King Saud University: Mathematics Department Math-254
Third Semester 1444 H Final Examination Solution
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————— **I.D. No.** ————————

Name of the Teacher:———————— **Section No.** ————————

Note: Check the total number of pages are Six (6).

(**15 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Q. No.	11	12	13	14	15
a,b,c,d					

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(Math)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	a	b	c	b	c	a	c	a	b	b	a	a	c	c	b

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MAth)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	c	a	b	a	b	c	a	b	c	c	b	c	a	b	c

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATh)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	b	c	a	c	a	b	b	c	a	a	c	b	b	a	a

Question 16: Use the following table to find the best approximation of $f(0.6)$ by using quadratic Lagrange interpolating polynomial for equally spaced data points:

x	0.15	0.2	0.3	0.5	0.55	0.8	1
$f(x)$	-0.0427	-0.0644	-0.1084	-0.1733	-0.1808	-0.1428	0.0000

The function tabulated is $f(x) = x^2 \ln x$. Compute the absolute error and an error bound (using error bound formula for equally spaced data points) for the approximation.

Solution. Given $x = 0.6$, so, the best three points for the quadratic Lagrange interpolating polynomial for equally spaced data points are, $x_0 = 0.3, x_1 = 0.55$ and $x_2 = 0.8$ with $h = 0.25$. Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{1}$$

$$f(0.6) \approx p_2(0.6) = L_0(0.6)(-0.1084) + L_1(0.6)(-0.1808) + L_2(0.6)(-0.1428). \tag{2}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned}
L_0(0.6) &= \frac{(0.6-0.55)(0.6-0.8)}{(0.3-0.55)(0.3-0.8)} = -2/25 = -0.08, \\
L_1(0.6) &= \frac{(0.6-0.3)(0.6-0.8)}{(0.55-0.3)(0.55-0.8)} = 24/25 = 0.96, \\
L_2(0.6) &= \frac{(0.6-0.3)(0.6-0.55)}{(0.8-0.3)(0.8-0.55)} = 3/25 = 0.12.
\end{aligned}$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-0.08)(-0.1084) + (0.96)(-0.1808) + (0.12)(-0.1428) = -0.1821,$$

which is the required approximation of the given exact solution $0.36 \ln 0.6 \approx -0.1839$.
The desired absolute error is

$$|f(0.6) - p_2(0.6)| = |0.36 \ln 0.6 - (-0.1821)| = |-0.1839 + 0.1821| = 0.0018.$$

To compute an error bound for the approximation of the given function in the interval $[0.3, 0.8]$, we use the following quadratic error formula

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}}.$$

As

$$M = \max_{0.3 \leq x \leq 0.8} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x \ln x + x, \qquad f''(x) = 2 \ln x + 3, \qquad f^{(3)}(x) = \frac{2}{x},$$

$$M = \max_{0.3 \leq x \leq 0.8} \left| \frac{2}{x} \right| = 20/3 = 6.6667.$$

Hence

$$|f(0.6) - p_2(0.6)| \leq \frac{(6.6667)(0.25)^3}{9\sqrt{(3)}} = 0.0067,$$

which is desired error bound. •

Question 17: Use best integration rule to find the absolute error for the approximation of $\int_0^{1.2} f(x)\,dx$ by using the following set of data points:

x	0.0	0.1	0.21	0.3	0.42	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$f(x)$	1.0000	1.0950	1.1880	1.2553	1.3331	1.3776	1.4253	1.4648	1.4967	1.5216	1.5403	1.5536	1.5624

The function tabulated is $f(x) = x + \cos x$. How many points approximate the given integral to within accuracy of 10^{-6} ?

Solution. We need only the equally spaced data points, which are as follows

$$x_0 = 0, x_1 = 0.3, x_2 = 0.6, x_3 = 0.9, x_4 = 1.2$$

gives, $h = 0.3$ and $n = (1.2 - 0)/0.3 = 4$, which means the best rule is Simpson's rule. Thus to select the following set of data points for Simpson's rule as

x	0.0	0.3	0.6	0.9	1.2
$f(x)$	1.0000	1.2553	1.4253	1.5216	1.5624

The composite Simpson's rule for five points can be written as

$$\int_0^{1.2} f(x)\,dx \approx S_4(f) = \frac{h}{3}\Big[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\Big],$$

$$\int_0^{1.2} f(x)\,dx \approx 0.1\Big[1.0000 + 4(1.2553 + 1.5216) + 2(1.4253) + 1.5624\Big] = 1.6521.$$

We can easily computed the exact value of the given integral as

$$I(f) = \int_0^{1.2} (x + \cos x)\,dx = (\frac{x^2}{2} + \sin x)\Big|_0^{1.2} = 1.6520.$$

Thus the absolute error $|E|$ in our approximation is given as

$$|E| = |I(f) - S_4(f)| = |1.6520 - 1.6521| = 0.0001.$$

The first four derivatives of the function $f(x) = x + \cos x$ can be obtain as

$$f'(x) = 1 - \sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x.$$

Since $\eta(x)$ is unknown point in $(0, 1.2)$, therefore, the bound $|f^{(4)}|$ on $[0, 1.2]$ is

$$M = \max_{0 \leq x \leq 1.2} |f^{(4)}| = \max_{0 \leq x \leq 1.2} |\cos x| = 1.0,$$

at $x = 0$. To find the minimum subintervals for the given accuracy, we use error bound formula of Simpson's rule

$$|E_{S_n}(f)| \leq \frac{|-(b-a)^5|}{180n^4} M \leq 10^{-6},$$

where $M = \max_{0 \leq x \leq 1.2} |f^{(4)}(x)| = \max_{0 \leq x \leq 1.2} |\cos(x)| = 1$, then solving for n, we obtain, $n \geq 10.8432$. Hence to get the required accuracy, we need 12 (even) subintervals which means $n + 1 = 13$ points. •