PHYSICS 201
Solutions 1st HOMEWORK
Dr. V. Lempesis

Hand in: Tuesday 8th of October March 2013

1. Solve the following matrix equation, $2(\mathbf{X}+\mathbf{B}) = 3(\mathbf{X}-\mathbf{A}) - 4\mathbf{B}$, if

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -3 & -1 & 0 \\ 7 & 0 & 3 \end{bmatrix}.$$

Solution: $2(\mathbf{X}+\mathbf{B}) = 3(\mathbf{X}-\mathbf{A}) - 4\mathbf{B} \Rightarrow 2\mathbf{X} + 2\mathbf{B} = 3\mathbf{X} - 3\mathbf{A} - 4\mathbf{B} \Rightarrow \mathbf{X} = 6\mathbf{B} + 3\mathbf{A}$

Thus

$$\mathbf{X} = 6\mathbf{B} + 3\mathbf{A} = 6\begin{bmatrix} -3 & -1 & 0 \\ 7 & 0 & 3 \end{bmatrix} + 3\begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -18 & -6 & 0 \\ 42 & 0 & 18 \end{bmatrix} + \begin{bmatrix} -3 & 3 & -9 \\ 6 & 0 & 18 \end{bmatrix} = \begin{bmatrix} -21 & -3 & -9 \\ 48 & 0 & 36 \end{bmatrix}$$

2. If,

$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}, \quad (n \in \mathrm{N}).$$

Solution: **a)** Show that holds for n=1

$$\mathbf{A}^1 = \begin{bmatrix} a^1 & 0 & 0 \\ 0 & b^1 & 0 \\ 0 & 0 & c^1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

b) Let's assume that it holds for n=k, thus

$$\mathbf{A}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

c) Let's check if it holds for n=k+1, thus

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a^k a + 0 + 0 & 0+0+0 & 0+0+0 \\ 0+0+0 & b^k b + 0 + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & c^k c + 0 + 0 \end{bmatrix} =$$

$$= \begin{bmatrix} a^{k+1} & 0 & 0 \\ 0 & b^{k+1} & 0 \\ 0 & 0 & c^{k+1} \end{bmatrix}$$

3. Show that $\left[\mathbf{A}, \mathbf{B}\right] = \mathbf{C}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: $\left[\mathbf{A}, \mathbf{B}\right] = \mathbf{C} \Rightarrow \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} = \mathbf{C}$

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot1+0\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot1+0\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot1+0\cdot0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot = \begin{bmatrix} 0\cdot0+0\cdot0+0\cdot0 & 0\cdot1+0\cdot0+0\cdot0 & 0\cdot0+0\cdot0+0\cdot0 \\ 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+0\cdot1 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot1+0\cdot0+0\cdot0 & 0\cdot0+0\cdot1+0\cdot0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{C}$

4. The three Pauli spin matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Calculate σ_i^2 for i =1, 2, 3.

These matrices were used by Pauli in the nonrelativistic theory of electron spin

Solution:

$$\sigma_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\sigma_2^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} (-i)i & 0 \\ 0 & i(-i) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\sigma_3^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0 \\ 0 & (-1)(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$