N(x) dx + \int_A f_N(x) dx \\ &\leq \frac{\varepsilon}{2} + N \sum_{k=1}^{m} b_k - a_k \leq \varepsilon. \end{aligned}$$

   Then F is absolutely continuous on $[a,b]$.

2. The second question.

   (a) The sequence $\left(\bigcap_{k=n}^{+\infty} E_k\right)_n$ is increasing then from the Monotone Convergence Theorem: $\mu\left(\bigcup_{n=1}^{+\infty} \bigcap_{k=n}^{+\infty} E_k\right) = \lim_{n \to +\infty} \mu\left(\bigcap_{k=n}^{+\infty} E_k\right)$. But $\mu\left(\bigcap_{k=n}^{+\infty} E_k\right) \leq \inf_{k \geq n} \mu(E_k)$, then $\mu(\liminf_{n \to +\infty} E_n) \leq \liminf_{n \to +\infty} \mu(E_n)$.

(b) We recall the total variation $|\mu|$ of μ is defined by:

$$|\mu|(A) = \sup \sum_{n=1}^{+\infty} |\mu(A_n)|,$$

where the supremum is taken over all measurable partitions $(A_n)_n$ of A.
The total variation $|\mu|$ is a finite measure.
We denote $E_+ = \{x \in X;\ f(x) > 0\}$ and $E_- = \{x \in X;\ f(x) < 0\}$.
For any subsets $F \subset E_+$ and $G \subset E_-$, $|\nu|(F) = \nu(F)$ and $|\nu|(G) = -\nu(G)$, indeed:
For any measurable partition $(F_n)_n$ of F, $\sum_{n=1}^{+\infty} |\nu(F_n)| = \sum_{n=1}^{+\infty} \nu(F_n) \leq \nu(F)$. Then $|\nu|(F) \leq \nu(F)$. The converse is trivial. The other inequality is obtained by the same reasons.
For any $A \in \mathscr{S}$, $A = (A \cap E_+) \cup (A \cap E_-)$,

$$|\nu|(A) = \nu(A \cap E_+) - \nu(A \cap E_-) = \int_X |f(x)| d|\mu(x)|$$

and

$$|\nu|(E) = \int_E g(x) d|\mu(x)| = \int_E |f(x)| d|\mu(x)|.$$

Then $g = |f|$ μ a.e.

(c)

$$\begin{aligned}\int_{[0,1]} \left(\int_{[0,1]} f(x,y)dx \right) dy &= \int_0^1 \left(\int_0^y \frac{1}{y^2} dx - \int_y^1 \frac{1}{x^2} dx \right) dy \\ &= \int_0^1 \frac{1}{y} + 1 - \frac{1}{y} dy = 1.\end{aligned}$$

$$\begin{aligned}\int_{[0,1]} \left(\int_{[0,1]} f(x,y)dy \right) dx &= \int_0^1 \left(-\int_0^x \frac{1}{x^2} dx + \int_x^1 \frac{1}{y^2} dy \right) dx \\ &= \int_0^1 -\frac{1}{x} - 1 + \frac{1}{x} dx = -1.\end{aligned}$$

$$\begin{aligned}\int_{[0,1]\times[0,1]} |f(x,y)| dxdy &= \int_0^1 \left(\int_0^x \frac{1}{x^2} dx + \int_x^1 \frac{1}{y^2} dy \right) dx \\ &= \int_0^1 \frac{1}{x} - 1 + \frac{1}{x} dx = +\infty.\end{aligned}$$

The function f is not integrable.

Ph.D Qualifying Examination
Analysis (General Paper)

October 2004

1. The first question.

   (a) i.State the definition of a measurable function?

   ii. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Prove that f is measurable if and only if $\arctan f$ $(\tan^{-1} \circ f)$ is measurable. ($\mathbb{R}$ is equipped with the Borel $\sigma-$algebra.)

   iii. Let f be a differentiable function everywhere on $[0,1]$. Prove that f' is Lebesgue measurable on $[0,1]$.

   (b) i. State the definition of the L^p space, $p \geq 1$, (including L^∞).

   ii. Let $(f_n)_n$ be a sequence of functions in $L^p(X,\mu)$, $p \geq 1$ such that:

   1) $(f_n)_n$ converges a.e. to f.

   2) $\lim\limits_{n\to+\infty} ||f_n||_p = ||f||_p$.

   Prove that $f_n \longrightarrow f$ in L^p as $n \longrightarrow +\infty$. (Hint: introduce the sequence $\varphi_n = 2^{1-p}(|f|^p + |f_n|^p) - |f - f_n|^p$. Prove that $\varphi_n \geq 0$ for all n and then use Fatou lemma.)

2. The second question.

   (a) i. State and prove the continuity of property of measure.

   ii. Let A be a measurable subset of $\mathbb{R}$ such that $\lambda(A) < \infty$, where $\lambda(A)$ is the Lebesgue measure of A. Show that the function $x \longmapsto \lambda(A \cap (-\infty, x])$ is continuous.

   (b) Let μ be a measure on an algebra $U \subset 2^X$. Assume that $\mu(X) = 1$. Prove that if for $A_1, \ldots, A_n \in U$ such that $\sum\limits_{k=1}^{n} \mu(A_k) > n-1$, then $\mu(\bigcap\limits_{k=1}^{n} A_k) > 0$. (Hint: Use the fact that $\mu(A_k^c) = 1 - \mu(A_k)$.

Answer Ph.D Qualifying Examination
Analysis (General Paper)

October 2004

1. The first question.

   (a) i. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces. A mapping $f\colon X \longrightarrow Y$ is called measurable if $f^{-1}(\mathscr{B}) \subset \mathscr{A}$.
   ii. The function $\tan\colon]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ is an homeomorphism. (Continuous and its inverse is continuous).
   If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is measurable, then $\tan^{-1} \circ f$ since $\tan^{-1}$ is measurable. In the other hand if $\tan^{-1} \circ f$ is measurable, then $\tan \circ \tan^{-1} \circ f = f$ is measurable.
   iii. For $x \in (0,1)$, $f'(x) = \lim\limits_{n \to +\infty} \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$. Then f' is measurable as limit of measurable functions.

   (b) i. Let $(X, \mathscr{A}, \mu)$ be a measure space and , $1 \leq p < +\infty$. The space $L^p(\mu)$ is the set of measurable functions $f\colon X \longrightarrow \bar{\mathbb{R}}$ such that $\displaystyle\int_X |f(x)|^p d\,\mu(x) < \infty$. (The functions are defined a.e.)
   For $p = +\infty$, we say that a function $f\colon X \longrightarrow \bar{\mathbb{R}}$ is essentially bounded over X with respect to the measure μ if f is measurable and there exists $M < +\infty$ such that $|f| \leq M$ a.e. on X.
   The space $L^\infty(\mu)$ is the set of all measurable functions $f\colon X \longrightarrow \bar{\mathbb{R}}$ which are essentially bounded over X with respect to the measure μ.
   ii. The function $x \longmapsto x^p$ is convex on the interval $]0, +\infty[$, then for all $x, y \in]0, +\infty[$, $\frac{1}{2^p}|x-y|^p \leq \frac{1}{2}x^p + \frac{1}{2}y^p$. Then $\varphi_n = 2^{1-p}(|f|^p + |f_n|^p) - |f - f_n|^p \geq 0$. The sequence $(\varphi_n)_n$ converges pointwise to $2^p|f|^p$. Then by Fatou lemma

$$2^p\|f\|_p^p \leq \underline{\lim}_{n \to +\infty} \int_X \varphi_n(x) d\mu(x) = 2^p\|f\|_p^p - \overline{\lim}_{n \to +\infty} \|f_n - f\|_p^p.$$

   Then $f_n \longrightarrow f$ in L^p as $n \longrightarrow +\infty$.

2. The second question.

   (a) i. State and prove the continuity of property of measure.
   ii. For $x < y$, $0 \leq \lambda(A \cap (-\infty, x]) - \lambda(A \cap (-\infty, y]) \leq |x - y|$. Then the function $x \longmapsto \lambda(A \cap (-\infty, x])$ is continuous.

(b) Since μ is finite, $\mu(A_k^c) = 1 - \mu(A_k)$. Moreover

$$\mu(\bigcap_{k=1}^{n} A_k)^c = \mu(\bigcup_{k=1}^{n} A_k^c) \leq \sum_{k=1}^{n} \mu(A_k^c) = n - \sum_{k=1}^{n} \mu(A_k) < 1.$$

Then $\mu(\bigcap_{k=1}^{n} A_k) > 0$.

Ph.D Qualifying Examination
Analysis-Measure (General Paper)
December 2014

Section A

Problem I:

1. State the Fubini Theorem.

   Let $\Omega = (0, +\infty) \times (0, +\infty)$.

2. Compute
$$\int_\Omega \frac{d\lambda(x, y)}{(1 + y)(1 + x^2 y)},$$
   where λ is the Lebesgue measure on $\mathbb{R}^2$.

3. Deduce the values of the following integrals
$$\int_0^{+\infty} \frac{\ln(x)}{1 - x^2} dx \quad \text{and} \quad \int_0^1 \frac{\ln(x)}{1 - x^2} dx.$$

4. Prove that
$$\int_0^1 \frac{\ln(x)}{1 - x^2} dx = \sum_{n=0}^{+\infty} \int_0^1 x^{2n} \ln(x) dx$$

5. Deduce the sum of the following series
$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{(2n + 1)^2}.$$

Problem II: [Note that parts 1) and 2) are independent]

1. (a) Prove that $\mu = \sum_{n=1}^{+\infty} \delta_{\frac{1}{n}}$ is a measure on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, where $\mathscr{B}_{\mathbb{R}}$ is the Borel $\sigma-$algebra on $\mathbb{R}$.

   (b) Consider the functions $f(x) = x$ and $g(x) = x \ln(1 + |x|)$ on $\mathbb{R}$. Give the values of $p, q \in [0, +\infty)$ for which $f \in L^p(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mu)$ and $g \in L^q(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mu)$.

2. (a) Prove that the function $f(x) = \dfrac{1}{\sqrt{x(1-x)}}$ is integrable on the interval $(0,1)$ and compute the following integral $\displaystyle\int_{(0,1)} \frac{d\lambda(x)}{\sqrt{x(1-x)}}$, with λ is the Lebesgue measure on $\mathbb{R}$.

   (b) Let $f\colon (a,b) \longrightarrow \mathbb{R}$ be a bounded Lebesgue integrable function and $\lim\limits_{t\to a^+} f(t) = c$.

   Prove that for any $t \in (a,b)$, the function $x \longmapsto \frac{f(x)}{\sqrt{(x-a)(t-x)}}$ is integrable on (a,t) and compute $\displaystyle\lim_{t\to a^+} \int_{(a,t)} \frac{f(x)}{\sqrt{(x-a)(t-x)}} d\lambda(x)$.

Solution of Ph.D Qualifying Examination
Analysis-measure (General Paper)
December 2014

Section A

Problem I:

1. (The Fubini's Theorem): Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$finite measure spaces, and let $(X, \mathscr{A}, \mu)$ be the product measure space. If $f \in L^1(X, d\mu)$, then $\int_{X_2} f(x,y)d\mu_2(y) \in L^1(X_1, \mu_1)$ and $\int_{X_1} f(x,y)d\mu_1(x) \in L^1(X_2, \mu_2)$ and

$$\int_{X_1\times X_2} f(x,y)\mu_1 \otimes \mu_2(x,y) = \int_{X_2}\left(\int_{X_1} f(x,y)d\mu_1(x)\right)d\mu_2(y) \quad (0.1)$$
$$= \int_{X_1}\left(\int_{X_2} f(x,y)d\mu_2(y)\right)d\mu_1(x)$$

Let $\Omega = (0, +\infty) \times (0, +\infty)$.

2. The function $(x,y) \longmapsto 1(1+y)(1+x^2y)$ is non negative continuous function, then by Fubini Tonelli Theorem

$$\int_\Omega \frac{d\lambda(x,y)}{(1+y)(1+x^2y)} = \int_0^{+\infty}\left(\int_0^{+\infty} \frac{dx}{(1+y)(1+x^2y)}\right)dy$$
$$= \int_0^{+\infty}\left(\int_0^{+\infty} \frac{dy}{(1+y)(1+x^2y)}\right)dx.$$

$\int_0^{+\infty} \frac{dx}{(1+x^2y)} = \frac{\pi}{2\sqrt{y}}$ and $\int_0^{+\infty} \frac{dy}{2\sqrt{y}(1+y)} \overset{y=t^2}{=} \frac{\pi^2}{2}$.

For $x \neq 1$, $\frac{1}{(1+y)(1+x^2y)} = \frac{A}{1+y} - \frac{x^2A}{1+x^2y}$, with $A = \frac{1}{1-x^2}$. Then $\int_0^{+\infty} \frac{dy}{(1+y)(1+x^2y)} = A\ln(\frac{1+y}{1+x^2y})\Big]_0^{+\infty} = -\frac{2\ln x}{1-x^2}$.

3. By Fubini Tonelli Theorem $\int_0^{+\infty} \frac{\ln(x)}{1-x^2}dx = -\frac{\pi^2}{4}$. Moreover by the change of variable $x = \frac{1}{t}$, $\int_0^1 \frac{\ln(x)}{1-x^2}dx = \int_1^{+\infty} \frac{\ln(x)}{1-x^2}dx = -\frac{\pi^2}{8}$.

4. $\dfrac{1}{1-x^2} = \sum_{n=0}^{+\infty} x^{2n}$ and by Monotone Convergence Theorem ($x^{2n}\ln(x) \leq 0$)

$$\int_0^1 \frac{\ln(x)}{1-x^2}dx = \sum_{n=0}^{+\infty}\int_0^1 x^{2n}\ln(x)dx.$$

5. By integration by parts $\int_0^1 x^{2n}\ln(x)dx - \dfrac{1}{(2n+1)^2}$. Then $\sum_{n=0}^{+\infty}\dfrac{1}{(2n+1)^2} = \dfrac{\pi^2}{8}$. $\sum_{n=1}^{+\infty}\dfrac{1}{n^2} = \dfrac{1}{4}\sum_{n=1}^{+\infty}\dfrac{1}{n^2} + \sum_{n=0}^{+\infty}\dfrac{1}{(2n+1)^2}$. Then $\sum_{n=1}^{+\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

Problem II:

1. (a) We know that if $(\mu_n)_n$ is an increasing sequence of measures on a measurable space $(X,\mathscr{A})$, the mapping $\mu\colon \mathscr{A} \longrightarrow [0,+\infty]$ defined by $\mu(A) = \lim_{n\to+\infty}\mu_n(A) = \sup_n \mu_n(A)$ for any $A\in\mathscr{A}$ is a measure on X.

   Indeed it is clear that $\mu(\emptyset) = 0 = \lim_{n\to+\infty}\mu_n(\emptyset)$, and if A, B are two disjoints measurable subsets, we have

$$\mu(A\cup B) = \lim_{n\to+\infty}\mu_n(A) + \lim_{n\to+\infty}\mu_n(B) = \mu(A)+\mu(B).$$

   Let now $(A_n)_n$ be an increasing sequence of $\mathscr{A}$ and $A = \bigcup_{n=1}^{+\infty} A_n$. We have $\mu_j(A_n) \leq \mu(A_n) \leq \mu(A)$. Then

$$\mu_j(A) = \lim_{n\to+\infty}\mu_j(A_n) \leq \lim_{n\to+\infty}\mu(A_n) \leq \mu(A)$$

   and then

$$\mu(A) = \lim_{j\to+\infty}\mu_j(A) \leq \lim_{n\to+\infty}\mu(A_n) \leq \mu(A).$$

   Then $\mu(A) = \lim_{n\to+\infty}\mu(A_n)$.

   Then $\mu_1 = \lim_{n\to+\infty}\sum_{k=1}^{n}\delta_{\frac{1}{k}}$ is a measure on the measurable space $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$.

   (b) $\int_{\mathbb{R}} f^p(x)d\mu_1(x) = \sum_{n=1}^{+\infty}\dfrac{1}{n^p}$. Then $f\in L^p(\mathbb{R},\mathscr{B}_{\mathbb{R}},\mu_1)$ if and only if $p>1$.

$\int_{\mathbb{R}} g^q(x)d\mu_1(x) = \sum_{n=1}^{+\infty} \frac{\ln^q(1+\frac{1}{n})}{n^q}$. Since $\frac{\ln^q(1+\frac{1}{n})}{n^q} \approx \frac{1}{n^{2q}}$, then $g \in L^q(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mu_1) \iff q > \frac{1}{2}$.

2. (a) In a neighborhood of 0, $f(x) \approx \frac{1}{\sqrt{x}}$, which is integrable and in a neighborhood of 1, $f(x) \approx \frac{1}{\sqrt{1-x}}$, which is integrable.

$$\int_{(0,1)} \frac{d\lambda(x)}{\sqrt{x(1-x)}} \overset{x=t^2}{=} \int_0^1 \frac{2dt}{\sqrt{1-t^2}} = \pi.$$

(b) In a neighborhood of a in (a,t), $\frac{1}{\sqrt{(x-a)(t-x)}} \approx \frac{1}{\sqrt{(x-a)(t-a)}}$, which is integrable and in a neighborhood of t in (a,t), $\frac{1}{\sqrt{(x-a)(t-x)}} \approx \frac{1}{\sqrt{(t-a)(t-x)}}$, which is integrable. Moreover since f is bounded then for any $t \in (a,b)$, the function $x \longmapsto \frac{f(x)}{\sqrt{(x-a)(t-x)}}$ is integrable on (a,t).

$$\int_{(a,t)} \frac{1}{\sqrt{(x-a)(t-x)}} d\lambda(x) \overset{x=st+(1-s)a}{=} \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi.$$

Since f is bounded, then by Dominated Convergence Theorem

$$\lim_{t\to a^+} \int_{(a,t)} \frac{f(x)}{\sqrt{(x-a)(t-x)}} d\lambda(x) \overset{x=st+(1-s)a}{=} \int_0^1 \frac{f(a+s(t-a))}{\sqrt{s(1-s)}} ds = \pi c.$$

Ph.D Qualifying Examination
Analysis (General Paper)

1424-1425

Question 5

1. Let Ω be a non-countable set. If $\mathcal{D}$ is the class of all singleton sets $\{x\}$. Find the σ-algebra generated by $\mathcal{D}$.
2. Let (ω_j) be a sequence in Ω and (p_j) be a sequence of positive real numbers. Suppose μ is the measure defined by $\mu(E) = \sum_{j,\omega_j\in E} p_j$ on the class of all subsets of Ω. Show that a function $f\colon \Omega \longrightarrow \bar{\mathbb{R}}$ is integrable with respect to μ if and only if $\sum_{j=1}^{\infty} p_j f(\omega_j)$ is absolutely convergent and that if f is integrable, then $\int_\Omega f(x)d\mu(x) = \sum_{j=1}^{\infty} p_j f(\omega_j)$.

Question 6

1. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$. Show that $D = \{(x,x); x \in [0,1]\}$ is measurable with respect to the σ-algebra $\mathcal{B} \oplus \mathcal{B}$.

   If μ is the counting measure on $\mathcal{B}$ (so that $\mu(\mathcal{B})$ is the number of elements of $\mathcal{B}$), λ is the Lebesgue measure and $h = \chi_D$, show that

   $$\int_0^1 \int_0^1 h(x,y)d\lambda(x)d\mu(y) \neq \int_0^1 \int_0^1 h(x,y)d\mu(y)d\lambda(x)$$

   why doesn't this contradict Fubini's theorem?

2. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. If $f \in L^1(\Omega, \mathcal{F}, \mu)$, use the Radon-Nykodym theorem to show the existence of a function $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that

   $$\int_E f(x)d\mu(x) = \int_E g(x)d\mu(x), \ \forall E \in \mathcal{G}.$$

Ph.D Qualifying Examination
Answer Analysis (General Paper)

1424-1425

Question 5

1. The σ-algebra generated by $\mathcal{D}$ is the set of countable subsets of Ω or their complement is countable.

2. $\mu(E) = \sum_{j=1}^{+\infty} p_j \chi_E(w_j)$. If f is a non negative simple function, $f = \sum_{j=1}^{m} \lambda_j \chi_{E_j}$,

$$\int_\Omega f(x)d\mu(x) = \sum_{j=1}^{m} \lambda_j \sum_{k=1}^{+\infty} p_k \chi_{E_j}(w_k) = \sum_{k=1}^{+\infty} p_k f(w_k).$$

If f is non negative measurable, there exists an increasing sequence of simple functions which converges to f, then by Monotone Convergence Theorem,

$$\int_\Omega f(x)d\mu(x) = \sum_{k=1}^{+\infty} p_k f(w_k).$$

Then f is integrable with respect to μ if and only if $\sum_{j=1}^{\infty} p_j f(\omega_j)$ is absolutely convergent and if f is integrable, then $\int_\Omega f(x)d\mu(x) = \sum_{j=1}^{\infty} p_j f(\omega_j)$.

Question 6

1. $D = \{(x,x); x \in [0,1]\}$ is a closed set, then D is measurable with respect to the σ-algebra $\mathcal{B} \oplus \mathcal{B}$.
   $\int_0^1 h(x,y)d\lambda(x) = 0$, then $\int_0^1 \left(\int_0^1 h(x,y)d\lambda(x)\right) d\mu(y) = 0$.
   $\int_0^1 h(x,y)d\mu(y) = 1$, then $\int_0^1 \left(\int_0^1 h(x,y)d\mu(y)\right) d\lambda(x) = \int_0^1 d\lambda(x) = 1$.
   This not contradict Fubini's theorem since μ is not a $\sigma-$finite measure.

2. The measure μ is finite ($\mu(\Omega) = 1$) and the measure $f\mu$ is absolutely continuous with respect to the measure μ on the measure space $(\Omega, \mathcal{F}, \mu)$. (If $A \in \mathcal{G}$ is a null set, then it is a null set in $\mathcal{F}$ and $\int_A f(x)d\mu(x) = 0$). In use of the Radon-Nykodym theorem there is a function $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that

$$\int_E f(x)d\mu(x) = \int_E g(x)d\mu(x), \ \forall E \in \mathcal{G}.$$

Ph.D Comprehensive Examination
Analysis

1425-1426

Question 5

1. Given a measure μ_0 on a ring $\mathcal{R}$, describe without proofs, how μ_0 can be extended to a measure on the σ-ring $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.

   Let $\Omega = \mathbb{Q} \cap [0,1)$, $\mathcal{R}$ be the ring of all finite disjoint unions of subsets of Ω of the form $\mathbb{Q} \cap [a,b)$ and μ_0 be the counting measure on $\mathcal{R}$.

   i) Show that $\sigma(\mathcal{R})$ is the class $\mathcal{P}(\Omega)$ of all subsets of Ω.

   ii) If μ_1 is the counting measure on $\mathcal{P}(\Omega)$ and $\mu_2 = 2\mu_1$, show that μ_1 and μ_2 are distinct σ-finite extensions of μ_0 to $\sigma(\mathcal{R})$. Why doesn't this contradict the uniqueness of the extension?

2. Given a measure space $(\Omega, \mathcal{F}, \mu)$ and a measurable function $f\colon \Omega \longrightarrow \bar{\mathbb{R}}$, describe without proofs how $\int_\Omega f d\mu$ is defined, when it exits.

   Let, for $\mathrm{i} = 1,2$, $(\Omega_\mathrm{i}, \mathcal{F}_\mathrm{i})$ be a measurable space, and suppose that $T\colon \Omega_1 \longrightarrow \Omega_2$ is measurable with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$. If μ is a measure on $\mathcal{F}_1$ and $g\colon \Omega_2 \longrightarrow \mathbb{R}$ is $\mathcal{F}_2$ measurable, show that μT^{-1} is a measure on $\mathcal{F}_2$ and

   $$\int_{\Omega_1} g \circ T(x) d\mu(x) = \int_{\Omega_1} g(x) d\mu T^{-1}(x)$$

   in the sense that either side exist, so does the other and the two are equal.

Question 6

1. $(a_{n,m})$ be a double sequence of non-negative numbers. Employing the counting measure on $\mathbb{N}$, use the Fubini-Tonelli theorem to prove that

   $$\sum_{n,m=1}^{\infty} a_{n,m} = \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} a_{n,m}\Big) = \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} a_{n,m}\Big)$$

   what can you say if we relax the requirement that $a_{n,m} \geq 0$, $\forall n,m \in \mathbb{N}$?

2. Let μ, ν and λ the signed measures on $(\Omega, \mathcal{F})$. If $\mu << \nu$ and $\nu << \lambda$, prove that

   $$\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu}.\frac{d\nu}{d\lambda}.$$

Answer Ph.D Comprehensive Examination
Analysis

1425-1426

Question 5

1. Given a measure μ_0 on a ring $\mathcal{R}$, for all $A \in \sigma(\mathcal{R})$, we define

$$\mu(A) = \inf\{\sum_{n=1}^{+\infty} \mu_0(A_n);\ A_n \in \mathcal{R},\ \forall n \in \mathbb{N},\ A \subset \cup_{n=1}^{+\infty} A_n\}.$$

Let $\Omega = \mathbb{Q} \cap [0,1)$, $\mathcal{R}$ the ring of all finite disjoint unions of subsets of Ω of the form $\mathbb{Q} \cap [a,b)$ and μ_0 the counting measure on $\mathcal{R}$.

i) For all $a \in \mathbb{Q}$, $\{a\} = \cap_{n=1}^{+\infty} \mathbb{Q} \cap [a, a+\frac{1}{n}[$. Then $\sigma(\mathcal{R}) = \mathcal{P}(\Omega)$.

ii) Since $\mu_1 \neq 0$, then $\mu_2 \neq \mu_1$. Moreover since $\mathbb{Q}$ is countable, μ_1 and μ_2 are σ-finite.

For every $A \in \mathcal{R}$, $A \neq \emptyset$, $\sigma_0(A) = \sigma_1(A) = \sigma_2(A) = +\infty$. Then μ_1 and μ_2 are extension of μ_0 on $\sigma(\mathcal{R})$. We don't have the uniqueness since μ_0 is not $\sigma-$finite on $\mathcal{R}$

2. We define the integral of non negative simple function $f = \sum_{j=1}^{m} c_j \chi_{A_j}$, where $c_j \neq c_k$ for $j \neq k$ and $(A_j)_j$ measurable subsets. We define

$$\int_\Omega f(x)d\mu(x) = \sum_{j=1}^{m} c_j \mu(A_j).$$

If f is a non-negative measurable function, there exists a sequence of non-negative simple functions $(f_j)_j$ which increases to f. We define

$$\int_\Omega f(x)d\mu(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\mu(x).$$

If f is a measurable function, we define $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$. If $\int_\Omega f^+(x)d\mu(x) < +\infty$ or $\int_\Omega f^-(x)d\mu(x) < +\infty$, we define

$$\int_\Omega f(x)d\mu(x) = \int_\Omega f^+(x)d\mu(x) - \int_\Omega f^-(x)d\mu(x).$$

We denote $\nu = \mu T^{-1}$.

Since $T^{-1}(\emptyset) = \emptyset$ and $\mu(\emptyset) = 0$, then $\nu(\emptyset) = 0$.

If $(A_n)_n$ is a sequence of $\mathcal{F}_2$ measurable sets,

$$\nu(\cup_{n=1}^{+\infty} A_n) = \mu(T^{-1}(\cup_{n=1}^{+\infty} A_n)) = \mu(\cup_{n=1}^{+\infty} T^{-1}(A_n)) = \lim_{n\to+\infty} \nu(A_n).$$

If g is a simple function,

$$\int_{\Omega_1} g \circ T(x) d\mu(x) = \int_{\Omega_1} g(x) d\mu T^{-1}(x).$$

If g is a non negative measurable function, the result is obtained by Monotone Convergence Theorem.

Question 6

1. If μ is the counting measure on $\mathbb{N}$ and g a non negative measurable function,

$$\int_{\mathbb{N}} g(x) d\mu(x) = \sum_{n=1}^{+\infty} g(n).$$

Define the function f on $\mathbb{N} \times \mathbb{N}$ by $f(n,m) = a_{n,m}$. By Fubini-Tonelli theorem

$$\sum_{n,m=1}^{\infty} a_{n,m} = \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} a_{n,m}\Big) = \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} a_{n,m}\Big)$$

If $a_{n,m}$ are not non negative, we use the Fubini theorem if $\sum_{n,m=1}^{\infty} |a_{n,m}| < +\infty$.

2. Since $\mu << \nu$, there is $f \in L^1(\nu)$ such that $\mu = f\nu$ and since $\nu << \lambda$, there is $g \in L^1(\lambda)$ such that $\nu = g\lambda$.

If A is a null set with respect to the measure λ, then since $\nu << \lambda$, A is a null set with respect to the measure ν and since $\mu << \nu$, A is a null set with respect to the measure μ. Then $\mu << \lambda$ and $\mu = f\nu = fg\lambda$. Then

$$\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} . \frac{d\nu}{d\lambda}.$$

Ph.D Comprehensive Examination
Analysis

1425-1426- Second semester

Question 3

a) Let $(X, \mathscr{M}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathscr{M} : \ \mu(N) = 0\}$ and $\overline{\mathscr{M}} = \{E \cup F, \ E \in \mathscr{M}$ and $F \subset N$ for some $N \in \mathcal{N}\}$.

i) Show that $\overline{\mathscr{M}}$ is a $\sigma-$algebra.

ii) Verify that the extension $\bar{\mu}$ of μ on $\overline{\mathscr{M}}$ is a complete measure.

b) i) State the definition of an outer measure.

ii) Let X be a space. We consider $\mathscr{M} \subset \mathscr{P}(X)$ an algebra of sets and f a non negative function defined on $\mathscr{M}$, such that $f(\emptyset) = 0$. For any $A \subset X$, define

$$\mu(A) = \inf\{\sum_{j=1}^{+\infty} f(E_j); \ E_j \in \mathscr{M} \text{ and } A \subset \cup_{j=1}^{+\infty} E_j\}.$$

Show that μ is an outer measure.

c) If $\mu_1, \ldots, \mu_n$ are measure on $(X, \mathscr{M})$ and $a_1, \ldots, a_n$ positive numbers. Prove that $\mu = \sum_{j=1}^{n} a_j \mu_j$ is a measure on $(X, \mathscr{M})$.

Question 4

a) Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be $\sigma-$finite measure spaces. Prove that if $E \in \mathscr{M} \otimes \mathscr{N}$, then the functions $x \in X \longrightarrow \nu(E_x)$ and $y \in Y \longrightarrow \mu(E^y)$ are measurable on X and Y respectively, and

$$\mu \otimes \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

(*recall that $E_x = \{y \in Y; \ (x, y) \in E\}$ and $E^y = \{x \in X; \ (x, y) \in E\}$.)

b) Let $X = [0, 1]$, $\mathscr{B}$ the Borel $\sigma-$algebra on $[0, 1]$, μ is the Lebesgue measure and ν the counting measure on $\mathscr{B}$ (if $B \in \mathscr{B}$, $\nu(B)$ is the number of elements of B). Let $D = \{(x, y) \in X \times X : x = y\}$.

i) Show that D is measurable with respect to the $\sigma-$algebra $\mathscr{B} \otimes \mathscr{B}$.

ii) Show that $\int_0^1 \int_0^1 \chi_D(x, y) d\mu(x) d\nu(y) \neq \int_0^1 \int_0^1 \chi_D(x, y) d\nu(y) d\mu(x)$. Explain why these integrals are not equal?

Answer Ph.D Comprehensive Examination Analysis

1425-1426- Second semester

Question 3

a) i) $\overline{\mathscr{M}}$ is closed under countable union. It remains to prove that it is closed under complementarity. Let $A' = A \cup N$ be an element of $\overline{\mathscr{M}}$. As N is a null set there exists a subset B of $\mathscr{M} \cap \mathcal{N}$ and $N \subset B$. We have

$$A'^c = (A \cup N)^c = (A \cup B)^c \cup (B \setminus (A \cup N)).$$

It follows that A'^c is an element of $\overline{\mathscr{M}}$ and $\overline{\mathscr{M}}$ is a $\sigma-$algebra.

ii) To show that $\overline{\mu}$ is a mapping on $\overline{\mathscr{M}}$, we must show that if $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathscr{M}$ and $N_1, N_2 \in \mathcal{N}$, then $\mu(A_1) = \mu(A_2)$. So we have $A_1 \setminus A_2 \subset N_2$, then it is a null set. If $B = A_1 \cap A_2$, then $A_1 = B \cup (A_1 \setminus A_2)$ and $\mu(B) = \mu(A_1)$. In the same way we have $\mu(B) = \mu(A_2)$, then $\mu(A_1) = \mu(A_2)$. Let prove now that $\overline{\mu}$ defines a measure on the σ-algebra $\overline{\mathscr{M}}$. If $(A'_n)_n$ is a sequence of disjoint elements of $\overline{\mathscr{M}}$, with $A'_n = A_n \cup N_n$, $A_n \in \mathscr{M}$ and $N_n \in \mathcal{N}$; $\forall n \in \mathbb{N}$. We have

$$\overline{\mu}(\bigcup_{n=1}^{+\infty} A'_n) = \overline{\mu}\Big((\bigcup_{n=1}^{+\infty} A_n) \cup (\bigcup_{n=1}^{+\infty} N_n)\Big) = \mu(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n) = \sum_{n=1}^{+\infty} \overline{\mu}(A'_n).$$

Finally the measure space $(X, \overline{\mathscr{M}}, \overline{\mu})$ is complete because the ν-null sets are elements of $\mathcal{N}$. It is evident that $\overline{\mu}$ is the smallest complete extension of the measure μ.

b) i) Let X be a non empty set. An outer measure or an exterior measure μ^* on X is a function $\mu^* \colon \mathscr{P}(X) \longrightarrow [0, \infty]$ which satisfies the following conditions:

- $\mu^*(\emptyset) = 0$.
- If $(A_n)_n$ is a sequence of subsets of X, then

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

- μ^* is increasing (i.e. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$).

ii) $\mu(\emptyset) \leq f(\emptyset) = 0$, then and μ^* is increasing.
Let $(A_n)_n$ be a sequence of subsets of X. We claim that

$$\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$

If there exists a subset A_n such that $\mu(A_n) = +\infty$, then the inequality is trivial. Assume now that $\forall n \in \mathbb{N}$, $\mu(A_n) < +\infty$.
For every $n \in \mathbb{N}$, and for every $\varepsilon > 0$, there exists a sequence $(A_{n,j})_j \in \mathscr{M}$, such that $\mu(A_n) \geq \sum_{j=1}^{+\infty} f(A_{n,j}) - \frac{\varepsilon}{2^n}$. The sequence $(A_{n,j})_{j,n\in\mathbb{N}}$ is a covering of the set $A = \bigcup_{j=1}^{+\infty} A_n$ and $\sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty} f(A_{n,j}) \leq \sum_{n=1}^{+\infty} \mu(A_n) + \varepsilon$. Then $\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n) + \varepsilon$, for all $\varepsilon > 0$ and thus $\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n)$, which proves that μ is an outer measure.

c) i) $\mu(\emptyset) = \sum_{j=1}^{n} a_j \mu_j(\emptyset) = 0$,

ii) If $A \cap B = \emptyset$ and $A, B \in \mathscr{M}$, $\mu(A \cup B) = \sum_{j=1}^{n} a_j \mu_j(A \cup B) = \sum_{j=1}^{n} a_j(\mu_j(A) + \mu_j(B)) = \mu(A) + \mu(B)$.

iii) If $(A_n)_n$ is an increasing sequence of the σ-algebra $\mathscr{M}$,

$$\mu(\bigcup_{k=1}^{+\infty} A_k) = \sum_{j=1}^{n} a_j \mu_j(\bigcup_{k=1}^{+\infty} A_k) = \sum_{j=1}^{n} a_j \lim_{k\to+\infty} \mu_j(A_k) = \lim_{k\to+\infty} \mu(A_k).$$

Then μ is a measure on $(X, \mathscr{M})$.

Question 4

a) Suppose in the first case that ν is finite and define

$$\mathscr{A} = \{C \in \mathscr{M} \otimes \mathscr{N}\,;\; x \longmapsto \nu(C_x) \text{ is measurable } \}.$$

$\mathscr{A}$ contains the measurable rectangles $C = A \times B$, with $A \in \mathscr{M}$ and $B \in \mathscr{N}$, since $\nu(C_x) = \chi_A(x)\nu(B)$. Moreover $\mathscr{A}$ is a monotone class: if $C \subset C'$, $\nu(C' \setminus C)_x = \nu(C'_x) - \nu(C_x)$ since ν is finite, and if $(C_n)_n$ is an increasing sequence

$$\nu(\cup_{k=1}^{+\infty} C_n)_x = \lim_{n\to+\infty} \nu(C_n)_x.$$

Then $\mathscr{A} = \mathscr{M} \otimes \mathscr{N}$.

If ν is $\sigma-$finite, we take a sequence $(B_n)_n$ such that $\nu(B_n) < +\infty$ for all $n \in \mathbb{N}$, $\nu(B_n) < +\infty$ and $X = \cup_{n=1}^{+\infty} B_n$. We define $\nu_{2,n}(B) = \nu(B \cap B_n)$. Then $\nu(C_x) = \lim_{n\to+\infty} \mu_{2,n}(C_x)$ which is measurable.
Define for all $E \in \mathscr{M} \otimes \mathscr{N}$,

$$\mu \otimes \nu(E) = \int_X \nu(E_x) d\mu(x).$$

To prove that $\mu \otimes \nu$ is a measure on $\mathscr{M} \otimes \mathscr{N}$, let $(C_n)_n$ be a sequence of disjoint measurable subsets in $\mathscr{M} \otimes \mathscr{N}$, the sequence $((C_n)_x)_n$ is disjoint for all $x \in X$ and

$$\begin{aligned}
\mu \otimes \nu(\cup_{n=1}^{+\infty} C_n) &= \int_X \nu(\cup_{n=1}^{+\infty} (C_n)_x) d\mu(x) \\
&= \int_X \sum_{n=1}^{+\infty} \nu((C_n)_x) d\mu(x) \\
&= \sum_{n=1}^{+\infty} \int_X \nu((C_n)_x) d\mu(x) = \sum_{n=1}^{+\infty} \mu \otimes \nu(C_n).
\end{aligned}$$

Moreover $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$.
In the same way, if we define

$$\mu \tilde{\otimes} \nu(C) = \int_Y \mu(C^y) d\nu(y).$$

$\mu \tilde{\otimes} \nu$ is a measure on $\mathscr{M} \otimes \mathscr{N}$ and fulfills $\mu \tilde{\otimes} \nu(A \times B) = \mu(A)\nu(B)$. We deduce that $\mu \otimes \nu = \mu \tilde{\otimes} \nu$ and

$$\mu \otimes \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

b) i) $D = \{(x, x); x \in [0, 1]\}$ is a closed set, then D is measurable with respect to the σ-algebra $\mathscr{B} \oplus \mathscr{B}$.

ii) $\int_0^1 h(x, y) d\lambda(x) = 0$, then $\int_0^1 \left(\int_0^1 h(x, y) d\lambda(x) \right) d\mu(y) = 0$.
$\int_0^1 h(x, y) d\mu(y) = 1$, then $\int_0^1 \left(\int_0^1 h(x, y) d\mu(y) \right) d\lambda(x) = \int_0^1 d\lambda(x) = 1$. This not contradict Fubini's theorem since μ is not $\sigma-$finite.

Ph.D Comprehensive Examination
Analysis (General Paper)

First semester 1426-1427

Section A

I) a) Let f be the function defined on $]0,+\infty[$ by: $f(x) = \dfrac{xe^{-ax}}{1-e^{-bx}}$, with a and b in $]0,+\infty[$.

Show that f is integrable on $[0,+\infty[$ and $\displaystyle\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \frac{1}{(a+nb)^2}$.

b) State the definition of the Borel σ-algebra on the real line $\mathbb{R}$.

II) a) Let $(X, \mathcal{B}, \mu)$ be a measure space and let f be a function defined on X. If f is μ-integrable, show that the set $\{x \in X;\ f(x) \neq 0\}$ is of σ-finite measure.

b) State the Fubini theorem with respect the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, where $X = Y = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non negative integers, $\mathcal{A} = \mathcal{B} = \mathcal{P}(\mathbb{N}_0)$ and $\mu = \nu$ the counting measure.

Answer Ph.D Comprehensive Examination Analysis (General Paper)

First semester 1426-1427

Section A

I) a) For $x > 0$, $f(x) = \dfrac{xe^{-ax}}{1-e^{-bx}} = \sum_{n=0}^{+\infty} xe^{-(a+nb)x}$.

f is continuous on $[0, +\infty[$ and non negative. ($f(0) = \lim_{x\to 0} f(x) = \dfrac{1}{b}$). Moreover $f(x) \le 2xe^{-ax}$ for x large, which is integrable. Then f is integrable.
By the Monotone Convergence Theorem or the Dominate Convergence Theorem, $\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \int_0^{+\infty} xe^{-(a+nb)x}dx = \sum_{n=0}^{+\infty} \dfrac{1}{(a+nb)^2}$.

b) The Borel σ-algebra on the real line $\mathbb{R}$ is the σ-algebra generated by the open subsets of $\mathbb{R}$.

II) a) For all $n \in \mathbb{N}$ define the set $E_n = \{x \in X;\ |f(x)| \ge \frac{1}{n}\}$.
$\mu(E_n) = \int_{E_n} d\mu(x) \le n \int_X |f(x)| d\mu(x) = n\|f\|_1 < +\infty$. Then the set $\{x \in X;\ f(x) \ne 0\}$ is σ-finite.

b) (The Fubini's Theorem): Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two $\sigma-$ finite measure spaces, and let $(X \times Y, \mathscr{A} \otimes \mathscr{B}, \mu \otimes \nu)$ be the product measure space. If $f \in L^1(X \times Y, d(\mu \otimes \nu))$, then $\int_Y f(x,y) d\,\nu(y) \in L^1(X, \mu)$ and $\int_X f(x,y) d\,\mu(x) \in L^1(Y, \nu)$ and

$$\begin{aligned}\int_{X\times Y} f(x,y) d\mu \otimes \nu(x,y) &= \int_Y \left(\int_X f(x,y) d\,\mu(x) \right) d\,\nu(y) \\ &= \int_X \left(\int_Y f(x,y) d\,\nu(y) \right) d\,\mu(x)\end{aligned}$$

Consider the special case where $X = Y = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non negative integers, $\mathcal{A} = \mathcal{B} = \mathcal{P}(\mathbb{N}_0)$ and $\mu = \nu$ the counting measure.
Let $(a_{m,n})_{m,n}$ be a sequence of real numbers. Then the Fubini-Tonelli theorem says that if $a_{m,n} \ge 0$ for all $m, n \in \mathbb{N}$, then

$$\sum_{m=0}^{+\infty}\left(\sum_{n=0}^{+\infty}a_{m,n}\right)=\sum_{n=0}^{+\infty}\left(\sum_{m=0}^{+\infty}a_{m,n}\right).$$

The Fubini theorem says that if $\sum_{m=0}^{+\infty}\left(\sum_{n=0}^{+\infty}|a_{m,n}|\right)<+\infty$, then

$$\sum_{m=0}^{+\infty}\left(\sum_{n=0}^{+\infty}a_{m,n}\right)=\sum_{n=0}^{+\infty}\left(\sum_{m=0}^{+\infty}a_{m,n}\right).$$

Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1429-1430 H Time 3 hours

Section B

III)

1. Let $(X, \mathscr{B}, \mu)$ be a measure space and let $(A_n)_n$ be a decreasing sequence of $\mathscr{B}$. Assume that μ is a finite.

   Prove that $\lim\limits_{n\to+\infty} \mu(A_n) = \mu(\lim\limits_{n\to+\infty} A_n)$.

2. Give an example of a measure space $(X, \mathscr{B}, \mu)$ and a decreasing sequence $(A_n)_n$ such that $\lim\limits_{n\to+\infty} \mu(A_n) \neq \mu(\lim\limits_{n\to+\infty} A_n)$.

3. a) Prove that for $n \geq 2$ and $x \geq 0$, we have $(1+\frac{x}{n})^n \geq \frac{x^2}{4}$, and find the following limit $\lim\limits_{n\to+\infty} \int_0^{+\infty} \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} dx$.

   b) Find the Lebesgue integral on $[0,1]$ of the function f defined by: $f(x) = \frac{1}{\sqrt{x}} + \chi_{\mathbb{Q}}(x)$, for $x \neq 0$ and $f(0) = 0$.

   c) Consider the function $g(x) = \frac{1}{(1+x^2)\sqrt{|\sin x|}}$, for $x \notin \pi\mathbb{N}$ and $g(n\pi) = 0$, for all $n \in \mathbb{N}$.
   Show that the following function g is Lebesgue integrable on $(0, +\infty)$.

IV)

1. Let $f\colon \mathbb{R} \longrightarrow [0,\infty)$ be defined as follows: $f(x) = \begin{cases} \frac{1}{x(\log x)^2} & \text{if } x \in (0, e^{-1}), \\ 0 & \text{if } x \notin (0, e^{-1}). \end{cases}$

   a) Check that $\int_{(0,x)} f(t)dt = \frac{-1}{\log x}$ for $x \in (0, e^{-1})$. Deduce that $f \in L^1(\mathbb{R})$.

b) Consider the maximal function $\mathcal{M}$ defined by $\mathcal{M}f(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)|dt$, ($I$ is an open interval and $|I|$ is the length of I).

Conclude that $\int_{(0,r)} \mathcal{M}f(x)dx = \infty$, for every $r > 0$.

2. Let $(\mathbb{X}, \mathscr{B}, \mu)$ be a measure space such that $\mu(\mathbb{X}) = 1$. Let L^p denote $L^p(\mathbb{X}, \mathscr{B}, \mu)$ for $1 \leq p \leq \infty$.

   a) Show that $L^q \subset L^p$ if $1 \leq p \leq q$.

   b) Use a) to show that $L^p \cap L^q \subset L^s$ if $1 \leq p \leq s \leq q \leq \infty$.

   c) Show that if $f \in L^\infty$, then $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

   d) Now, suppose that $\mu(\mathbb{X})$ is not necessarily finite. Put $s = tp + (1-t)q$ for $t \in [0,1]$ and apply Hölder's inequality to $\int_{\mathbb{X}} |f|^s d\mu$, to show that $||f||_s \leq ||f||_p^\Phi ||f||_q^{1-\Phi}$, where $\Phi = \frac{tp}{s} \in [0,1]$. Deduce again that $L^p \cap L^q \subset L^s$.

Answer Ph.D Comprehensive Examination Analysis (General Paper)

Second semester 1429-1430 H

Section B

III)

1. The sequence $(A_n^c)_n$ is increasing, then $\lim_{n\to+\infty} \mu(A_n^c) = \mu(X \setminus \lim_{n\to+\infty} A_n)$. As μ is finite $\mu(A_n^c) = \mu(X) - \mu(A_n)$ and $\mu(X \setminus \lim_{n\to+\infty} A_n) = \mu(X) - \mu(\lim_{n\to+\infty} A_n)$. Then $\lim_{n\to+\infty} \mu(A_n) = \mu(\lim_{n\to+\infty} A_n)$.

2. We can take $A_n = [n, +\infty[\subset \mathbb{R}$ and μ the Lebesgue measure on $\mathbb{R}$. $\mu(A_n) = +\infty$, $\lim_{n\to+\infty} A_n = \emptyset$.

3. a) $(1+\frac{x}{n})^n = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} \geq 1+x+\frac{(n-1)x^2}{2n} \geq \frac{x^2}{4}$ for $n \geq 2$ and $x \geq 0$.

   The function $\frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}$ is dominated by the function $\frac{4}{x^{2+\frac{1}{n}}}$ on the interval $[1, +\infty[$ which is integrable on $[1, +\infty[$, and it is dominated by the integrable function $\frac{1}{x^{\frac{1}{n}}}$ on the interval $]0, 1]$. Furthermore $\lim_{n\to+\infty} \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} = e^{-x}$. Then by the dominated convergence theorem $\lim_{n\to+\infty} \int_0^{+\infty} \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} dx = \int_0^{+\infty} e^{-x} dx = 1$.

   b) $\mathbb{Q}$ is a Lebesgue null set, $\frac{1}{\sqrt{x}}$ is continuous on $]0, +\infty[$, then the Lebesgue integral on $[0, 1]$ of the function f is the Riemann integral of the function $g(x) = \frac{1}{\sqrt{x}}$, and $\int_0^1 \frac{dx}{\sqrt{x}} = 2$

   c) By the Monotone Convergence Theorem

$$\begin{aligned}\int_0^{+\infty}\frac{dx}{(1+x^2)\sqrt{|\sin x|}} &= \sum_{n=0}^{+\infty}\int_{n\pi}^{(n+1)\pi}\frac{dx}{(1+x^2)\sqrt{|\sin x|}}\\ &= \sum_{n=0}^{+\infty}\int_0^{\pi}\frac{dx}{(1+(x+n\pi)^2)\sqrt{|\sin x|}}\\ &\le \sum_{n=0}^{+\infty}\frac{1}{(1+n^2\pi^2)}\int_0^{\pi}\frac{dx}{\sqrt{|\sin x|}}.\end{aligned}$$

$\int_0^{\pi}\frac{dx}{\sqrt{|\sin x|}} = 2\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}$ and on the interval $[0,\frac{\pi}{2}]$, $\sin x \ge \frac{2x}{\pi}$. Then the function $\frac{1}{\sqrt{|\sin x|}}$ is Lebesgue integrable on the interval $(0,\frac{\pi}{2})$, then the function g is Lebesgue integrable on the interval $(0,+\infty)$.

IV)

1. a) For $x \in (0,e^{-1})$, $\int_{(0,x)} f(t)dt = \int_{(0,x)}\frac{dt}{t(\log t)^2} \overset{s=\log t}{=} \int_{(-\infty,\log x)}\frac{ds}{s^2} = \frac{-1}{\log x}$.

   Since $f(x)=0$ for $x \notin (0,e^{-1})$, $f \ge 0$ for $x \in (0,e^{-1})$ and $\int_{(0,e^{-1})} f(t)dt = 1$, then $f \in L^1(\mathbb{R})$.

   b) To prove the result, we can consider $0 < r < e^{-1}$. For $x > 0$,

$$\mathcal{M}f(x) := \sup_{x\in I}\frac{1}{|I|}\int_I |f(t)|dt \ge \frac{1}{2x}\int_{(0,x)} f(t)dt = \frac{-1}{2x\log x}.$$

   Then

$$\int_{(0,r)}\mathcal{M}f(x)dx \ge \int_{(0,r)}\frac{-dx}{2x\log x} = +\infty.$$

2. a) We consider $p < q < +\infty$, $r = \frac{q}{p} > 1$ and $\frac{1}{r}+\frac{1}{s} = 1$. If $f \in L^q$, by Hölder's inequality

$$\int_{\mathbb{X}} |f(x)|^p d\mu(x) \le \left(\int_{\mathbb{X}} |f(x)|^q d\mu(x)\right)^{\frac{p}{q}} (\mu(\mathbb{X}))^{\frac{1}{s}} < +\infty.$$

Then $L^q \subset L^p$ if $1 \leq p \leq q$.

If $q = +\infty$, $\|f\|_p \leq \|f\|_\infty (\mu(\mathbb{X}))^{\frac{1}{p}}$.

b) If $1 \leq p \leq s \leq q \leq \infty$, then $L^p \cap L^q \subset L^p \subset L^s$ since If $1 \leq p \leq s$.

c) If $f \in L^\infty$, $\|f\|_p \leq \|f\|_\infty (\mu(\mathbb{X}))^{\frac{1}{p}}$ for all $p \geq 1$, then $\overline{\lim}_{p\to\infty} \|f\|_p \leq \|f\|_\infty$.

Consider for $t \in [0, \|f\|_\infty)$ the measurable set $A_t = \{x \in \mathbb{X};\ |f(x)| > t\}$.

$$\|f\|_p \geq \left(\int_{A_t} |f(x)|^p d\mu(x) \right)^p \geq t(\mu(A_t))^{\frac{1}{p}}.$$

Then $\underline{\lim}_{p\to\infty} \|f\|_p = \|f\|_\infty$ and $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

d) By Hölder's inequality

$$\begin{aligned} \int_{\mathbb{X}} |f(x)|^s d\mu(x) &= \int_{\mathbb{X}} |f(x)|^{tp} |f(x)|^{(1-t)q} d\mu(x) \\ &\leq \left(\int_{\mathbb{X}} |f(x)|^p d\mu(x) \right)^t \left(\int_{\mathbb{X}} |f(x)|^q d\mu(x) \right)^{1-t} \end{aligned}$$

Then

$$\left(\int_{\mathbb{X}} |f(x)|^s d\mu(x) \right)^{\frac{1}{s}} \leq \left(\int_{\mathbb{X}} |f(x)|^p d\mu(x) \right)^{\frac{t}{s}} \left(\int_{\mathbb{X}} |f(x)|^q d\mu(x) \right)^{\frac{1-t}{s}}$$

and

$$||f||_s \leq ||f||_p^{\Phi} ||f||_q^{1-\Phi},$$

where $\Phi = \frac{tp}{s} \in [0,1]$.

It results that if $f \in L^p \cap L^q$ then $f \in L^s$.

Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1996 Time 3 hours

Section I

1. (a) Let $(E_n)_n$ be a sequence of Borel sets in $\mathbb{R}$ and μ the Lebesgue measure. Show that $\mu(\liminf\limits_{n\to+\infty} E_n) \leq \liminf\limits_{n\to+\infty} \mu(E_n)$.

   (b) Construct a sequence $(f_n)_n$, $f_n \geq 0$ of Lebesgue measurable functions on $\mathbb{R}$, for which $\liminf_{n\to+\infty} \int_{\mathbb{R}} f_n(x)d\mu(x) > \int_{\mathbb{R}} \liminf_{n\to+\infty} f_n(x)d\mu(x)$.

   (c) Check whether the continuous function $x^2 \sin \dfrac{1}{x^2}$ is a function of bounded variation in $[-1, 1]$.

2. (a) Let f be a function on (a, b) such that $|f(x) - f(y)| \leq 2|x - y|$ for all x and y in the interval. Show that f is absolutely continuous on (a, b).

   (b) Give an example of a function $f(x)$ on $[a, b]$ for which $\displaystyle\int_a^b f'(x)dx$ exists, but $\displaystyle\int_a^b f'(x)dx \neq f(b) - f(a)$.

   (c) Let $g(x) = \dfrac{1}{\sqrt[4]{x}}$, $0 < x < 1$. Show that $A(f) = \displaystyle\int_0^1 f(x)g(x)dx$ defines a bounded linear functional on L^3; find the value of $\|A\|$.

3. Let X and Y be the unit interval $[0, 1]$ and $\mathcal{B}$ be the class of Borel sets in $[0, 1]$. For $E \in \mathcal{B}$, let $\mu(E)$ be the Lebesgue measure of E and $\mu(E)$ be the number of points in E. Let $D = \{(x, y) : \ x = y\}$ be the diagonal of $X \times Y$.

   (a) Show that D is a measurable subset of $X \times Y$.

   (b) If $D_x = \{y : \ (x, y) \in D\}$, show that $\displaystyle\int_X \nu(D_x)d\mu(x) = 1$.

   (c) If $D^y = \{x : \ (x, y) \in D\}$, show that $\displaystyle\int_Y \mu(D^y)d\nu(y) = 0$.

(d) Using the above results, show that it may happen for some function $f(x, y)$ and some measures λ_1 and λ_2 that $\int \int f(x, y) d\lambda_1(x) d\lambda_2(y) \neq \int \int f(x, y) d\lambda_2(y) d\lambda_1(x)$.

Answer Ph.D Comprehensive Examination
Analysis (General Paper)

Second semester 1996 Time

3 hours

Section I

1. (a) The sequence $\left(\bigcap_{k=n}^{+\infty} E_k\right)_n$ is increasing then from the Monotone Convergence Theorem: $\mu\left(\bigcup_{n=1}^{+\infty}\bigcap_{k=n}^{+\infty} E_k\right) = \lim_{n\to+\infty}\mu\left(\bigcap_{k=n}^{+\infty} E_k\right)$. But $\mu\left(\bigcap_{k=n}^{+\infty} E_k\right) \leq \inf_{k\geq n}\mu(E_k)$, then $\mu(\liminf_{n\to+\infty} E_n) \leq \liminf_{n\to+\infty}\mu(E_n)$.

   (b) Take $f_n = \chi_{[n,+\infty[}$, $\liminf_{n\to+\infty} f_n = 0$ but $\liminf_{n\to+\infty}\int_{\mathbb{R}} f_n(x)d\mu(x) = +\infty$.

   (c) $\int_{-1}^{1}|f'(x)|dx = \int_{-1}^{1}|2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}|dx$.
   $\int_{-1}^{1}|2x\sin\frac{1}{x^2}|dx \leq 4$, but $\int_{-1}^{1}|\frac{2}{x}\cos\frac{1}{x^2}|dx = 2\int_{0}^{1}\frac{2}{x}|\cos\frac{1}{x^2}|dx = 2\int_{1}^{+\infty}\frac{|\cos t|}{t}dt = +\infty$. Then the function $f(x) = x^2\sin\frac{1}{x^2}$ is not of bounded variation in $[-1,1]$.

2. (a) Let $\varepsilon > 0$ and let $]a_k, b_k[,\ k = 1, \ldots, n$ be a finite number of mutually disjoint subintervals of $]a, b[$ such that $\sum_{k=1}^{n}(b_k - a_k) \leq \frac{\varepsilon}{2}$, then $\sum_{k=1}^{n}|f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} 2(b_k - a_k) \leq \varepsilon$. Then f is absolutely continuous on $]a, b[$.

   (b) The function f defined on $[-1,1]$ by $f(x) = 1$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. $f' = 0$ a.e. and $f(1) - f(-1) = 1$.

   (c) Let $f \in L^3$, $|A(f)| \leq \int_0^1 |f(x)||g(x)|dx \leq \|f\|_3\|g\|_{\frac{3}{2}} = (\frac{8}{5})^{\frac{2}{3}}\|f\|_3$. Then A defines a bounded linear functional on L^3 and $\|A\| = \|g\|_{\frac{3}{2}} = (\frac{8}{5})^{\frac{2}{3}}$.

3. (a) $D = \{(x,x); x \in [0,1]\}$ is a closed set, then D is measurable with respect to the σ-algebra $\mathcal{B} \oplus \mathcal{B}$.

   (b) $D_x = \{y : (x,y) \in D\} = \{x\}$, then $\nu(D_x) = 1$ and $\int_X \nu(D_x) d\mu(x) = 1$.

   (c) $D^y = \{x : (x,y) \in D\} = \{y\}$, then $\mu(D^y) = 0$ and $\int_Y \mu(D^y) d\nu(y) = 0$.

   (d) Let $f(x,y) = \chi_D(x,y)$. $\int_Y \int_X f(x,y) d\mu(x) d\nu(y) = 1 \neq \int_X \int_Y f(x,y) d\nu(y) d\mu(x) = 0$.

Ph.D Comprehensive Examination
Analysis (General Paper)

Autumn 1997 Time 3 hours

Section I

1. (a) Let $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be two functions such that f is measurable and g is continuous. Is $f \circ g$ measurable?

   (b) Describe a non measurable set A on $\mathbb{R}$. Suppose A is a non measurable set. Define

$$f(x) = \begin{cases} e^{x} & \text{if} x \in A \\ e^{-x} & \text{if} x \notin A \end{cases}.$$

   Show that for any c, $\{x;\ f(x) = c\}$ is measurable, but f is not a measurable function.

2. (a) Let f be a monotonic function on $[a, b]$. Show that f can be written as $f = h + g$, where h is absolutely continuous and g is monotonic for which $g'(x) = 0$ a.e.

   (b) Construct two measures μ and σ on $\mathbb{R}$ such that $\mu << \sigma$, but there exist no function f such that $\frac{d\mu}{d\sigma} = f$.

3. (a) State Tonelli theorem.

   (b) Let $f\colon \Omega \longrightarrow \mathbb{R}$ defined by: $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \text{ and} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   with $\Omega = \{(x,y);\ -1 \leq x \leq 1,\ -1 \leq x \leq 1\}$.

   Is $\displaystyle\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dx \right) dy = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dy \right) dx$?

   What can you say about the double integral $\displaystyle\int\int_{\Omega} f(x,y)dxdy$?.

Answer Ph.D Comprehensive Examination
Analysis (General Paper)

Autumn 1997 Time 3 hours

Section I

1. (a) If $\mathscr{B}$ is the Borel $\sigma-$algebra, f and g are measurable, then

$$(f \circ g)^{-1}(\mathscr{B}) = g^{-1}(f^{-1}(\mathscr{B})) \subset f^{-1}(\mathscr{B}) \subset \mathscr{B}.$$

Then $f \circ g$ is measurable.

(b) We consider on $E = [0, 1]$ the equivalence relationship $\mathcal{R}$, defined by $x\mathcal{R}y \iff x - y \in \mathbb{Q}$. We choose a representative of each class, and we denote A the set of these representatives. The set A is not countable and non measurable set.
If $c > 0$, $\{x;\ f(x) = c\} = \{\ln c\} \cap A$ is measurable.
If $c < 0$, $\{x;\ f(x) = c\} = \{\ln -c\} \cap A^c$ is measurable.
If $c = 0$, $\{x;\ f(x) = 0\} = \emptyset$ is measurable.
f is not a measurable function since $\{x;\ f(x) > 0\} = A$ which is not measurable.

2. (a) Since f is monotonic, then it is of bounded variation, f is a.e differentiable. The function h defined $h(x) = \displaystyle\int_a^x f'(t)dt$ is absolutely continuous. The function $g = f - f_a$ is singular i.e $g' = 0$ a.e and monotonic.

(b) Consider λ the Lebesgue measure and δ the Dirac measure.
Construct two measures μ and σ on $\mathbb{R}$ such that $\mu << \sigma$, but there exist no function f such that $\dfrac{d\mu}{d\sigma} = f$.

3. (a) The Fubini Tonelli theorem: Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$ finite measure spaces. Let f be a non negative measurable function on $(X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2, \mu_1 \otimes \mu_2)$. Then the functions

$$x \longmapsto g(x) = \int_{X_2} f(x,y)d\,\mu_2(y) \quad \text{and} \quad y \longmapsto h(y) = \int_{X_1} f(x,y)d\,\mu_1(x)$$

are respectively measurable on X_1 and X_2. Moreover

$$\int_{X_1 \times X_2} f(x,y) \mu_1 \otimes \mu_2(x,y) = \int_{X_2} \left(\int_{X_1} f(x,y) d\, \mu_1(x) \right) d\, \mu_2(y)$$
$$= \int_{X_1} \left(\int_{X_2} f(x,y) d\, \mu_2(y) \right) d\, \mu_1(x).$$

(b) $\int_{-1}^{1} f(x,y)dx = 2-2y\tan^{-1}(\frac{1}{y})$ for $y \neq 0$ and $\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dx \right) dy =$ $4\int_{0}^{1} 1-y\tan^{-1}(\frac{1}{y})dy = 2$. Since $f(x,y) = -f(y,x)$, then $\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dy \right) dx =$ -2.

Ph.D Qualifying Examination
Analysis (General Paper)

Dhu Al-Hijjah 1425, October 2014

Section A

Problem I:

1. State the Fubini Theorem.

   Let $\Omega = (0, +\infty) \times (0, +\infty)$.

2. Compute
$$\int_{\Omega} \frac{d\lambda(x,y)}{(1+y)(1+x^2y)},$$
   where λ is the Lebesgue measure on $\mathbb{R}^2$.

3. Deduce the values of the following integrals
$$\int_0^{+\infty} \frac{\ln(x)}{1-x^2}dx \quad \text{and} \quad \int_0^1 \frac{\ln(x)}{1-x^2}dx.$$

4. Prove that
$$\int_0^1 \frac{\ln(x)}{1-x^2}dx = \sum_{n=0}^{+\infty} \int_0^1 x^{2n}\ln(x)dx$$

5. Deduce the sum of each of the following series
$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2}.$$

Problem II: [Note that parts 1) and 2) are independent]

1. (a) Prove that $\mu_1 = \sum_{n=1}^{+\infty} \delta_{\frac{1}{n}}$ is a measure on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, where $\mathscr{B}_{\mathbb{R}}$ is the Borel $\sigma-$algebra on $\mathbb{R}$.

   (b) Consider the functions $f(x) = x$ and $g(x) = x\ln(1+|x|)$ on $\mathbb{R}$. Give the values of $p, q \in [0, +\infty)$ for which $f \in L^p(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and $g \in L^q(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.

2. (a) Prove that the function $f(x) = \dfrac{1}{\sqrt{x(1-x)}}$ is integrable on the interval $(0,1)$ and compute the following integral $\displaystyle\int_{(0,1)} \frac{d\lambda(x)}{\sqrt{x(1-x)}}$, with λ is the Lebesgue measure on $\mathbb{R}$.

   (b) Let $f\colon (a,b) \longrightarrow \mathbb{R}$ be a bounded Lebesgue integrable function and $\lim\limits_{t\to a^+} f(t) = c$.

   Prove that for any $t \in (a,b)$, the function $x \longmapsto \frac{f(x)}{\sqrt{(x-a)(t-x)}}$ is integrable on (a,t) and compute $\displaystyle\lim_{t\to a^+} \int_{(a,t)} \frac{f(x)}{\sqrt{(x-a)(t-x)}} d\lambda(x)$.

Answer Ph.D Qualifying Examination
Analysis (General Paper)

Dhu Al-Hijjah 1425, October 2014

Section A

Problem I:

1. (The Fubini's Theorem): Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$ finite measure spaces, and let $(X, \mathscr{A}, \mu)$ be the product measure space. If $f \in L^1(X, d\,\mu)$, then $\displaystyle\int_{X_2} f(x,y)d\,\mu_2(y) \in L^1(X_1, \mu_1)$ and $\displaystyle\int_{X_1} f(x,y)d\,\mu_1(x) \in L^1(X_2, \mu_2)$ and

$$\begin{aligned}\int_{X_1\times X_2} f(x,y)\mu_1\otimes\mu_2(x,y) &= \int_{X_2}\left(\int_{X_1} f(x,y)d\,\mu_1(x)\right)d\,\mu_2(y)\\ &= \int_{X_1}\left(\int_{X_2} f(x,y)d\,\mu_2(y)\right)d\,\mu_1(x)\end{aligned}$$

Let $\Omega = (0,+\infty)\times(0,+\infty)$.

2. The function $(x,y) \longmapsto 1(1+y)(1+x^2y)$ is non negative continuous function, then by Fubini Tonelli Theorem

$$\begin{aligned}\int_\Omega \frac{d\lambda(x,y)}{(1+y)(1+x^2y)} &= \int_0^{+\infty}\left(\int_0^{+\infty}\frac{dx}{(1+y)(1+x^2y)}\right)dy\\ &= \int_0^{+\infty}\left(\int_0^{+\infty}\frac{dy}{(1+y)(1+x^2y)}\right)dx.\end{aligned}$$

$\displaystyle\int_0^{+\infty}\frac{dx}{(1+x^2y)} = \frac{\pi}{2\sqrt{y}}$ and $\displaystyle\int_0^{+\infty}\frac{dy}{2\sqrt{y}(1+y)} \overset{y=t^2}{=} \frac{\pi^2}{2}$.

For $x\neq 1$, $\displaystyle\frac{1}{(1+y)(1+x^2y)} = \frac{A}{1+y} - \frac{x^2A}{1+x^2y}$, with $A = \frac{1}{1-x^2}$. Then

$\displaystyle\int_0^{+\infty}\frac{dy}{(1+y)(1+x^2y)} = A\ln(\frac{1+y}{1+x^2y})\Big]_0^{+\infty} = -\frac{2\ln x}{1-x^2}$.

3. By Fubini Tonelli Theorem $\int_0^{+\infty} \frac{\ln(x)}{1-x^2}dx = -\frac{\pi^2}{4}$. Moreover by the change of variable $x = \frac{1}{t}$, $\int_0^1 \frac{\ln(x)}{1-x^2}dx = \int_1^{+\infty} \frac{\ln(x)}{1-x^2}dx = -\frac{\pi^2}{8}$.

4. For $|x| < 1$, $\frac{1}{1-x^2} = \sum_{n=0}^{+\infty} x^{2n}$ and by Monotone Convergence Theorem $(x^{2n}\ln(x) \leq 0)$

$$\int_0^1 \frac{\ln(x)}{1-x^2}dx = \sum_{n=0}^{+\infty} \int_0^1 x^{2n}\ln(x)dx.$$

5. By integration by parts $\int_0^1 x^{2n}\ln(x)dx = -\frac{1}{(2n+1)^2}$. Then $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{1}{n^2} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$. Then $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Problem II:

1. (a) We know that if $(\mu_n)_n$ is an increasing sequence of measures on a measurable space $(X, \mathscr{A})$, the mapping $\mu\colon \mathscr{A} \longrightarrow [0, +\infty]$ defined by $\mu(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_n \mu_n(A)$ for any $A \in \mathscr{A}$ is a measure on X.

   Indeed it is clear that $\mu(\emptyset) = 0 = \lim_{n\to+\infty} \mu_n(\emptyset)$, and if A, B are two disjoints measurable subsets, we have

$$\mu(A \cup B) = \lim_{n\to+\infty} \mu_n(A) + \lim_{n\to+\infty} \mu_n(B) = \mu(A) + \mu(B).$$

   Let now $(A_n)_n$ be an increasing sequence of $\mathscr{A}$ and $A = \bigcup_{n=1}^{+\infty} A_n$. We have $\mu_j(A_n) \leq \mu(A_n) \leq \mu(A)$. Then

$$\mu_j(A) = \lim_{n\to+\infty} \mu_j(A_n) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A).$$

   Moreover

$$\mu(A) = \lim_{j\to+\infty} \mu_j(A) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A).$$

   Then $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$.

Then $\mu_1 = \lim_{n\to+\infty}\sum_{k=1}^{n}\delta_{\frac{1}{k}}$ is a measure on the measurable space $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$.

(b) $\int_{\mathbb{R}} f^p(x)d\mu_1(x) = \sum_{n=1}^{+\infty}\frac{1}{n^p}$. Then $f \in L^p(\mathbb{R},\mathscr{B}_{\mathbb{R}},\mu_1)$ if and only if $p > 1$.

$\int_{\mathbb{R}} g^q(x)d\mu_1(x) = \sum_{n=1}^{+\infty}\frac{\ln^q(1+\frac{1}{n})}{n^q}$. Since $\frac{\ln^q(1+\frac{1}{n})}{n^q} \approx \frac{1}{n^{2q}}$, then $g \in L^q(\mathbb{R},\mathscr{B}_{\mathbb{R}},\mu_1) \iff q > \frac{1}{2}$.

2. (a) In a neighborhood of 0, $f(x) \approx \frac{1}{\sqrt{x}}$, which is integrable and in a neighborhood of 1, $f(x) \approx \frac{1}{\sqrt{1-x}}$, which is integrable.

$$\int_{(0,1)}\frac{d\lambda(x)}{\sqrt{x(1-x)}} \overset{x=t^2}{=} \int_0^1 \frac{2dt}{\sqrt{1-t^2}} = \pi.$$

(b) In a neighborhood of a in (a,t), $\frac{1}{\sqrt{(x-a)(t-x)}} \approx \frac{1}{\sqrt{(x-a)(t-a)}}$, which is integrable and in a neighborhood of t in (a,t), $\frac{1}{\sqrt{(x-a)(t-x)}} \approx \frac{1}{\sqrt{(t-a)(t-x)}}$, which is integrable. Moreover since f is bounded then for any $t \in (a,b)$, the function $x \longmapsto \frac{f(x)}{\sqrt{(x-a)(t-x)}}$ is integrable on (a,t).

$$\int_{(a,t)}\frac{d\lambda(x)}{\sqrt{(x-a)(t-x)}} \overset{x=st+(1-s)a}{=} \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi.$$

Since f is bounded, then by Dominated Convergence Theorem

$$\lim_{t\to a^+}\int_{(a,t)}\frac{f(x)}{\sqrt{(x-a)(t-x)}}d\lambda(x) \overset{x=st+(1-s)a}{=} \lim_{t\to a^+}\int_0^1 \frac{f(a+s(t-a))}{\sqrt{s(1-s)}}ds = \pi c.$$

Ph.D Comprehensive Examination
Analysis (Special Paper)

Second semester 28-29

Section B

Problem III

1. State the Dominate Convergence Theorem. Prove that if f is integrable on $[0,1]$, then $\lim_{n\to+\infty}\int_0^1 x^n f(x)dx = 0$.

2. We consider the function F defined on $[0,+\infty[$ by $F(x) = \displaystyle\int_0^{+\infty} \frac{e^{-xt}}{1+t^2}dt$.

   a) Find $\lim_{x\to+\infty} F(x)$ and $\lim_{x\to 0} F(x)$

   b) Prove that F is of class $\mathcal{C}^2$ for $x>0$ and find $F''(x)$.

3. Show that $\displaystyle\int_0^1 \sin x \ln x dx = \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n)(2n)!}$.

Problem IV

1. Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the measure space, with μ the counting measure. Let $f\colon \mathbb{N} \longrightarrow [0,+\infty]$ be a function.

   a) Show that $\displaystyle\int_{\mathbb{N}} f(x)d\mu(x) = \sum_{n=1}^{+\infty} f(n)$.

   b) Let $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection. Show that

   $$\sum_{n=1}^{+\infty} f(n) = \sum_{n=1}^{+\infty} f(\sigma(n)).$$

   c) Let $(u_{j,k})_{j,k}$ be a sequence of non negative numbers. Deduce

   $$\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} u_{j,k} = \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} u_{j,k}.$$

   d) Find $\displaystyle\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} a^j b^k$, with $0 \le a, b < 1$.

e) Give an example of sequence $(u_{j,k})_{j,k}$ for which the result of c) is false.

2. a) Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be $\sigma-$finite measure spaces. Prove that if $E \in \mathscr{M} \otimes \mathscr{N}$, then the functions $x \in X \longmapsto \nu(E_x)$ and $y \in Y \longmapsto \mu(E^y)$ are measurable on X and Y respectively with $E_x = \{y \in Y;\ (x,y) \in E\}$ and $E^y = \{x \in X;\ (x,y) \in E\}$.

   b) Let $X = [0,1]$, $\mathscr{B}$ the Borel $\sigma-$algebra on $[0,1]$.

   Show that $D = \{(x,y) \in X \times X;\ x - y = 0\}$ is measurable with respect to the $\sigma-$algebra $\mathscr{B} \otimes \mathscr{B}$.

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Section B

Problem III

1. The Dominate Convergence Theorem:

   Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathscr{A}, \mu)$. We assume that:

   i) the sequence $(f_n)_n$ converges almost everywhere on X to a measurable function f definite almost everywhere.

   ii) There exist a non-negative integrable function g such that: $|f_n| \leq g$ almost everywhere for all n. Then the sequence $(f_n)_n$ and the function f are integrable and we have:

$$\int_X f(x)\ d\,\mu(x) \ = \lim_{n \longrightarrow +\infty} \int_X f_n(x) d\,\mu(x).$$

   If f is integrable on $[0,1]$, the sequence $(f_n)_n$ defined by $f_n(x) = x^n f(x)$ is dominated by $|f|$ and $\lim\limits_{n\to+\infty} f_n = 0$ a.e, the by the Dominate Convergence Theorem $\lim\limits_{n\to+\infty} \int_0^1 x^n f(x)dx = 0$.

2. a) We have $f(x,t) = \dfrac{e^{-xt}}{1+t^2} \leq \dfrac{1}{1+t^2}$ which is integrable and $\lim\limits_{x\to+\infty} f(x,t) = 0$. Then by the Dominate Convergence Theorem $\lim\limits_{x\to+\infty} F(x) = 0$.

   We have also $\lim\limits_{x\to 0} f(x,t) = \dfrac{1}{1+t^2}$. Then by the Dominate Convergence Theorem, $\lim\limits_{x\to 0} F(x) = \dfrac{\pi}{2}$.

   b) $x \longmapsto f(x,t)$ is C^∞, $\dfrac{\partial f}{\partial x}(x,t) = \dfrac{-te^{-xt}}{1+t^2}$ and $\dfrac{\partial^2 f}{\partial x^2}(x,t) = \dfrac{t^2 e^{-xt}}{1+t^2}$. For $a > 0$, $\left|\dfrac{\partial f}{\partial x}(x,t)\right| \leq \dfrac{te^{-at}}{1+t^2}$ and $\left|\dfrac{\partial^2 f}{\partial x^2}(x,t)\right| \leq \dfrac{t^2 e^{-at}}{1+t^2}$ for all $x \in [a, +\infty[$. Since the functions $t \longmapsto \dfrac{te^{-at}}{1+t^2}$ and $t \longmapsto \dfrac{t^2 e^{-at}}{1+t^2}$ are integrable, the function F is of class $\mathcal{C}^2$ on $[0, +\infty[$ and $F''(x) = \int_0^{+\infty} \frac{t^2 e^{-xt}}{1+t^2} dt = \frac{1}{x} - F(x)$.

3. We have $\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\forall x \in \mathbb{R}$. By Dominate Convergence Theorem

$$\begin{aligned}
\int_0^1 \sin x \ln x dx &= \sum_{n=0}^{+\infty} \int_0^1 \frac{(-1)^n x^{2n+1} \ln x}{(2n+1)!} dx \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{2n+1} \ln x dx \\
&= \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n)(2n)!}.
\end{aligned}$$

Problem IV

1. a) $\int_{\mathbb{N}} f(x) d\mu(x) \overset{\text{M.C.T}}{=} \lim_{n\to+\infty} \int_{[1,n]} f(x) d\mu(x) = \lim_{n\to+\infty} \sum_{k=1}^{n} f(k) = \sum_{k=1}^{+\infty} f(k).$

   b) If $A_n = \sigma([1,n])$, then $\bigcup_{n=1}^{+\infty} A_n = \mathbb{N}$. The sequence $(A_n)_n$ is increasing. It follows from the Monotone Convergence Theorem that

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(\sigma(k)) = \lim_{n\to+\infty} \int_{A_n} f(x) d\mu(x) = \int_{\mathbb{N}} f(x) d\mu(x) = \sum_{k=1}^{+\infty} f(k).$$

   c) a) If $f_n(m) = \sum_{k=1}^{n} u_{k,m}$, then $\int_{\mathbb{N}} f_n(x) d\mu(x) = \sum_{m=1}^{+\infty} \sum_{k=1}^{n} u_{k,m}$. Since the sequence $(f_n)_n$ is increasing then

$$\begin{aligned}
\int_{\mathbb{N}} \lim_{n\to+\infty} f_n(x) d\mu(x) &= \lim_{n\to+\infty} \int_{\mathbb{N}} f_n(x) d\mu(x) \\
&= \sum_{m=1}^{+\infty} \sum_{k=1}^{+\infty} u_{k,m} = \lim_{n\to+\infty} \sum_{k=1}^{n} \sum_{m=1}^{+\infty} u_{k,m} \\
&= \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} u_{k,m}.
\end{aligned}$$

d) $\sum_{k=1}^{+\infty} a^j = \frac{a}{1-a}$, then $\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} a^j b^k = \frac{ab}{(1-a)(1-b)}$.

e) Let $a_{j,k} = \frac{1}{kj(j+1)}$, for $j \geq 2$ and $a_{1,k} = \frac{-1}{2k}$.

$\sum_{j=1}^{+\infty} \frac{1}{j(j+1)} = 1$ and $\sum_{j=1}^{+\infty} a_{j,k} = 0$. So $\sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} a_{j,k} = 0$ and $\sum_{k=1}^{+\infty} a_{j,k} = +\infty$, for $j \geq 2$ and $\sum_{k=1}^{+\infty} a_{1,k} = -\infty$.

2. a) Suppose that ν is finite and define

$$\mathscr{A} = \{E \in \mathscr{M} \otimes \mathscr{N};\ x \longmapsto \nu(E_x) \text{ is measurable }\}.$$

$\mathscr{A}$ contains the measurable rectangles $E = A \times B$ since $\nu(E_x) = \chi_A(x)\nu(B)$. Moreover $\mathscr{A}$ is a monotone class: if $E \subset E'$, $\nu(E' \setminus E)_x = \nu(E'_x) - \nu(E_x)$ since ν is finite, and if $(E_n)_n$ is an increasing sequence

$$\nu(\cup_{k=1}^{+\infty} E_n)_x = \lim_{n\to+\infty} \nu(E_n)_x.$$

Then $\mathscr{A} = \mathscr{M} \otimes \mathscr{N}$.

In the general case where ν is $\sigma-$finite, we take an increasing sequence $(B_n)_n$ such that $\nu(B_n) < +\infty$ and $X = \bigcup_{n=1}^{+\infty} B_n$. Define $\nu_n(B) = \nu(B \cap B_n)$. Then $\nu(E_x) = \lim_{n\to+\infty} \nu_n(E_x)$ which is measurable.

By the same arguments, $y \in Y \longmapsto \mu(E^y)$ is measurable on Y.

b) $D = \{(x,y) \in X \times X;\ x - y = 0\}$ is closed then it is measurable with respect to the $\sigma-$algebra $\mathscr{B} \otimes \mathscr{B}$.

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Solve five (5) problems.

Section B

Problem III

1. a) Give the definitions of a measure and an outer measure.

   b) Let $(X, \mathscr{B})$ be a measurable space and $(\mu_n)_n$ be a sequence of measures on X such that $\mu_n(X) = 1$ for all $n \in \mathbb{N}_0$. For any $A \in \mathscr{B}$, define

$$\mu(A) = \sum_{n=0}^{+\infty} \frac{\mu_n(A)}{2^{n+1}}.$$

   Prove that μ defines a probability measure on $(X, \mathscr{B})$.

2. a) Let $(X, \mathscr{B})$ be a measurable space. Give the definition of a measurable function on X.

   Let $f_n \colon X \longrightarrow \mathbb{R}$ be a sequence of measurable functions. Prove that $\{x \in X;\ (f_n(x))_n \text{ converges }\}$ is measurable.

3. a) Let $(X, \mathscr{B}, \mu)$ be a measure space and f an integrable function on X. Suppose that $\int_E f(x)d\mu(x) = 0$ for any measurable set E. Show that $f = 0$ almost every where.

Problem IV

1. Let λ be the Lebesgue measure on $\mathbb{R}$. Evaluate the following integrals:

   a) $\displaystyle\int_{[0,\pi]} f(x)d\lambda(x)$, where $f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \cap [0,\pi], \\ \cos x & x \in [0,\pi] \setminus \mathbb{Q}, \end{cases}$.

   b) $\displaystyle\int_{[0,1]} \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)d\lambda(x)$. (Recall $\chi_{\mathbb{R}\setminus\mathbb{Q}}(x) = 1$ if $x \in \mathbb{R}\setminus\mathbb{Q}$ and 0 otherwise.)

2. a) State the Monotone Convergence Theorem.

   b) Let $f(x) = \dfrac{xe^{-x}}{1-e^{-x}}$. Prove that f is integrable on $[0, +\infty)$ and

$$\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \frac{1}{(1+n)^2}.$$

3. a) Let f be an integrable function on a measure space $(X, \mathscr{B}, \mu)$. Prove that $\{x \in X;\ f(x) = \pm\infty\}$ is a null set.

   b) Let f be an integrable function on $\mathbb{R}$ and $\alpha > 0$. Prove that $\dfrac{f(nx)}{n^\alpha} \longrightarrow 0$ as $n \longrightarrow +\infty$ almost every where. (Hint: prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{|f(nx)|}{n^\alpha}$ is integrable.)

Answer Ph.D Comprehensive Examination
Analysis (Special Paper)

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Problem III

1. a) • Let $(X, \mathscr{A})$ be a measurable space. A measure on X is a set function $\mu\colon \mathscr{A} \to [0, \infty]$ such that:

   i) $\mu(\emptyset) = 0$;

   ii) For any disjoint sequence $(A_n)_n \in \mathscr{A}$,

$$\mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n). \tag{0.2}$$

   • Let X be a non empty set. An outer measure μ^* on X is a set function $\mu^*\colon \mathscr{P}(X) \longrightarrow [0, \infty]$ which satisfies the following conditions:

   i) $\mu^*(\emptyset) = 0$.

   ii) If $(A_n)_n$ is a sequence of subsets of X, then

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

   iii) μ^* is increasing (i.e. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$).

   b) Let $A \in \mathscr{B}$, the series $\sum_{n\geq 0} \frac{\mu_n(A)}{2^{n+1}}$ is convergent. Then μ is well defined.

   $\mu_n(\emptyset) = 0$, then $\mu(\emptyset) = 0$.

   If A and B are measurable and disjoint, then $\mu_n(A \cup B) = \mu_n(A) + \mu_n(B)$ and $\mu(A \cup B) = \mu(A) + \mu(B)$.

   Let $(A_n)_n \in \mathscr{B}$ be a disjoint sequence and $A = \bigcup_{n=0}^{+\infty} A_n$.

$$\begin{aligned}
\mu(A) &= \lim_{m\to+\infty}\sum_{n=0}^{m}\frac{\mu_n(A)}{2^{n+1}}\\
&= \lim_{m\to+\infty}\sum_{n=0}^{m}\sum_{k=0}^{+\infty}\frac{\mu_n(A_k)}{2^{n+1}}\\
&= \lim_{m\to+\infty}\sum_{k=0}^{+\infty}\sum_{n=0}^{m}\frac{\mu_n(A_k)}{2^{n+1}}\\
&\leq \sum_{k=0}^{+\infty}\mu(A_k), \quad \forall p\in\mathbb{N}.
\end{aligned}$$

Then $\mu(A) \leq \sum_{n=0}^{+\infty}\mu(A_n)$.

Moreover for all $m\in\mathbb{N}$, $\mu(A)\geq\sum_{n=0}^{m}\mu(A_n)$. Then $\mu(A)\geq\sum_{n=0}^{+\infty}\mu(A_n)$. Which proves that $\mu(A)=\sum_{n=0}^{+\infty}\mu(A_n)$.

It is obvious that μ defines a probability measure on $(X, \mathscr{B})$.

2. a) A function $f\colon X\longrightarrow\mathbb{R}$ is called measurable if the σ-algebra $f^{-1}(\mathscr{B}_{\mathbb{R}})\subset\mathscr{B}$.

   Let $C=\{x\in X;\ (f_n(x))_n \text{ converges }\}$ and Let $D=C^c$, $D=\{x\in X;\ \underline{\lim}_{n\to+\infty}f_n(x)<\overline{\lim}_{n\to+\infty}f_n(x)\}$. If we set $g=\underline{\lim}_{n\to+\infty}f_n$ and $h=\overline{\lim}_{n\to+\infty}f_n$. For each rational r, let

   $$D_r=\{x\in X;\ g(x)<r<h(x)\}=\{x\in X;\ g(x)<r\}\cap\{x\in X;\ h(x)>r\}$$

   which is measurable. $D=\bigcup_{r\in\mathbb{Q}}D_r$ which proves the measurability of D.

3. a) Let $E^+=\{x\in X;\ f(x)>0\}$ and $E^-=\{x\in X;\ f(x)<0\}$. Since $\chi_{E^+}f\geq 0$, $\chi_{E^-}f\leq 0$, $\int_{E^+}f(x)d\mu(x)=0$ and $\int_{E^-}f(x)d\mu(x)=0$, then $\chi_{E^-}f=0$ and $\chi_{E^+}f=0$ almost every where, which proves that $f=0$ almost every where.

Problem IV

1. a) $\displaystyle\int_{[0,\pi]} f(x)d\lambda(x) = \int_0^{\pi} \cos(x)dx = 0.$

   b) $\displaystyle\int_{[0,1]} \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)d\lambda(x) = \int_0^{1} dx = 1.$

2. a) The Monotone Convergence Theorem:

   Let $(f_n)_n$ be an increasing sequence of non-negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$, then

$$\int_X \lim_{n\to+\infty} f_n(x)d\,\mu(x) = \lim_{n\to+\infty} \int_X f_n(x)d\,\mu(x).$$

   b) f is a continuous non negative function on $]0, +\infty[$. Moreover $\lim_{x\to 0} f(x) = 1$. Then f is integrable on $[0, +\infty[$ if and only if the improper integral $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent. For x large enough, $f(x) \leq 2xe^{-x}$ which is integrable on $[0, +\infty)$.

   For $x > 0$, $f(x) = \displaystyle\sum_{n=0}^{+\infty} xe^{-(n+1)x}$. Then by Monotone Convergence Theorem

$$\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \int_0^{+\infty} xe^{-(n+1)x}dx = \sum_{n=0}^{+\infty} \frac{1}{(1+n)^2}.$$

3. a) $\{x \in X;\ f(x) = \pm\infty\} = \{x \in X;\ |f(x)| = \infty\} = \cap_{n=1}^{+\infty}\{x \in X;\ |f(x)| \geq n\}$. If $E_n = \{x \in X;\ |f(x)| \geq n\}$,

$$\int_X |f(x)|d\mu(x) \geq \int_{E_n} |f(x)|d\mu(x) \geq n\mu(E_n).$$

   Then $\{x \in X;\ f(x) = \pm\infty\}$ is a null set.

   b) By Monotone Convergence Theorem

$$\int_{\mathbb{R}} \sum_{n=1}^{+\infty} \frac{|f(nx)|}{n^\alpha}dx = \sum_{n=1}^{+\infty} \int_{\mathbb{R}} \frac{|f(nx)|}{n^\alpha}dx = \sum_{n=1}^{+\infty} \frac{\|f\|_1}{n^{\alpha+1}}.$$

   Then $\displaystyle\lim_{n\to+\infty} \frac{f(nx)}{n^\alpha} = 0$ as $n \longrightarrow +\infty$ almost every where.