# Analysis II

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# Real Analysis II

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# PREFACE

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# CHAPTER I

# THE RIEMANN INTEGRAL

## 1 Definition of The Riemann Integral

**Definition 1.1**

1. A finite ordered set $\sigma = \{x_0, \dots, x_n\}$ is called a partition of the interval $[a, b]$ if $a = x_0 < \dots < x_n = b$. The interval $[x_j, x_{j+1}]$ is called the $j^{\text{th}}$ subinterval of $\sigma$.

2. If $\sigma = \{x_0, \dots, x_n\}$ is a partition of the interval $[a, b]$, we define the norm of $\sigma$ by:

$$||\sigma|| = \sup_{0 \leq j \leq n-1} x_{j+1} - x_j.$$

3. A partition $\sigma_n = (x_0, \dots, x_n)$ of the interval $[a, b]$ is called uniform if $(x_k = a + k\dfrac{b-a}{n})$. In this case $\|\sigma\| = \dfrac{b-a}{n}$.

4. A partition $\sigma_1 = \{x_0, \dots, x_n\}$ is called finer than a partition $\sigma_2 = \{y_0, \dots, y_m\}$ if $\{y_0, \dots, y_m\} \subset \{x_0, \dots, x_n\}$ and we denote $\sigma_2 < \sigma_1$.

5. If $\sigma_1 = \{x_0, \dots, x_n\}$ and $\sigma_2 = \{y_0, \dots, y_m\}$ are two partitions of the interval $[a, b]$, we define the partition $\sigma_1 \cup \sigma_2$ defined by ordering the points $\{y_0, \dots, y_m, x_0, \dots, x_n\}$.

**Definition 1.2**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Define

$$M_j = \sup_{x\in[x_j,x_{j+1}]} f(x), \qquad m_j = \inf_{x\in[x_j,x_{j+1}]} f(x),$$

$$U(f,\sigma) = \sum_{j=0}^{n-1} M_j(x_{j+1}-x_j), \quad L(f,\sigma) = \sum_{j=0}^{n-1} m_j(x_{j+1}-x_j) \tag{1.1}$$

The sums $U(f,\sigma)$ and $L(f,\sigma)$ are called respectively the upper and the lower sums of $f$ on the partition $\sigma$. (Note that $L(f,\sigma) \leq U(f,\sigma)$.)

**Lemma 1.3**

Let $\sigma_1 = \{x_0, \dots, x_n\}$ be a partition of the interval $[a,b]$, $\sigma_2 = \{a,y,b\}$ with $y \in ]a,b[$ and $f\colon [a,b] \longrightarrow \mathbb{R}$ a bounded function, then

$$L(f,\sigma_1) \leq L(f,\sigma) \leq U(f,\sigma) \leq U(f,\sigma_1), \tag{1.2}$$

where $\sigma = \sigma_1 \cup \sigma_2$.

**Proof** .
The proof is obvious if $y \in \sigma_1$. Suppose now that $y \in ]x_j, x_{j+1}[$, we have
$L(f,\sigma_1) = \sum_{i=0}^{j-1}(x_{i+1}-x_i)m_i + (x_{j+1}-x_j)m_j + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)m_i$,
$U(f,\sigma_1) = \sum_{i=0}^{j-1}(x_{i+1}-x_i)M_i + (x_{j+1}-x_j)M_j + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)M_i$ and

$$\begin{aligned} L(f,\sigma) &= \sum_{i=0}^{j-1}(x_{i+1}-x_i)m_i + (y-x_j)\inf_{x\in]x_j,y[} f(x) \\ &+ (x_{j+1}-y)\inf_{x\in]y,x_{j+1}} f(x) + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)m_i. \end{aligned}$$

$$\begin{aligned} U(f,\sigma) &= \sum_{i=0}^{j-1}(x_{i+1}-x_i)M_i + (y-x_j)\sup_{x\in]x_j,y[} f(x) \\ &+ (x_{j+1}-y)\sup_{x\in]y,x_{j+1}} f(x) + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)M_i. \end{aligned}$$

But $m_j \leq \inf_{x\in]x_j,y[} f(x)$, $m_j \leq \inf_{x\in]y,x_{j+1}[} f(x)$, $M_j \geq \sup_{x\in]x_j,y[} f(x)$ and $M_j \geq \sup_{x\in]y,x_{j+1}[} f(x)$. This yields that $L(f,\sigma_1) \leq L(f,\sigma)$ and $U(f,\sigma) \leq U(f,\sigma_1)$. □

**Corollary 1.4**

If $\sigma_1$ is finer than $\sigma_2$ and $f\colon [a,b] \longrightarrow \mathbb{R}$ is a bounded function, then

$$L(f,\sigma_2) \leq L(f,\sigma_1) \leq U(f,\sigma_1) \leq U(f,\sigma_2) \tag{1.3}$$

**Proof** .

□

**Theorem 1.5**

If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a bounded function and $\sigma_1, \sigma_2$ are two partitions of the interval $[a,b]$, then $L(f,\sigma_1) \leq U(f,\sigma_2)$.

**Proof** .
$L(f,\sigma_1) \leq L(f,\sigma_1\cup\sigma_2) \leq U(f,\sigma_1\cup\sigma_2) \leq U(f,\sigma_2)$. □

**Definition 1.6**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function, $P([a,b])$ the set of partitions of $[a,b]$, then we define respectively the upper and the lower integral of $f$ on the interval $[a,b]$ by:

$$U(f) = \inf_{\sigma\in P([a,b])} U(f,\sigma), \qquad L(f) = \sup_{\sigma\in P([a,b])} L(f,\sigma).$$

$U(f)$ and $L(f)$ are called respectively the upper and the lower Darboux sums of $f$ on the interval $[a,b]$.

**Definition 1.7**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The function $f$ is called Riemann integrable on the interval $[a,b]$ if $U(f) = L(f)$.

If $f$ is Riemann integrable on the interval $[a,b]$, we denote $\displaystyle\int_a^b f(x)dx = U(f) = L(f)$ and called the integral of $f$ on the interval $[a,b]$.
The set of Riemann integrable functions on the interval $[a,b]$ is denoted by $\mathscr{R}([a,b])$.

**Remark 1 :**
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. If there exists a partition $\sigma$ of $[a,b]$ such that $U(f,\sigma) = L(f,\sigma)$, then $f$ is Riemann integrable on $[a,b]$ and $\displaystyle\int_a^b f(x)dx = U(f,\sigma)$.
This is because $L(f,\sigma) \leq U(f)$ and $L(f) \leq U(f,\sigma)$.

**Example 1 :**

1. Any step function on an interval $[a,b]$ is Riemann integrable. Indeed let $\sigma = (x_0 = a, \ldots, x_n = b)$ be a partition of $[a,b]$ associated to $f$. If $f(x) = c_j$ on $]x_j, x_{j+1}[$, then $M_j = m_j = c_j$ and $U(f,\sigma) = L(f,\sigma)$ and $f$ is is Riemann integrable.
2. Let $f$ be the caracteristic function of $\mathbb{Q} \cap [0,1]$. For any partition $\sigma$ of $[0,1]$, $L(f,\sigma) = 0$ and $U(f,\sigma) = 1$. Then $f$ is not Riemann integrable.

**Theorem 1.8**

[Riemann's Criterion]
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent

1. $f$ is Riemann-integrable.
2. $\forall \varepsilon > 0$; there exists a partition $\sigma$ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$.

**Proof** .
NC: If $U(f) = L(f)$, then $\forall\ \varepsilon > 0$, there exists a partition $\sigma$ such that $0 \leq L(f) - L(f,\sigma) \leq \frac{\varepsilon}{2}$ and there exists a partition $\sigma'$ such that $0 \leq U(f,\sigma') - U(f) \leq \frac{\varepsilon}{2}$. Then $0 \leq U(f,\sigma\cup\sigma') - U(f) \leq U(f,\sigma') - U(f) \leq \frac{\varepsilon}{2}$. Also $0 \leq L(f) - L(f,\sigma\cup\sigma') \leq L(f) - L(f,\sigma) \leq \frac{\varepsilon}{2}$. Then $U(f,\sigma\cup\sigma') - L(f,\sigma\cup\sigma') \leq \varepsilon$.
SC: $L(f,\sigma) \leq L(f) \leq U(f,\sigma)$ and $L(f,\sigma) \leq U(f) \leq U(f,\sigma)$, then $0 \leq U(f) - L(f) \leq U(f,\sigma) - L(f,\sigma) \leq \varepsilon$, for all $\varepsilon > 0$. Hence $U(f) = L(f)$. $\square$

**Proposition 1.9**

A function $f$ is Riemann integrable if and only if $\forall\ \varepsilon > 0$, there are two step functions on $[a,b]$ $f_\varepsilon$ and $g_\varepsilon$ such that $f_\varepsilon \leq f \leq g_\varepsilon$ and $\displaystyle\int_a^b (g_\varepsilon - f_\varepsilon)(x)\ dx \leq \varepsilon$.

**Proof** .

1. If $f$ is Riemann integrable, then $\forall\ \varepsilon > 0$, there exists a partition $\sigma$ of $[a,b]$ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$. We take $f_\varepsilon = m_i$ and $g_\varepsilon = M_i$ on $]x_i, x_{i+1}[$ and $f_\varepsilon(x_i) = g_\varepsilon(x_i) = f(x_i)$ for all $0 \leq i \leq n-1$.

2. Conversely: Let $\varepsilon > 0$ and $\sigma$ a partition of $[a,b]$ associated to both $f_\varepsilon$ and $g_\varepsilon$. $f_\varepsilon \leq f \leq g_\varepsilon$.

   $0 \leq U(f,\sigma) - L(f,\sigma) \leq U(g_\varepsilon,\sigma) - L(f_\varepsilon,\sigma) = \displaystyle\int_a^b (g_\varepsilon - f_\varepsilon)\ dx \leq \varepsilon$. So $f$ is Riemann integrable.

$\square$

**Theorem 1.10**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. We denote $\mathscr{S}([a,b])$ the set of step functions on $[a,b]$. We have the following:

$$L(f) = \sup\{\int_a^b g(x)dx:\ g \leq f,\ g \in \mathscr{S}([a,b])\}, \tag{1.4}$$

$$L(f) = \inf\{\int_a^b g(x)dx:\ f \leq g,\ g \in \mathscr{S}([a,b])\}. \tag{1.5}$$

**Proof** .

For any partition $\sigma = \{a_0, \dots, a_n\}$ of $[a,b]$, $L(f,\sigma) = \displaystyle\int_a^b f_\sigma(x)dx$ and $U(f,\sigma) = \displaystyle\int_a^b F_\sigma(x)dx$, where $f_\sigma$ and $F_\sigma$ are the step functions defined by $f_\sigma(x) = \inf_{t\in[a_{k-1},a_k)} f(t)$ and $F_\sigma(x) = \sup_{t\in[a_{k-1},a_k)} f(t)$, for $x \in [a_{k-1}a_k)$, $k = 0, \dots, n$. If $g \leq f$ and $g \in \mathscr{S}([a,b])$, then there exists a partition $\sigma = \{a_0, \dots, a_n\}$ of $[a,b]$ such that $g$ is constants on any interval $(a_{k-1}, a_k)$. In this case

$\int_a^b g(x)dx \leq L(f,\sigma)$. Then

$$\sup\{\int_a^b g(x)dx: \ g \leq f, \ g \in \mathscr{S}([a,b])\} \leq L(f).$$

Moreover for any partition $\sigma$, there exists $g \leq f, \ g \in \mathscr{S}([a,b])$ such that $L(f,\sigma) = \int_a^b g(x)dx$. Then

$$L(f) \leq \sup\{\int_a^b g(x)dx: \ g \leq f, \ g \in \mathscr{S}([a,b])\}.$$

The same method for the upper sum. $\square$

**Theorem 1.11**

[Darboux's Criterion]
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent

1. $f$ is Riemann-integrable,
2. For all $\varepsilon > 0$; there exists $\delta > 0$ such that for all partition of the interval $[a,b]$ such that if $||\sigma|| \leq \delta$ then $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$.

Recall the notion of oscillation of a function on an interval.

**Definition 1.12**

[Oscillation of a function]
The Oscillation of a function $f\colon I \longrightarrow \mathbb{R}$ at a point $a \in I$ is defined by

$$w_a(f) = \lim_{r\to 0} \sup\{|f(y) - f(z)|; \ y,z \in ]a-r, a+r[\cap I\}.$$

If $f$ is bounded, the oscillation of $f$ on the interval $[a,b]$ denoted by $O(f,[a,b])$ is defined by $\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$.

Note that $w_a(f) \geq 0$ and $f$ is continuous at $a$ if and only if $w_a(f) = 0$. Moreover, if $f$ is bounded then $w_a(f) \leq O(f,[a,b])$.

**Proof** .
The condition is obviously sufficient.

NC: Let $f$ be a Riemann integrable function (we assume that $f$ is not constant), so $\forall\ \varepsilon > 0$ there is a partition $\sigma = (x_0 = a, \dots, x_n = b)$ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$. We set $M = O(f,[a,b])$ the oscillation of $f$ on the interval $[a,b]$, $\alpha_1 = \dfrac{\varepsilon}{nM}$, $\alpha_2 = \inf\limits_{0\leq j\leq n-1}(x_{j+1} - x_j)$ and $\alpha = \min(\alpha_1, \alpha_2)$. That is $\sigma' = (y_0 = a, \dots, y_m = b)$ a partition of $[a,b]$ such that $|\sigma'| < \alpha$. There are at most $n$ intervals $]y_{j-1}, y_j[$ which contain $x_i$. The others are contained in the intervals $]x_{k-1}, x_k[$. We denote

$$\begin{aligned} M'_j &= \sup_{x\in ]y_j,y_{j+1}[} f(x), \quad M_j = \sup_{x\in ]x_j,x_{j+1}[} f(x), \\ m'_j &= \inf_{x\in ]y_j,y_{j+1}[} f(x) \quad \text{et } m_j = \inf_{x\in ]x_j,x_{j+1}[} f(x). \end{aligned}$$

$$\begin{aligned} U(f,\sigma') - L(f,\sigma') &= \sum_{]y_j,y_{j+1}[\subset ]x_i,x_{i+1}[} (y_{j+1} - y_j)(M'_j - m'_j) \\ &+ \sum_{x_i\in ]y_j,y_{j+1}[} (y_{j+1} - y_j)(M'_j - m'_j) \end{aligned}$$

It follows that

$$\begin{aligned} U(f,\sigma') - L(f,\sigma') &\leq \sum_{i=0}^{n-1} (x_{i+1} - x_i)(M_i - m_i) + n\alpha M \\ &= U(f,\sigma) - L(f,\sigma) + n\alpha M \leq 2\varepsilon. \end{aligned}$$

□

**Proposition 1.13**

Let $f$ be a Riemann integrable function and $I = \displaystyle\int_a^b f(x)dx$. Then $\forall\ \varepsilon > 0$ there exists $\alpha > 0$ such that for all partition $\sigma$ of $[a,b]$ with $\|\sigma\| < \alpha$, $|U(f,\sigma) - I| \leq \varepsilon$ and $|L(f,\sigma) - I| \leq \varepsilon$.

**Theorem 1.14**

Any monotone function on an interval $[a,b]$ is Riemann integrable.

**Proof** .

Suppose that $f$ is increasing. Let $\sigma = (x_0 = a, \ldots, x_n = b)$ be a partition of $[a, b]$ and $\alpha = \|\sigma\| = \sup_{0 \le j \le n-1}(x_{j+1} - x_j)$.
$U(f, \sigma) - L(f, \sigma) \le \alpha[(M_0 - m_0) + \ldots + (M_{n-1} - m_{n-1})]$.
$M_j = \sup_{x \in ]x_j, x_{j+1}[} f(x) \le f(x_{j+1})$ and $m_j = \inf_{x \in ]x_j, x_{j+1}[} f(x) \ge f(x_j)$. Then

$$U(f, \sigma) - L(f, \sigma) \le \alpha \sum_{j=0}^{n-1} (f(x_{j+1}) - f(x_j)) \le \alpha(f(b) - f(a)).$$

For $\varepsilon > 0$, we take a partition $\sigma = (x_0 = a, \ldots, x_n = b)$ of $[a, b]$ such that

$$(f(b) - f(a)) \sup_{0 \le j \le n-1} (x_{j+1} - x_j) \le \varepsilon.$$

We get: $U(f, \sigma) - L(f, \sigma) \le \varepsilon$. Then $f$ is Riemann integrable. □

**Theorem 1.15**

Any continuous function on an interval $[a, b]$ is Riemann-integrable.

**Proof** .
Let $f$ be a continuous function on an interval $[a, b]$, then $f$ is uniformly continuous. Hence $\forall\, \varepsilon > 0$, $\exists\, \alpha > 0$ such that $|f(x) - f(x')| < \frac{\varepsilon}{b-a}$ for all $|x - x'| < \alpha$. Let $\sigma = (x_0 = a, \ldots, x_m = b)$ be a partition of $[a, b]$ such that $\sup_{0 \le j \le n-1}(x_{j+1} - x_j) < \alpha$. As $f$ is continuous on $[a, b]$, there exists $x'_j$ and $x''_j$ in $[x_j, x_{j+1}]$ such that $M_j = f(x'_j)$ and $m_j = f(x''_j)$; $|x'_j - x''_j| \le |x_{j+1} - x_j| < \alpha$, then $M_j - m_j \le \frac{\varepsilon}{b-a}$. We deduce that

$$0 \le U(f, \sigma) - L(f, \sigma) \le \sum_{j=0}^{n-1} (x_{j+1} - x_j)(M_j - m_j) \le \frac{\varepsilon}{b - a} \sum_{j=0}^{n-1} (x_{j+1} - x_j) = \varepsilon.$$

□

**Definition 1.16**

Let $\sigma = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$. We say that $\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ is a mark of $\sigma$ if $\forall 0 \le j \le n - 1$, $\alpha_j \in [x_j, x_{j+1}]$.
We define

$$U(f,\sigma,\alpha)=\sum_{j=0}^{n-1} f(\alpha_j)(x_{j+1}-x_j)$$

called the Riemann sum of $f$ on the partition $\sigma$ with respect to the mark $\alpha$.

**Remark 2 :**

1. Let $f$ be a Riemann integrable function on the interval $[a,b]$. If $\sigma = \{x_0, \ldots, x_n\}$ a partition of $[a,b]$ and $\tau = (\lambda_1, \ldots, \lambda_n)$ a mark on $\sigma$, then the sum $R(f,\sigma,\tau) = \sum_{j=0}^{n-1}(x_{j+1}-x_j)f(\lambda_j)$ verifies
$$U(f,\sigma) \leq R(f,\sigma,\tau) \leq L(f,\sigma).$$
Then $\forall\, \varepsilon > 0\ \exists \alpha > 0$ such that for all partition $\sigma$ such that $\|\sigma\| < \alpha$ and for all $\tau = (\lambda_1, \ldots, \lambda_n)$ a mark on $\sigma$, we have: $|R(f,\sigma,\tau) - I| \leq \varepsilon$.

2. The same result is obtained if we replace $f(\lambda_j)$ by any constant $\mu_j$, with $m_j \leq \mu_j \leq M_j$.

3. If $f$ is Riemann integrable on the interval $[a,b]$, the sequence $(S_n)_n$ defined by:
$$S_n = \frac{b-a}{n}\sum_{k=1}^{n} f(a + k\frac{b-a}{n})$$
converges to $\int_a^b f(x)dx$.

# 2 Properties of the Riemann Integral

## 2.1 Basic Properties

**Properties 2.1**

1. Linearity: $\int_a^b \alpha(f+\beta g)(x)dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$

2. If $f \geq 0$, then $\int_a^b f(x)dx \geq 0.$

3. If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
4. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.
5. If $m \leq f(x) \leq M$, for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

## 2.2 The Chasles Indentity

**Proposition 2.2**

If $f$ is Riemann integrable on $[a,b]$, it is also interval on any interval $[c,d] \subset [a,b]$.

**Proof** .
Lett $\varepsilon > 0$. there exist $g \in \mathscr{S}([a,b])$ $g \leq f$ and $h \in \mathscr{S}([a,b])$ $f \leq h$ such that $0 \leq \int_a^b (h-g)(x)dx < \varepsilon$. From the Chasles identity, we have

$$\int_a^b (h-g)(x)dx = \underbrace{\int_a^c (h-g)(x)dx}_{\geq 0} + \int_c^d (h-g)(x)dx + \underbrace{\int_d^b (h-g)(x)dx}_{\geq 0} \geq \int_a^b (h-g)(x)dx.$$

Then $0 \leq \int_c^d (h-g)(x)dx \leq \varepsilon$ and hence $f$ is integrable on $[c,d]$. □

**Theorem 2.3**

A bounded function on an interval $[a,b]$ is Riemann-integrable if and only if it is Riemann-integrable on $[a,c]$ and on $[c,b]$, for all $c \in [a,b]$. Moreover if $f$ is Riemann-integrable on $[a,b]$, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx. \tag{2.6}$$

(This identity is called the Chasles identity)

**Proof** .
Assume that $f$ is Riemann-integrable on $[a, b]$, so $\forall\, \varepsilon > 0$, there exists a partition $\sigma$ of $[a, b]$ such that $U(f, \sigma) - L(f, \sigma) \leq \varepsilon$. Let $\sigma' = \sigma \cup \{c\}$; then $U(f, \sigma') - L(f, \sigma') \leq U(f, \sigma) - L(f, \sigma) \leq \varepsilon$. Consider $\sigma' = \sigma_1 \cup \sigma_2$, with $\sigma_1$ a partition of $[a, c]$ formed from the points of $\sigma'$ in $[a, c]$ and $\sigma_2$ a partition of $[c, b]$ formed from the points of $\sigma'$ in $[c, b]$. It follows that $U(f, \sigma_1) - L(f, \sigma_1) \leq \varepsilon$ and $U(f, \sigma_2) - L(f, \sigma_2) \leq \varepsilon$. So $f$ is separately Riemann-integrable over $[a, c]$ and $[c, b]$.

If $f$ is separately Riemann-integrable over $[a, c]$ and $[c, b]$, so $\forall\, \varepsilon > 0$, there is a partition $\sigma_1$ of $[a, c]$ and a partition $\sigma_2$ of $[c, b]$ such that $U(f, \sigma_1) - L(f, \sigma_1) \leq \varepsilon$ and $U(f, \sigma_2) - L(f, \sigma_2) \leq \varepsilon$. The set $\sigma = \sigma_1 \cup \sigma_2$ is a partition of $[a, b]$ and $U(f, \sigma) - L(f, \sigma) \leq 2\varepsilon$, which proves that $f$ is Riemann-integrable on $[a, b]$.

Consider for a Riemann-integrable function $f$ on $[a, b]$ the numbers: $I = \displaystyle\int_a^b f(x)\, dx$, $I_1 = \displaystyle\int_a^c f(x)\, dx$ and $I_2 = \displaystyle\int_c^b f(x)\, dx$.
$\forall\, \varepsilon > 0$, there exists $\alpha$ >0 such that for any partitions $\sigma$ of $[a, b]$, $\sigma_1$ of $[a, c]$ and $\sigma_2$ of $[c, b]$, with $(\|\sigma\| < \alpha$, $\|\sigma_1\| < \alpha$ and $\|\sigma_2\| < \alpha$ we have: $|U(f, \sigma) - I| \leq \varepsilon$, $|U(f, \sigma_1) - I_1| \leq \varepsilon$ and $|U(f, \sigma_2) - I_2| \leq \varepsilon$. We consider the partition $\sigma' = \sigma_1 \cup \sigma_2$, $\|\sigma'\| < \alpha$, $|U(f, \sigma') - I| \leq \varepsilon$; similarly $|U(f, \sigma') - I_1 - I_2| \leq |U(f, S_1) - I_1| + |U(f, S_2) - I_2| \leq 2\varepsilon$. So $I = I_1 + I_2$.

$\square$

**Remark 3 :**

By convention if $b < a$, we set $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

**Exercise 1 :**
Compute the following integrals:

1. $F(x) = \displaystyle\int_0^\pi |x - t| \sin t dt$ for $x \in \mathbb{R}$.

2. $F(x) = \displaystyle\int_0^\pi |x - t| \sin t dt$ for $x \in \mathbb{R}$.

Solution

1. I $x \leq 0$, $F(x) = \displaystyle\int_0^\pi (t - x) \sin t dt = \pi - 2x$.

   If $0 \leq x \leq \pi$, then $F(x) = \displaystyle\int_0^x (x - t) \sin t dt + \int_x^\pi (t - x) \sin t dt = \pi - 2 \sin x$.

   If $x \geq \pi$, then $F(x) = \displaystyle\int_0^\pi (x - t) \sin t dt = 2x - \pi$.

2. If $x \le 0$, then $F(x) = \displaystyle\int_0^\pi (t-x)\sin t dt = \pi - 2x$.

   If $0 \le x \le \pi$, then $F(x) = \displaystyle\int_0^x (x-t)\sin t dt + \int_x^\pi (t-x)\sin t dt = \pi - 2\sin x$.

   If $x \ge \pi$, then $F(x) = \displaystyle\int_0^\pi (x-t)\sin t dt = 2x - \pi$.

**Proposition 2.4: (Chasles Indentity for Lower and Uper sums)**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function and let $c \in [a,b]$. Then

$$L_{[a,b]}(f) = L_{[a,c]}(f) + L_{[c,b]}(f), \quad \text{and} \quad U_{[a,b]}(f) = U_{[a,c]}(f) + U_{[c,b]}(f). \tag{2.7}$$

**Proof** .
The identities are trivially true for $c = a$ or $b$. Let $c \in ]a,b[$ and $g \in \mathscr{S}([a,b])$, $g \le f$. Consider $g_1$ and $g_2$ the restrictions respectively of $g$ on $[a,c]$ and $[c,b]$ respectively. $g_1$ and $g_2$ are step functions. Using the Chasles identity for the step function $g$, we get:

$$\int_a^b g(x)dx = \int_a^c g_1(x)dx + \int_c^b g_2(x)dx \le L_{[a,c]}(f) + L_{[c,b]}(f).$$

Then $L_{[a,b]}(f) \le L_{[a,c]}(f) + L_{[c,b]}(f)$.
Inversely let $g_1 \in \mathscr{S}([a,c])$, $g \le f$ and $g_2 \in \mathscr{S}([c,b])$, $g_2 \le f$. Define the function $g$ on the interval $[a,b]$ by $g = g_1$ on $[a,c]$ and $g = g_2$ on $]c,b]$. The function $g$ is a step function and $g \le f$ on $[a,b]$. We have:

$$\int_a^c g_1(x)dx + \int_c^b g_2(x)dx \le L_{[a,b]}(f).$$

We fix $g_2$ and take the sup on $g_1$, we get

$$L_{[a,c]}(f) + \int_c^b g_2(x)dx \le L_{[a,b]}(f)$$

and if we take the sup on $g_2$ we get

$$L_{[a,c]}(f) + L_{[c,b]}(f)dx \le L_{[a,b]}(f).$$

We deduce that

$$L_{[a,c]}(f) + L_{[c,b]}(f)dx = L_{[a,b]}(f).$$

□

**Proposition 2.5**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function and let $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x$ in the open interval $]a,b[$. Then we have:

$$(b-a)m \leq L_{[a,b]}(f) \leq U_{[a,b]}(f) \leq (b-a)M.$$

**Proof** .
Let $g$ (resp. $h$ ) be the step functions on $[a,b]$ defined by $g(a) = f(a) = h(a)$, $g(b) = f(b) = h(b)$ and $g(x) = m$ (resp. $h(x) = M$) for all $x \in ]a,b[$. Then $g \in \mathscr{S}([a,b])$, $g \leq f$ and $h \in \mathscr{S}([a,b])$, $h \geq f$ and therefore

$$(b-a)m = \int_a^b g(x)dx \leq L_{[a,b]}(f) \leq U_{[a,b]}(f) \leq \int_a^b h(x)dx = (b-a)M.$$

$\square$

**Remark 4 :**
The lower and upper integrals are not linear: for two bounded functions $f, g\colon [a,b] \longrightarrow \mathbb{R}$ we can show that $L_{[a,b]}(f) + L_{[a,b]}(g) \leq L_{[a,b]}(f+g)$ and $U_{[a,b]}(f+g) \leq U_{[a,b]}(f) + U_{[a,b]}(g)$, but these inequalities can be strict. For example, if $f, g : [0,1] \longrightarrow \mathbb{R}$ are defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise, and $g(x) = 1 - f(x)$, then $L_{[a,b]}(f) = 0 = L_{[a,b]}(g)$ and $U_{[a,b]}(f) = 1 = U_{[a,b]}(g)$, while $L_{[a,b]}(f+g) = 1 = U_{[a,b]}(f+g)$.

## 2.3 Examples of Riemann Integrable Functions

**Definition 2.6**

A function $f$ defined on an interval $[a,b]$ is said to be piecewise continuous if there is a partition $\sigma = (x_0 = a, \ldots, x_n = b)$ of $[a,b]$ such that $f$ is continuous on each open interval $]x_i, x_{i+1}[$ and $f$ admits a right limit of $x_i$ for all $0 \leq i \leq n-1$ and a left limit of $x_{i+1}$ for all $1 \leq i \leq n$.

**Exercise 1 :**
Show that any piecewise continues function on an interval $[a,b]$ is Riemann integrable.

**Theorem 2.7**

The space of Riemann-integrable functions on $[a, b]$ is a vector space on $\mathbb{R}$.

**Theorem 2.8**

If $f$ is Riemann-integrable on an interval $[a, b]$, then $|f|$ is too.

**Proof** .
Let $[c, d] \subset [a, b]$.
- If $f$ is non negative on $[c, d]$, then $\sup\limits_{[c,d]} |f| = \sup\limits_{[c,d]} f$ and $\inf\limits_{[c,d]} |f| = \inf\limits_{[c,d]} f$.
- If $f$ is non positive on $[c, d]$, then $\sup\limits_{[c,d]} |f| = -\inf\limits_{[c,d]} f$ and $\inf\limits_{[c,d]} |f| = -\sup\limits_{[c,d]} f$.
- If $f$ has no constant sign on $[c, d]$, then $\sup\limits_{[c,d]} f \geq 0$ and $\inf_{[c,d]} f \leq 0$.

It follows that $\sup\limits_{[c,d]} |f| = \max(\sup\limits_{[c,d]} f, -\inf\limits_{[c,d]} f)$. We deduce that in all cases $\sup\limits_{[c,d]} |f| - \inf\limits_{[c,d]} |f| \leq \sup\limits_{[c,d]} f - \inf\limits_{[c,d]} f$, which gives that $U(|f|, \sigma) - L(|f|, \sigma) \leq U(f, \sigma) - L(f, \sigma)$, for any partition $\sigma$ of $[a, b]$. It results that $|f|$ is Riemann-integrable. □

**Proposition 2.9**

If two functions $f$ and $g$ are Riemann-integrable on a interval $[a, b]$, then $\sup(f, g)$ and $\inf(f, g)$ are Riemann-integrable.

**Proof** .
$\sup(f, g) = \frac{1}{2}(f + g + |f - g|)$ and $\inf(f, g) = \frac{1}{2}(f + g - |f - g|)$. □

**Theorem 2.10**

The product of two Riemann-integrable functions is a Riemann-integrable function.

**Proof** .

It suffices to prove the result for two non negative functions. Let $f$ and $g$ be two non negative Riemann-integrable functions on $[a,b]$. Let $M$ be an upper bound of $f$ and $g$ over $[a,b]$. For any a partition $\sigma$ of $[a,b]$, $U(fg,\sigma) - L(fg,\sigma) \leq M(U(f,\sigma) - L(f,\sigma)) + M(U(g,\sigma) - L(g,\sigma))$. It follows that $f.g$ is Riemann-integrable. $\square$

**Theorem 2.11**

Let $f$ be a non negative Riemann-integrable function on $[a,b]$. Then for all $\alpha > 0$, the function $f^\alpha(x)$ is Riemann-integrable.

**Proof** .
Let $\varepsilon > 0$, there is a partition $\sigma = (x_0 = a, x_1, \ldots, x_n = b)$ such that:

$$\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i) < \varepsilon,$$

with

$$M_i = \sup_{x\in]x_i,x_{i+1}[} f(x) \quad \text{et } m_i = \inf_{x\in]x_i,x_{i+1}[} f(x).$$

Note that $\forall\, t \in [0,1];\ 1 - t^\alpha \leq (1-t)\sup(1,\alpha)$, which gives that

$$M_i^\alpha - m_i^\alpha \leq (M_i - m_i)M_i^{\alpha-1}\sup(\alpha, 1).$$

If $\alpha > 1$: $M_i^{\alpha-1} \leq M^{\alpha-1}$, with $M = \sup\limits_{x\in[a,b]} f(x)$. In this case, we have:

$$\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i^\alpha - m_i^\alpha) < \alpha\varepsilon M^{\alpha-1},$$

which gives the result in this case.
If $\alpha < 1$ and if $M_i \leq \varepsilon$ we have: $M_i^\alpha - m_i^\alpha \leq \varepsilon^\alpha$ and if $M_i > \varepsilon$ we have: $M_i^{\alpha-1} < \varepsilon^{\alpha-1}$ which yields

$$\begin{aligned}\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i^\alpha - m_i^\alpha) &\leq \sum_{i=0}^{n-1}(x_{i+1} - x_i)\varepsilon^\alpha + \sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i)\varepsilon^{\alpha-1} \\ &= (b-a)\varepsilon^\alpha + \varepsilon^{\alpha-1}\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i) \leq \varepsilon^\alpha(b-a+1).\end{aligned}$$

In general, we have the following theorem:

**Theorem 2.12**

Let $f\colon [a,b] \longrightarrow [c,d]$ be a Riemann integrable function and $\varphi\colon [c,d] \longrightarrow \mathbb{R}$ a continuous function. Then $\varphi \circ f$ is Riemann integrable.

**Proof** .
Let $\varepsilon > 0$, we will construct a partition $\sigma = (x_0 = a, x_1, \ldots, x_n = b)$ of $[a,b]$ such that: $U(\varphi \circ f, \sigma) - L(\varphi \circ f, \sigma) < \varepsilon$.
The function $\varphi$ is uniformly continuous on $[c,d]$ and bounded, then there is $M > 0$ such that $|\varphi(x)| \leq M$, $\forall x \in [c,d]$ and if $\varepsilon' = \dfrac{\varepsilon}{2M + (b-a)}$, there is $0 < \alpha < \varepsilon'$ such that for $|x - y| < \alpha$, $|\varphi(x) - \varphi(y)| \leq \varepsilon'$, for all $x, y \in [c,d]$.
Since $f$ is Riemann-integrable on $[a,b]$, there exist a partition $\sigma = (x_0 = a, x_1, \ldots, x_n = b)$ of $[a,b]$ such that:

$$U(f,\sigma) - L(f,\sigma) < \alpha^2. \tag{2.8}$$

Let $M_j = \sup\{f(x);\ x \in [x_j, x_{j+1}]\}$, $m_j = \inf\{f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{M}_j = \sup\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{m}_j = \inf\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$.
We denote $J_1 = \{0 \leq j \leq n-1;\ M_j - m_j < \alpha$ et $J_2 = \{0 \leq j \leq n-1;\ M_j - m_j \geq \alpha$.
If $j \in J_1$, then by the uniform continuity of $\varphi \circ f$, we have $|\varphi \circ f(x) - \varphi \circ f(y)| < \varepsilon'$ for all $x, y \in [x_j, x_{j+1}]$, which yields $\tilde{M}_j - \tilde{m}_j \leq \varepsilon'$, then

$$\sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'(b-a). \tag{2.9}$$

By (1.1),

$$\alpha^2 > \sum_{j \in J_2} (M_j - m_j)(x_{j+1} - x_j) \geq \alpha \sum_{j \in J_2} (x_{j+1} - x_j).$$

Then $\sum\limits_{j \in J_2} (x_{j+1} - x_j) < \alpha < \varepsilon'$ and since $\tilde{M}_j - \tilde{m}_j \leq 2M$, we have:

$$\sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq 2M \sum_{j \in J_2} (x_{j+1} - x_j) < 2M\varepsilon'. \tag{2.10}$$

It results by (1.2) and (1.5) that

$$U(\varphi \circ f, \sigma) - L(\varphi \circ f, \sigma) = \sum_{j=0}^{n-1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'((b-a) + 2M) = \varepsilon.$$

□

**Remark 5 :**

1. The integral of a non negative Riemann-integrable function is a non negative real number.

2. If $f$ is Riemann-integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx \leq (b-a) \sup_{x\in[a,b]} |f(x)|.$$

**Corollary 2.13**

If $f$ is Riemann-integrable on $[a, b]$, then the function $F(x) = \displaystyle\int_a^x f(t)\ dt$ is continuous on $[a, b]$.

**Proof** .
$F(x) - F(y) = \displaystyle\int_y^x f(t)\ dt$. Since $f$ is bounded on $[a, b]$, there exist $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$. □

**Corollary 2.14**

Let $f$ be a Riemann-integrable function on $[a, b]$. If $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$, there exist $\lambda \in [m, M]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \lambda.$$

**Proof** .
We have: $m(b-a) \leq \displaystyle\int_a^b f(x)\ dx \leq M(b-a)$, then $\displaystyle\frac{1}{b-a}\int_a^b f(x)\ dx \in [m, M]$.

**Corollary 2.15**

[First Mean Value Formula]
Let $f$ and $g$ be two Riemann-integrable functions on an interval $[a, b]$. Assume that $f$ is continuous and $g$ has a constant sign on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)\ dx.$$

**Proof** .
If $\displaystyle\int_a^b g(x)\ dx = 0$, then $\displaystyle\int_a^b f(x)g(x)\ dx = 0$.
If $\displaystyle\int_a^b g(x)\ dx \neq 0$, we set $g_1 = \dfrac{1}{\int_a^b g(x)\ dx} g$, then $\displaystyle\int_a^b g_1(x)\ dx = 1$ and if $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$, there is $\lambda \in [m, M]$ such that $\displaystyle\int_a^b f(x)g_1(x)dx = \lambda$.

□

## 2.4 The Fundamental Theorem of Calculus

The following theorem can be called the "fundamental theorem of integral calculus", although we usually reserve this terminology for the particular case where $f$ is assumed to be continuous.

**Theorem 2.16**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Let $c \in ]a,b[$ and suppose that $f$ has a limit on the left at $c$ denoted by $f(c-)$, (respectively a limit on the right of $c$ denoted by $f(c+)$). Then the functions $x \longmapsto L_{[a,x]}(f)$ and $x \longmapsto U_{[a,x]}(f)$ are left (resp right) differentiable at $c$, with left derivative $f(c-)$ (resp with right derivative $f(c+)$).

**Proof** .
Let $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in ]c-\delta, c[$ we have $f(c-) - \varepsilon < f(x) < f(c-) + \varepsilon$ and therefore we have

$$(c-x)(f(c-)-\varepsilon) \leq L_{[x,c]}(f) \leq U_{[x,c]}(f) \leq (c-x)(f(c-)+\varepsilon).$$

Now, according to the Chasles identity, for all $x \in [a, c[$ we have:

$$L_{[a,c]}(f) - L_{[a,x]}(f) = L_{[x,c]}(f) \quad \text{and} \quad U_{[a,c]}(f) - U_{[a,x]}(f) = U_{[x,c]}(f)$$

We deduce that

$$(f(c-)-\varepsilon) \leq \frac{L_{[a,c]}(f) - L_{[a,x]}(f)}{c-x} \leq (f(c-)+\varepsilon)$$

and

$$(f(c-)-\varepsilon) \leq \frac{U_{[a,c]}(f)-U_{[a,x]}(f)}{c-x} \leq (f(c-)+\varepsilon).$$

Using the same method for the right derivative. $\square$

**Theorem 2.17**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. We suppose that at any point $x \in ]a,b[$, $f$ has a limit on the left, denoted $f(x-)$, and a limit on right, denoted $f(x+)$. So:

1. $f$ is Riemann integrable on $[a,b]$.
2. The function $F\colon [a,b] \longrightarrow \mathbb{R}$ defined by $F(x) = \displaystyle\int_a^x f(t)dt$ is continuous on $[a,b]$, $F(a) = 0$, and for all $x \in ]a,b[$, $F$ is left and right differentiable at $x$, with left derivative $f(x-)$ and with right derivative $f(x+)$.
3. If $f$ has a right limit $f(a+)$ at $a$, then $F$ is right differentiable at $a$ with right derivative $f(a+)$, and likewise if $f$ has a left limit $f(b-)$ at $b$.

**Proof** .
For all $x \in [a,b]$, let $G(x) = U_{[a,x]}(f) - L_{[a,x]}(f)$. We have $G(a) = 0$, $G$ is continuous on $[a,b]$ and for all $x \in ]a,b[$, $G$ is left differentiable at $x$ with left derivative zero, and also right differentiable at $x$ with zero right derivative. Therefore, $G$ is differentiable at every point $x \in ]a,b[$, with derivative $G'(x) = 0$. It follows that $G$ is constant on $[a,b]$, with value $G(a) = 0$. So $0 = G(b) = U_{[a,b]}(f) - L_{[a,b]}(f)$. This proves that $f$ is integrable on $[a,b]$.

Moreover, the proof also gives that for all $x \in [a,b]$, we have $0 = G(x) = U_{[a,x]}(f) - L_{[a,x]}(f)$, which proves that $U_{[a,x]}(f) = L_{[a,x]}(f)$ and that $f$ is integrable on $[a,x]$. So the function $F\colon [a,b] \longrightarrow \mathbb{R}$, defined by $F(x) = \displaystyle\int_a^x f(t)dt = U_{[a,x]}(f) = L_{[a,x]}(f)$ is well defined, and it is continuous on $[a,b]$, zero at $a$, and at all $x \in ]a,b[$ it is differentiable on the left of derivative left $F'_\ell(x) = f(x-)$ and right differentiable from right derivative $F'_r(x) = f(x+)$, and we also have $F'_r(a) = f(a+)$ (resp. $F'_\ell(b) = f(b-)$) if the limit $f(a+)$ (resp. $f(b-)$) exists. $\square$

In the particular case where $f$ is continuous on $[a,b]$ (therefore bounded on $[a,b]$), we obtain:

**Theorem 2.18**

[The Fundamental Theorem of Calculus]
Let $f\colon [a,b] \longrightarrow [c,d]$ be continuous function, then the function $F$ defined by

$$F(x) = \int_a^x f(t)dt$$

is differentiable and $F'(x) = f(x)$.

**Proof** .
For $x \in [a,b]$ and $h \in \mathbb{R}^*$ such that $x+h \in [a,b]$.

$$\begin{aligned}\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\Big(\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\Big)\\ &= \frac{1}{h}\int_x^{x+h} f(t)\ dt = f(c)\end{aligned}$$

where $c \in [x, x+h]$ or $c \in [x+h, x]$. Since $f$ is continuous, $\lim_{h\to 0} f(c) = f(x)$. Then $F'(x) = f(x)$. □

**Corollary 2.19**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable function and $f'$ is Riemann integrable, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

**Theorem 2.20**

[The Cauchy-Schwarz Inequality]
Let $f$ and $g$ be two Riemann-integrable functions on an interval $[a,b]$, then

$$\Big(\int_a^b f(x)g(x)\ dx\Big)^2 \le \int_a^b f^2(x)\ dx \int_a^b g^2(x)\ dx.$$

**Proof** .

Let $\lambda$ be a real number.

$$P(\lambda) = \int_a^b (f(x)+\lambda g(x))^2 \, dx = \lambda^2 \int_a^b g^2(x) \, dx + 2\lambda \int_a^b f(x)g(x) \, dx + \int_a^b f(x)^2 \, dx$$

If $\int_a^b g^2(x) \, dx > 0$, $P(\lambda)$ is a non negative polynomial. It follows that its discriminant is non positive, which gives the desired inequality.
If $\int_a^b g^2(x) \, dx = 0$, $P(\lambda) \geq 0$, then $\int_a^b (fg)(x) \, dx = 0$ and the inequality holds.
□

**Corollary 2.21**

[Minkowsky Inequality]
Let $f$ and $g$ be two Riemann-integrable functions on an interval $[a, b]$, then
$(\int_a^b (f(x) + g(x))^2 \, dx)^{\frac{1}{2}} \leq (\int_a^b f^2(x) \, dx)^{\frac{1}{2}} + (\int_a^b g^2(x) \, dx)^{\frac{1}{2}}$.

**Proof** .
$\int_a^b (f(x)+g(x))^2 \, dx = \int_a^b f^2(x) \, dx + \int_a^b g^2(x) \, dx + 2\int_a^b f(x)g(x) \, dx$. By the Cauchy-Schwarz inequality we have

$$(\int_a^b (f(x) + g(x))^2 \, dx)^{\frac{1}{2}} \leq (\int_a^b f^2(x) \, dx)^{\frac{1}{2}} + (\int_a^b g^2(x) \, dx)^{\frac{1}{2}}.$$

**Remark 6 :**

If $f$ is a non negative Riemann-integrable function and $\int_a^b f(x) \, dx = 0$, then $\int_a^b f(x)g(x) \, dx = 0$ for all Riemann-integrable function $g$. In particular $\int_a^b f^\alpha(x) \, dx = 0$, $\forall \alpha > 0$.

**Theorem 2.22**

[Hölder Inequality for Integrals]
Let $f$ and $g$ be two non negative Riemann-integrale functions on an interval $[a, b]$. Then for all conjugate positive numbers $p, q$, $(\frac{1}{p} + \frac{1}{q} = 1)$

we have:

$$\int_a^b f(x)g(x)\,dx \le \Big(\int_a^b f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)\,dx\Big)^{\frac{1}{q}}.$$

**Proof** .

If $\int_a^b f^p(x)\,dx = 0$ or $\int_a^b g^q(x)\,dx = 0$, the result is trivial.

If $\int_a^b f^p(x)\,dx \neq 0$ and $\int_a^b g^q(x)\,dx \neq 0$, we set $f_1(x) = \dfrac{f(x)}{(\int_a^b f^p(t)\,dt)^{1/p}}$ and $g_1(x) = \dfrac{g(x)}{(\int_a^b g^q(t)\,dt)^{1/q}}$, we get $\int_a^b f_1^p(x)\,dx = \int_a^b g_1^q(x)\,dx = 1$. From the convexity of the function $t \longmapsto t^p$ on $]0,+\infty[$, for $p > 1$, we get $f_1^{\frac{1}{p}} g_1^{\frac{1}{q}} \le \dfrac{1}{p} f_1 + \dfrac{1}{q} g_1$. We deduce the desired result.

$$\int_a^b f(x)g(x)\,dx \le (\int_a^b f^p(x)\,dx)^{\frac{1}{p}}(\int_a^b g^q(x)\,dx)^{\frac{1}{q}}.$$

□

**Theorem 2.23**

[Second Mean Value Formula]
Let $f$ be a decreasing non negative continuous function on the interval $[a,b]$ and let $g$ be a Riemann-integrable function on $[a,b]$. Then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx.$$

**Proof** .

Consider the function $G(x) = \int_a^x g(t)\,dt$. $G$ is continuous on $[a,b]$. Let $m = \inf_{x\in[a,b]} G(x)$ and $M = \sup_{x\in[a,b]} G(x)$. To prove the theorem it suffices to prove that $mf(a) \le \int_a^b f(x)g(x)\,dx \le Mf(a)$. Let $\sigma_n = (x_0 = a, \dots, x_n)$ be the uniform partition of $[a,b]$ i.e. $x_{i+1}-x_i=\dfrac{b-a}{n}$, $x_j=a+j\frac{b-a}{n}$. We set $\lambda_i=\dfrac{G(x_{i+1})-G(x_i)}{x_{i+1}-x_i}$.

$$\lim_{n\to+\infty}\sum_{i=0}^{n-1}(x_{i+1}-x_i)(fg)(x_i)=\int_a^b f(x)g(x)\,dx.$$

$$\begin{aligned}|\sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_i)(g(x_i)-\lambda_i)| &\le f(a)\sum_{i=0}^{n-1}(x_{i+1}-x_i)(M_i-m_i)\\ &= f(a)(U(g,\sigma_n)-L(g,\sigma_n)\underset{n\to+\infty}{\longrightarrow}0,\end{aligned}$$

with $M_i=\sup\limits_{t\in]x_i,x_{i+1}[}g(t)$ and $m_i=\inf\limits_{t\in]x_i,x_{i+1}[}g(t)$. It results that

$$\lim_{n\to+\infty}\sum_{i=0}^{n-1}f(x_i)(G(x_{i+1})-G(x_i))=\int_a^b f(x)g(x)\,dx.$$

$$\begin{aligned}\sum_{i=0}^{n-1}f(x_i)(G(x_{i+1})-G(x_i)) &= \sum_{i=0}^{n-1}f(x_i)G(x_{i+1})-\sum_{i=0}^{n-1}f(x_i)G(x_i)\\ &= \sum_{i=0}^{n-1}(f(x_{i-1})-f(x_i))G(x_i)+f(x_{n-1})G(b).\end{aligned}$$

Since $f$ is decreasing and non negative, we deduce

$$\begin{aligned}m\big[f(x_{n-1})+\sum_{i=0}^{n-1}(f(x_{i-1})-f(x_i))\big] &\le \sum_{i=0}^{n-1}f(x_i)(G(x_{i+1})-G(x_i))\\ &\le M\big[f(x_{n-1})+\sum_{i=0}^{n-1}(f(x_{i-1})-f(x_i))\big].\end{aligned}$$

Then

$$mf(a)\le\int_a^b f(x)g(x)\,dx\le Mf(a).$$

**Corollary 2.24**

Let $f$ be a monotone continuous function on an interval $[a,b]$ and let $g$

be a Riemann-integrable function, then there exist $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a)\int_a^c g(x)\ dx + f(b)\int_c^b g(x)\ dx.$$

**Proof .**
We can assume that $f$ is increasing. We use the previous theorem to the functions $h(x) = f(b) - f(x)$ and $g$. □

**Theorem 2.25**

Let $f$ be a Riemann-integrable function on the interval $[a,b]$.

1. If $\lim\limits_{x\to t,(x>t)} f(x) = s$ exists, then the function $F(x) = \displaystyle\int_a^x f(t)\ dt$ is differentiable at the right of $t$ and $F'(t+) = s$.
2. If $\lim\limits_{x\to t,(x<t)} f(x) = s$ exists, then the function $F$ is differentiable at the left of $t$ and $F'(t-) = s$.

**Proof .**

1. For $\varepsilon > 0$, there exists $\alpha > 0$ such that $|f(x) - s| \le \varepsilon$, for all $x \in ]t, t+\alpha[$. If $u \in [t, t+\alpha]$, then $|\displaystyle\int_t^u (f(x) - s)\ dx| \le \varepsilon(u-t)$ and $|F(u) - F(t) - s(u-t)| \le \varepsilon(u-t)$. Hence $\frac{F(u)-F(t)}{u-t} - s| \le \varepsilon$.
2. With the same arguments we get the result.

□

**Theorem 2.26**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function and $u\colon I \longrightarrow [a,b]$ a differentiable function. Then the function $F(x) = \displaystyle\int_a^{u(x)} f(t)\ dt$ is differentiable on $I$ and $F'(x) = u'(x)f(u(x))$, for all $x \in I$.

**Proof .**

Let $G$ be an antiderivative of $f$ such that $G(a) =$. Then $G \circ u = F$ and

$$F'(x) = (G \circ u)'(x) = G'(u(x)).u'(x) = f(u(x))u'(x).$$

□

**Theorem 2.27**

[Integral by Substitution]
If $g$ is continuously differentiable ($C^1$) on $[a, b]$, and if $f$ is continuous on $g([a, b]$. Then

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f \circ g(t) g'(t)\ dt.$$

**Proof** .
Let $F(t) = \displaystyle\int_{g(a)}^{g(t)} f(x)\ dx$, $G(t) = \displaystyle\int_a^t f \circ g(x) g'(x)\ dx$, $G'(t) = g'(t) f \circ g(t)$, $F(a) = G(a) = 0$ and $F'(t) = g'(t) f \circ g(t)$. Then $F' = G'$ on the interval $[a, b]$ and $F = G$ on the interval $[a, b]$. □

**Example 2 :**

1. If the function $f$ is even, then $\displaystyle\int_{-a}^{a} f(t)\ dt = 2\int_0^a f(t)\ dt$ and if $f$ is odd, then $\displaystyle\int_{-a}^{a} f(t)\ dt = 0$

2. If the function $f$ is $T-$periodic, with $T > 0$ on $\mathbb{R}$. Then $\displaystyle\int_a^{a+T} f(t)\ dt = \int_0^T f(t)\ dt$, for all $a \in \mathbb{R}$.
$\displaystyle\int_a^{a+T} f(t)\ dt = \int_a^0 f(t)\ dt + \int_0^T f(t)\ dt + \int_T^{a+T} f(t)\ dt$. Then $\displaystyle\int_T^{a+T} f(t)\ dt = \int_0^a f(t)\ dt$, (substitution $t = T + x$).

**Theorem 2.28**

[Integration by Parts]
Let $f$ and $g$ be two continuously differentiable functions ($\mathrm{C}^1$) on an

interval $I$, then

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx.$$

Moreover if $[a,b] \subset I$, then

$$\int_a^b f(x)g'(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\ dx.$$

**Example 3 :**

$$\int_0^1 x\tan^{-1} x dx = \frac{x^2}{2}\tan^{-1} x\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx = \frac{\pi}{4} - \frac{1}{2}.$$

**Theorem 2.29**

Let $f$ and $g$ be two functions of class $C^n$ on an interval $I$, then

$$\int f(x)g^{(n)}(x)dx = \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-1-p)}(x) + (-1)^n \int g(x)f^{(n)}(x)dx.$$

**Proof** .

$$\begin{aligned}
\Big(\sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-1-p)}(x)\Big)' &= \sum_{p=0}^{n-1}(-1)^p f^{(p+1)}(x)g^{(n-1-p)}(x) \\
&\quad + \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&= \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&\quad - \sum_{p=1}^{n}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&= f(x)g^{(n)}(x) - (-1)^n g(x)f^{(n)}(x).
\end{aligned}$$

□

**Theorem 2.30**

[Taylor Formula with integral Reminder]
Let $f$ be function of class $\mathcal{C}^{n+1}$ defined on an interval $I$ in $\mathbb{R}$. For $a$ and $x$ in $I$, we have:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{(n)!} f^{(n+1)}(t)dt.$$

**Proof** .
We apply the theorem (2.4) to the function $f$ and the function $g(t) = \dfrac{(x-t)^{n-1}}{(n-1)!}$.
$\square$

## 2.5 The Lebesgue Theorem

**Definition 2.31**

A subset $E \subset \mathbb{R}$ is said to be a null set (or a set of zero measure or a negligible set or zero set) if for any $\varepsilon > 0$ there is a countable number of open intervals $(]a_n, b_n[)_n$ such that $\displaystyle\sum_{n=1}^{+\infty}(b_n - a_n[< \varepsilon$ and $E \subset \cup_{n=1}^{+\infty}]a_n, b_n)$.

**Theorem 2.32**

[Lebesgue's Theorem on Riemann Integrable Functions]
A bounded function $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points of $f$ is a null set.

**Proof** .
Let $D = \{x \in [a,b] : f$ is discontinuous at $x\}$. We have

$$D = \{x \in [a,b];\ w_x(f) > 0\} = \bigcup_{n=1}^{+\infty} \{x \in [a,b];\ w_x(f) \geq \frac{1}{n}\}.$$

Let $D_n = \{x \in [a,b];\ w_x(f) \geq \frac{1}{n}\}$. Note that $D$ is a null set if and only if each $D_n$ is a null set.

Now assume that $f$ be Riemann integrable on $[a, b]$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. Since $f$ is Riemann integrable, there exists a partition $\sigma = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ such that

$$U(f, \sigma) - L(f, \sigma) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \varepsilon,$$

Let $J_k = \{j : ]x_{j-1}, x_j[ \cap D_k \neq \emptyset\}$.
If $J_k = \emptyset$, then $D_k \subset \{x_0, x_1, \cdots, x_n\}$, hence $D_k$ is finite and then it is a null set. Otherwise, for each $j \in J_k$, there exists $t \in D_k \cap ]x_{j-1}, x_j[$ and hence $\frac{1}{k} \leq w_t(f) \leq M_j - m_j$. Thus we have

$$\sum_{j \in J_k} \frac{1}{k}(x_j - x_{j-1}) \leq \sum_{j \in J_k} (M_j - m_j)(x_j - x_{j-1}) < \varepsilon$$

and hence $\sum_{j \in J_k} (x_j - x_{j-1}) < k\varepsilon$. Then $D_k \setminus \sigma \subset \bigcup_{j \in J_k} ]x_{j-1}, x_j[$, where $\sum_{j \in J_k} (x_j - x_{j-1}) < k\varepsilon$. Since $\varepsilon$ is arbitrary, $D_k \setminus \sigma$ is a null set. Thus $D_k \subset (D_k \setminus \sigma) \cup \sigma$ is a null set.
Conversely if $D$ is a null set, to show $f$ is Riemann integrable, we take an arbitrary $\varepsilon > 0$. Since $D$ is a null set, there is a countable family of open intervals $(I_j = ]a_j, b_j[)_j$ such that $\sum_{j=1}^{+\infty} (b_j - a_j) < \varepsilon$ and $D \subset \cup_{j=1}^{+\infty} I_j$. For all $x \in [a, b] \setminus D$, $w_x(f) = 0$ and hence by definition there exists an open interval $J_x$ containing $x$ such that $\sup\{|f(y) - f(z)|;\ y, z \in J_x \cap [a, b]\} < \varepsilon$.
The set $\mathcal{F} = \{I_j;\ j \in \mathbb{N}\} \cup \{J_x;\ x \in [a, b] \setminus D\}$ is an open cover of the compact set $[a, b]$. So $\mathcal{F}$ has a finite subcover $\mathcal{F}' = \{I_j;\ j = 1, \ldots, m\} \cup \{J_{x_j};\ j = 1, \ldots, p\}$.
Let $\sigma = \{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$ be the partition of $[a, b]$ determined by those endpoints of $(I_j)_{1 \leq j \leq m}$ and $(J_{x_j})_{1 \leq j \leq p}$ which are inside $[a, b]$. Also let $M_j = \sup_{t \in [t_{j-1},\ t_j]} f(t)$, $m_j = \inf_{t \in [t_{j-1},\ t_j]} f(t)$ and $\delta_j = t_j - t_{j-1}$, $j = 1, 2, \ldots, n$ and $|f(x)| \leq M$.
Then for each $j \in \{1, 2, \ldots, n\}$ the interval $]t_{j-1}, t_j[$ is contained in some $I_k$, $1 \leq k \leq m$ or some $J_{x_k}$, $1 \leq k \leq p$ and let $\mathcal{J} = \{j;\ ]t_{j-1}, t_j) \subset I_k$ for some $k = 1 \leq m\}$.
Note that if $j \notin \mathcal{J}$ then $]t_{j-1}, t_j[ \subseteq J_{x_k}$ for some $k = 1, 2, \ldots, p$ and
$M_k - m_k \leq \sup\{|f(t) - f(s)|;\ t, s \in J_{x_k} \cap [a, b]\} < \varepsilon$. Then

$$
\begin{aligned}
U(f,\sigma)-L(f,\sigma) &= \sum_{j=1}^{n}(M_j-m_j)\delta_j \\
&= \sum_{j\in\mathcal{J}}(M_j-m_j)\delta_j+\sum_{j\notin\mathcal{J}}(M_j-m_j)(t_j-t_{j-1}) \\
&\leq \sum_{j\in\mathcal{J}}2M(t_j-t_{j-1})+\sum_{j\notin\mathcal{J}}\varepsilon(t_j-t_{j-1}) \\
&\leq \sum_{j\in\Lambda}2M(b_j-a_j)+(b-a)\varepsilon \\
&\leq \sum_{j\in\mathbb{N}}2M(b_j-a_j)+(b-a)\varepsilon \\
&< 2M\varepsilon+(b-a)\varepsilon=(2M+b-a)\varepsilon.
\end{aligned}
$$

can be made arbitrary small. Hence $f$ is Riemann integrable on $[a,b]$.

□

# 3 Improper Integrals

## 3.1 Presentation of the Improper Integral

**Definition 3.1**

1. Let $f$ be a piecewise continuous function on the interval $[a,b[$, where $a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$.

   We say that the integral of $f$ on the interval $[a,b[$ is convergent if the function $F(x)=\displaystyle\int_a^x f(t)dt$ defined on $[a,b[$ has a finite limit when $x$ tends to $b$ $(x<b)$. This limit is called the improper integral of $f$ on $[a,b[$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

2. Let $f$ a piecewise continuous function on the interval $]a,b]$, where $a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}$.

   We say that the integral of $f$ on the interval $]a,b]$ is convergent if the function $G(x)=\displaystyle\int_x^b f(t)dt$ defined on $]a,b]$ has a finite limit when $x$ tends to $a$ $(x>a)$. This limit is called the improper integral of $f$ on $]a,b]$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

3. Let $f$ be a piecewise continuous function on the interval $]a, b[$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$.

   We say that the integral of $f$ on the interval $]a, b[$ is convergent if the integral of $f$ is convergent on $]a, c]$ and on $[c, b[$ for any $c$ in $]a, b[$.

4. Let $f$ be a piecewise continuous function on an interval $I$. The function is called integrable on $I$ (or the integral is absolutely convergent) if the integral of $|f|$ on the interval $I$ is convergent.

**Example 4 :**

1. $\displaystyle\int_0^{+\infty} \frac{dx}{1+x}$ is divergent, $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2$.

2. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^a \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.

3. For $\beta \in \mathbb{R}$ and $a \in ]1, +\infty[$, we set

$$F_\beta(x) = \int_a^x \frac{dt}{t(\ln t)^\beta},$$

   for $x \geq a$. In taking the change of variable $u = \ln t$, we get:
   $F_1(x) = \ln(\ln x) - \ln(\ln a)$ and for $\beta \neq 1$;
   $F_\beta(x) = \displaystyle\int_{\ln a}^{\ln x} \frac{du}{u^\beta} = \frac{1}{1-\beta}[\frac{1}{(\ln x)^{\beta-1}} - \frac{1}{(\ln a)^{\beta-1}}]$. Thus the integral $\displaystyle\int_a^{+\infty} \frac{dx}{x(\ln x)^\beta}$ is convergent if and only if $\beta > 1$.

**Definition 3.2**

Let $f$ be a locally Riemann integrable function on an interval $I$. The intgeral of $f$ on $I$ is called absolutely convergent if the integral of $|f|$ on $I$ is convergent.

**Proposition 3.3**

Let $f$ be a locally Riemann integrable function on the interval $[a, b[$.

1. If the integral $\int_a^b f(x)dx$ is absolutely convergent, then $\int_a^b f(x)dx$ is convergent.
2. If there exists a non negative piecewise continuous function $g$ on $[a, b[$, such that $\int_a^b g(x)dx$ converges and $|f(x)| \leq g(x)$, then $\int_a^b f(x)dx$ is absolutely convergent.

**Remark 7 :**
If $\int_a^b f(x)dx$ is convergent, then $\int_a^b f(x)dx$ is not in general absolutely convergent.
Consider the function $\frac{\sin x}{x}$ on the interval $[1, +\infty[$.
By integration by parts, $\int_1^s \frac{\sin x}{x}dx = \cos 1 - \frac{\cos s}{s} - \int_1^s \frac{\cos x}{x^2}dx$; this shows that the integral of the function $\frac{\sin x}{x}$ is convergent on $[1, +\infty[$. (we can also use the second mean value formula theorem 2.4). Moreover

$$\begin{aligned}
\int_\pi^{n\pi} \frac{|\sin x|}{x}dx &= \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx \\
&\geq \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|dx \\
&= \sum_{k=1}^{n-1} \frac{2}{(k+1)\pi}
\end{aligned}$$

As the sequence $(v_n)_n$ defined by $v_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is divergent, then the integral of $f$ is not absolutely convergent.
Another proof: we remark that $|\sin x| \geq \sin^2 x = \frac{1-\cos 2x}{2}$. As the integral $\int_1^{+\infty} \frac{\cos(2x)}{2x}dx$ is convergent, the integral $\int_1^{+\infty} \frac{|\sin x|}{x}dx$ is divergent.

## 3.2 Convergence Tests of Improper Integrals

**Theorem 3.4**

[The Cauchy Test]
Let $f$ be a piecewise continuous function on $[a, b[$, $b \in \mathbb{R} \cup \{+\infty\}$.
$\displaystyle\int_a^b f(x)dx$ converges if and only if

$$\forall \varepsilon > 0, \exists\ c \ \text{ tel que } \forall x, y \in ]c, b[;\ \left|\int_x^y f(t)dt\right| \leq \varepsilon.$$

(We can suppose only $f$ locally Riemann integrable function).

**Corollary 3.5**

Let $f\colon [a, b[ \longrightarrow \mathbb{R}$ a bounded function and $a, b \in \mathbb{R}$. If $f$ is piecewise continuous on $[a, b[$, then the integral of $f$ on $[a, b[$ is convergent.

**Example 5 :**
The integral of the function $f(t) = \dfrac{\sin t}{t}$ is convergent on $]0, 1]$.
Also the function $g(t) = \sin \dfrac{1}{t}$ on $]0, 1]$.

**Theorem 3.6**

Let $f$ be a non negative locally Riemann integrable function on $[a, b[$.
The integral $\displaystyle\int_a^x f(t)dt$ converges if and only if there exists $M > 0$ such that $\forall x \in [a, b[$; $\displaystyle\int_a^x f(t)dt \leq M$.

**Corollary 3.7**

Let $f$ and $g$ be two non negative locally Riemann integrable functions on $[a, b[$. Assume that $f(t) \leq g(t)$; $\forall t \in [a, b[$. Then

If $\int_a^b g(x)dx$ converges; the integral $\int_a^b f(x)dx$ converges.
If $\int_a^b f(x)dx$ diverges, the integral $\int_a^b g(x)dx$ diverges.

**Corollary 3.8**

Let $f$ be a non negative locally Riemann integrable function on the interval $[a,b[$ and let $\mathcal{E} = \{(x_n)_n \in [a,b[;\ \lim_{n\to+\infty} x_n = b\}$. For any $x \in [a,b[$, we define $F(x) = \int_a^x f(t)dt$. Then following properties are equivalent

1. The integral of $f$ on $[a,b[$ is convergent.
2. $\{F(x);\ x \in [a,b[\}$ is bounded.
3. For any sequence $(x_n)_n \in \mathcal{E}$, the sequence $(F(x_n))_n$ is convergent.
4. There exists a sequence $(x_n)_n \in \mathcal{E}$ such that the sequence $(F(x_n))_n$ is convergent.

**Example 6 :**

1. $f(t) = e^{-t^2}, t \in [0,+\infty[$, we have $f(t) \leq e^{-t}$ and $\int_0^{+\infty} e^{-x}dx = 1$, thus $\int_0^{+\infty} e^{-x^2}dx$ is convergent.
2. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin\ x}$ diverges because $\frac{1}{\sin x} \geq \frac{1}{x}\ \forall\ x\ \in ]0,\frac{\pi}{2}]$.

**Proposition 3.9**

Let $I$ be an interval and $f\colon I \longrightarrow \mathbb{R}^+$ a non negative locally Riemann integrable function. The integral of $f$ on $I$ converges if and only if there exists an increasing sequence of intervals $([a_n,b_n])_n$ which covers $I$ and a real $M \geq 0$ such that $\int_{a_n}^{b_n} f(x)dx \leq M$, for any $n \in \mathbb{N}$. In this case

$$\int_I f(x)dx = \sup_{n\in\mathbb{N}} \int_{a_n}^{b_n} f(x)dx.$$

**Theorem 3.10**

Let $f\colon [a,b[\longrightarrow \mathbb{R}$ and $g\colon [a,b[\longrightarrow \mathbb{R}^+$ be two locally Riemann integrable functions. Assume that there exists $\ell \in \mathbb{R}\setminus\{0\}$ such that $f \approx \ell g$ (when $t$ tends to $b^-$). Then $\displaystyle\int_a^b f(x)dx$ converges if and only if $\displaystyle\int_a^b g(x)dx$ converges.

**Proof** .
If $f \approx \ell g$ (when $t$ tends to $b^-$), then there exists a function $h$ such that $f(t) = \ell h(t)g(t)$ and $\lim\limits_{t\to b^-} h(t) = 1$. Thus $f(t)-\ell g(t) = (h(t)-1)\ell g(t)$ and, thus there exists $c$ such that $\forall t \in ]c,b[,\ |f(t)-\ell g(t)| \leq g(t)$, let $|f(t)| \leq (1+|\ell|)g(t)$.
If the integral $\displaystyle\int_a^b g(x)dx$ converges, then the integral $\displaystyle\int_a^b f(x)dx$ converges absolutely.
If the integral $\displaystyle\int_a^b f(x)dx$ converges, as $\ell \neq 0$, there exists $c$ such that $\forall t \in$ ]c,b[ ; $|f(t)-\ell g(t)| \leq \dfrac{|\ell|}{2}g(t)$. If $x < y \in ]c,b[$, we have: $\left|\displaystyle\int_x^y f(t)-\ell g(t)dt\right| \leq \dfrac{|\ell|}{2}\displaystyle\int_x^y g(t)dt$, thus $\dfrac{|\ell|}{2}\displaystyle\int_x^y g(t)dt \leq \left|\displaystyle\int_x^y f(t)dt\right| \underset{x,y\to b}{\longrightarrow} 0$.

**Remark 8 :**
If $g$ change of sign the previous result is not true. It suffices to take the function $f(t) = \dfrac{|\sin t|}{t} + \dfrac{\sin t}{\sqrt{t}}$ and $g(t) = \dfrac{\sin t}{\sqrt{t}}$, for $t \in [1,+\infty[$. The integral of the function $g$ is convergent on $[1,+\infty[$, it suffices to use the Cauchy test and the second Mean Value Formula. The integral of the function $f$ is divergent.

**Theorem 3.11**

Let $f\colon [1,+\infty[\longrightarrow \mathbb{R}^+$ be a piecewise continuous function.

1. If there exists $\alpha > 1$ such that $\lim\limits_{x\to+\infty} x^\alpha f(x) = 0$, then the integral of $f$ is convergent on $[1,+\infty[$.

2. If there exists $\alpha < 1$ such that $\lim_{x \to +\infty} x^\alpha f(x) = +\infty$, then the integral of $f$ is not convergent on $[1, +\infty[$.

**Theorem 3.12**

Let $a, b \in \mathbb{R}$ and $f\colon ]a, b] \longrightarrow \mathbb{R}+$ be a locally Riemann integrable function.

1. If there exists $\alpha < 1$ such that $\lim_{x \to a^+} (x-a)^\alpha f(x) = 0$, then the integral of $f$ is convergent on $]a, b]$.
2. If there exists $\alpha > 1$ such that $\lim_{x \to +\infty} (x-a)^\alpha f(x) = +\infty$, then the integral of $f$ is not convergent on $]a, b]$.

**Theorem 3.13**

[Abel's Theorem]
Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, and $f$ and $g$ be two continuous functions on the interval $[a, b[$. Assume that:
i) there exists $M \geq 0$ such that $\left| \int_x^y f(t)dt \right| \leq M$ for any $x, y$ in $[a, b[$.
ii) $g$ is monotonic on $[a, b[$ and $\lim_{t \to b} g(t) = 0$.
Then $\int_a^b f(x)g(x)dx$ converges.

**Proof** .
We can assume that $g$ is decreasing. By second mean value formula, theorem 2.4, for any $x < y$ in $[a, b[$,

$$\begin{aligned} \left| \int_x^y f(t)g(t)dt \right| &= g(x) \left| \int_x^c f(t)dt \right| \\ &\leq Mg(x) \underset{x \to b^-}{\longrightarrow} 0. \end{aligned}$$

$\square$

**Example 7 :**

1. Let $f$ be a non negative continuous function, decreasing and $\lim_{x\to+\infty} f(x) = 0$, then the integral $\displaystyle\int_0^{+\infty} e^{\mathrm{i}\lambda x} f(x)dx$ converges for $\lambda \neq 0$.

2. Let $f\colon [a,+\infty[\longrightarrow [0,+\infty[$ be a decreasing continuous function. Define for all $n\in\mathbb{N}$; $x_n = \displaystyle\sum_{k=0}^{n} f(a+k)$ and $y_n = x_n - \displaystyle\int_a^{a+k+1} f(x)dx$. Then

   i) the sequence $(y_n)_n$ is convergent,

   the integral $\displaystyle\int_a^{+\infty} f(x)dx$ converges if and only if the sequence $(x_n)_n$ converges.

   Indeed:

$$\begin{aligned} f(a+n+1) &= \int_{a+n}^{a+n+1} f(a+n+1)dx \\ &\leq \int_{a+n}^{a+n+1} f(x)dx \leq \int_{a+n}^{a+n+1} f(a+n)dx = f(a+n) \end{aligned}$$

   $y_n = \displaystyle\sum_{k=0}^{n}(f(a+k) - \int_{a+k}^{a+k+1} f(x)dx)$, thus the sequence $(y_n)_n$ is non negative and increasing. Moreover $y_n \leq \displaystyle\sum_{k=0}^{n}(f(a+k)-f(a+k+1)) \leq f(a)$, thus the sequence $(y_n)_n$ is convergent.

   The sequence $(x_n)_n$ is increasing and $\displaystyle\int_a^{a+n+1)} f(x)dx \leq x_n$ and $x_{n+1} \leq f(a) + \displaystyle\int_a^{a+n+1} f(x)dx$, thus the sequence $(x_n)_n$ converges if and only if the integral $\displaystyle\int_a^{+\infty} f(x)dx$ converges.

   As application the sequence $z_n = (\displaystyle\sum_{k=1}^{n}\frac{1}{k}) - \ln n$ is convergent. Its limit is called the Euler constant.

# CHAPTER II

## INFINITE SERIES

## 1 Tests of Convergence of Infinite Series

**Definition 1.1**

1. Let $(u_n)_n$ be a sequence of real numbers (eventually complex numbers). Consider the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$.

   If the sequence $(S_n)_n$ is convergent, we say that the series $\sum_{n\geq 1} u_n$ is convergent.

   The limit of the sequence $(S_n)_n$ if it exists is denoted by $\sum_{n=1}^{+\infty} u_n$ .

2. The series $\sum_{n\geq 1} u_n$ is called divergent if the sequence $(S_n)_n$ is divergent.

**Remark 9 :**

1. If the series $\sum_{n\geq 1} u_n$ converges, then $\lim_{n\longrightarrow+\infty} u_n = 0$. ($u_n = S_n - S_{n-1}$.)

2. The condition $\lim_{n\longrightarrow+\infty} u_n = 0$ is not, however, sufficient to ensure the convergence of the series $\sum_{n\geq 1} u_n$. For instance, the series

$\sum_{n\geq 1} \sqrt{n+1}-\sqrt{n}$ is divergent because $S_n = \sqrt{n+1}-1$, for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} u_n = 0$.

**Theorem 1.2**

[Cauchy Criterion]
Let $(u_n)_n$ be a sequence of real numbers. The series $\sum_{n\geq 1} u_n$ converges if and only if,

$$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N};\ |\sum_{n=p}^{q} u_n| \leq \varepsilon, \qquad \forall\, q \geq p \geq N_\varepsilon. \tag{1.1}$$

**Definition 1.3**

A series $\sum_{n\geq 1} u_n$ is called absolutely convergent if the series $\sum_{n\geq 1} |u_n|$ is convergent.

**Remark 10 :**

Every absolutely convergent series is convergent but the converse is false, it suffices to take the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$. Indeed, if $S_n = \sum_{p=1}^{n} \frac{(-1)^{p+1}}{p}$, then $S_{2n+1} - S_{2n} = \frac{-1}{2n+1} \xrightarrow[p\to+\infty]{} 0$. To prove that the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is convergent, it suffices to prove that the sequence $(S_{2n})_n$ is convergent.
We have: $S_{2n+2}-S_{2n} = \dfrac{1}{2n+2}-\dfrac{1}{2n+1} \leq 0$ and $S_{2n+1}-S_{2n-1} = \frac{1}{2n}-\frac{1}{2n+1} \geq 0$, then the sequences $(S_{2n})_n$ and $(S_{2n+1})_n$ are adjacent, which shows that the sequence $(S_n)_n$ is convergent.
We remark also that $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{n}{2n} = \frac{1}{2}$, then the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent.
There are several standard tests for convergence of a series of non negative terms. These tests are based primarily on the fact that an increasing sequence is convergent if, and only, if, it is bounded above. It follows that a series $\sum_{n\geq 1} u_n$ with non negative terms is convergent if, and only, if, the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$ is bounded.

## 1.1 Comparison Test

**Theorem 1.4**

[Comparison Test]
Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative real numbers. Assume that there exists an integer $k \in \mathbb{N}$ such that $u_n \leq v_n$, for every $n \geq k$. Then if the series $\sum_{n\geq 1} v_n$ is convergent, the series $\sum_{n\geq 1} u_n$ is also convergent.

**Proof** .
Let $S_n = \sum_{j=k}^{n} u_j$ and $T_n = \sum_{j=k}^{n} v_j$. We have $S_n \leq T_n$. The series $\sum_{n\geq 1} v_n$ is convergent if and only if the sequence $(T_n)_n$ is bounded above, which gives the result.
The result can also be deduced by the Cauchy Criterion (1). $\square$

**Corollary 1.5**

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists $a > 0$ and $b > 0$ such that $au_n \leq v_n \leq bu_n$ for every $n \geq k$, then the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge together.

**Corollary 1.6**

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that $\lim_{n\to+\infty} \frac{u_n}{v_n} = \ell$.

1. If $\ell > 0$, the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge together..
2. If $\ell = 0$, the convergence of the series $\sum_{n\geq 1} v_n$ involves the convergence of the series $\sum_{n\geq 1} u_n$.

3. If $\ell = +\infty$, the convergence of the series $\sum\limits_{n\geq 1} u_n$ involves the convergence of the series $\sum\limits_{n\geq 1} v_n$.

**Theorem 1.7**

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of positive numbers. If there exists $m \in \mathbb{N}$ such that, $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$, whenever $n \geq m$, then the convergence of the series $\sum\limits_{n\geq 1} v_n$ involves the convergence of the series $\sum\limits_{n\geq 1} u_n$.

**Proof** .
Let $N \in \mathbb{N}$ be large enough such that $\forall n \geq N$, $\dfrac{u_{n+1}}{u_n} \leq \dfrac{v_{n+1}}{v_n}$. Thus $\dfrac{u_{n+1}}{v_{n+1}} \leq \dfrac{u_n}{v_n}$ for $n \geq N$. The sequence $\left(\dfrac{u_n}{v_n}\right)_{n\geq N}$ is decreasing and $\dfrac{u_n}{v_n} \leq \dfrac{u_N}{v_N} = M \in \mathbb{R}_+^*$, $\forall n \geq N$. Then $u_n \leq M v_n$ for all $n \geq N$, which yields the result. □

## 1.2 Integral Test

**Theorem 1.8**

[Integral Test]
Let $f$ be a decreasing continuous function on $[1, +\infty[$. We define $u_n = f(n)$, for all $n \in \mathbb{N}$. Then:

$$\int_1^{+\infty} f(x)dx \text{ is convergent } \iff \sum_{n\geq 1} u_n \text{ is convergent.}$$

**Proof** .
Let $S_n = \sum\limits_{k=0}^{n} u_k$ and $v_n = \int_1^n f(t)dt$. We have: $f(n+1) \leq \int_n^{n+1} f(t)dt \leq f(n)$, thus

$$\sum_{k=1}^{n} f(k+1) \leq \int_1^{n+1} f(t)dt \leq \sum_{k=1}^{n} f(k).$$

If the sequence $(S_n)_n$ is convergent, then it is bounded above. Hence the sequence $(v_n)_n$ is also bounded above, and since it is increasing it is convergent.

Conversely if the sequence $(v_n)_n$ is convergent, the sequence $(S_n)_n$ is bounded above and then it is convergent. $\square$

**Corollary 1.9**

[Convergence of Riemann series]
The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is convergent if and only if, $\alpha > 1$.

**Proposition 1.10**

[Application: Comparison with Riemann series]
Let $(u_n)_n$ be a sequence with non negative real numbers. Assume that there exist $0 < a < b$ such that $0 < a \leq n^\alpha u_n \leq b < +\infty$ for every $n$ large enough, then the series $\sum_{n\geq 1} u_n$ is convergent if and only if, $\alpha > 1$.

This proposition results from Theorem (1.1)

**Exercise 2 :**
Show that the Bertrand series $\displaystyle\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ is convergent if and only if $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.
**Solution**
If $\alpha \leq 0$, $\displaystyle\lim_{n\to+\infty} \frac{n}{n^\alpha (\ln n)^\beta} = +\infty$, then the series is divergent.

If $0 < \alpha < 1$, we take $\alpha < \gamma < 1$ and consider the sequence $v_n = \dfrac{1}{n^\gamma}$. $\displaystyle\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = +\infty$, then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is divergent.

If $\alpha > 1$, we take $1 < \gamma < \alpha$ and consider the sequence $v_n = \dfrac{1}{n^\gamma}$, $\displaystyle\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = 0$, then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is convergent.

If $\alpha = 1$, we consider the sequence $u_n = \dfrac{1}{n \ln^\beta n}$ and $f(x) = \dfrac{1}{x \ln^\beta x}$, for $x \geq 2$. The function $f$ is decreasing for $x$ large. Then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n(\ln n)^\beta}$ is convergent if and only if $\displaystyle\int_2^\infty \frac{dx}{x \ln^\beta x}$.
The integral

$$\int_2^\infty \frac{dx}{x \ln^\beta x} \overset{t=\ln x}{=} \int_{\ln 2}^\infty \frac{dt}{t^\beta}$$

is convergent if and only if $\beta > 1$. $\square$

## 1.3 Root Test or the Cauchy Test

**Theorem 1.11**

[Root Test or the Cauchy Test]
Let $(u_n)_n$ be a sequence of real numbers and $\ell = \overline{\lim}_{n\to+\infty} \sqrt[n]{|u_n|}$.

1. If $\ell < 1$, the series $\sum\limits_{n\geq 1} u_n$ is absolutely convergent.
2. If $\ell > 1$, the general term of the series does not tends to 0 and the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude about the convergence of the series.

**Proof** .

1. Let $\alpha$ be such that $\ell < \alpha < 1$, there exists $N \in \mathbb{N}$ such that $\sqrt[n]{|u_n|} < \alpha$, for every $n \geq N$. Then $u_n \leq \alpha^n$. Since the series $\sum_{n\geq 1} \alpha^n$ is convergent, the series $\sum_{n\geq 1} u_n$ is convergent.
2. Let $1 < \beta < \ell$, there exists an increasing sequence of integers $(n_k)_k$ such that $\lim\limits_{k\to+\infty} |u_{n_k}|^{1/n_k} = \ell > \beta$. Hence there exists $k_0 \in \mathbb{N}$ such $|u_{n_k}| \geq \beta^{n_k}$, for all $k \geq k_0$. It follows that $\lim\limits_{k\longrightarrow+\infty} |u_{n_k}| = +\infty$ and the series $\sum\limits_{n\geq 1} u_n$ is divergent.
3. We know that the series $\sum\limits_{n\geq 1} \dfrac{1}{n}$ is divergent and $\sum\limits_{n\geq 1} \dfrac{1}{n^2}$ is convergent, but in the two cases $\lim\limits_{n\to+\infty} n^{-\frac{1}{n}} = \lim\limits_{n\to+\infty} n^{-\frac{2}{n}} = 1$.

$\square$

## 1.4 The Ratio Test or the D'Alembert's Test

**Proposition 1.12**

Let $(u_n)_n$ be a sequence of real numbers. Assume that $\lim\limits_{n\longrightarrow+\infty} |\dfrac{u_{n+1}}{u_n}| = \ell$. Then

1. If $\ell < 1$, the series $\sum\limits_{n\geq 1} u_n$ is absolutely convergent.
2. If $\ell > 1$ the series $\sum\limits_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude about the convergence of the series.

We prove that is this case $\lim\limits_{n\longrightarrow+\infty} \sqrt[n]{|u_n|} = \ell$.

**Proof** .

1. Let $\alpha$ be a real number such that $\ell < \alpha < 1$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\dfrac{|u_{n+1}|}{|u_n|} < \alpha$, then $u_n \leq \alpha^n \dfrac{|u_N|}{\alpha^N}$. Since the series $\sum\limits_{n\geq 1} \alpha^n$ is convergent, the series $\sum\limits_{n\geq 0} u_n$ is absolutely convergent.

2. Let $1 < \beta < \ell$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\dfrac{|u_{n+1}|}{|u_n|} \geq \beta$, then $u_n \geq \beta^n \dfrac{|u_N|}{\alpha^N}$. Since the series $\sum_{n\geq 1} \beta^n$ is divergent, the series $\sum_{n\geq 0} u_n$ is not convergent.

3. We know that $\sum_{n\geq 1} \frac{1}{n}$ diverges and $\sum\limits_{n\geq 1} \dfrac{1}{n^2}$ converges, but in the two cases $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 1$.

Assume $\lim\limits_{n\longrightarrow+\infty} |\dfrac{u_{n+1}}{u_n}| = \ell$ and $0 < \ell < +\infty$.

For $0 < \alpha < \ell < \beta < +\infty$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $\alpha < \frac{|u_{n+1}|}{|u_n|} < \beta$. It follows that

$$\alpha^n \frac{|u_N|}{\beta^{N-1}} \leq \alpha|u_n| \leq |u_{n+1}| \leq \beta|u_n| \leq \beta^{n-N+1}|u_N| = \beta^n \frac{|u_N|}{\beta^{N-1}}, \ \forall n \geq N.$$

We deduce that

$$\alpha = \lim_{n\to+\infty} \alpha^{1-N/n} \sqrt[n]{|u_N|} \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \lim_{n\to+\infty} \beta^{1-N/n} \sqrt[n]{|u_N|} = \beta.$$

Thus $\alpha \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \beta$ for every $0 < \alpha < \ell < \beta < +\infty$, this which yields that $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = \ell$.

If $\ell = +\infty$ and $0 < \alpha$. The above proof yields that $\alpha \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|}$, then $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = +\infty$.
If $\ell = 0$ and $0 < \beta$. The above proof yields that $\lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \beta$, then $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = 0$.

□

**Examples 8 :**

1. Let $z \in \mathbb{C}$, the series $\sum_{n\geq 0} \frac{z^n}{n!}$ is absolutely convergent on $\mathbb{C}$, because for every $z \in \mathbb{C}$; $|\dfrac{u_{n+1}}{u_n}| = \dfrac{|z|}{n+1} \xrightarrow[n\to+\infty]{} 0$. We denote $e^z$ the sum of this series. $e^z = \displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$.

2. For $|z| < 1$, the series $\sum_{n\geq 1} \frac{z^n}{n}$ is absolutely convergent.

## 1.5 The Abel's Criterion

**Theorem 1.13**

[Abel's Criterion]
Let $(u_n)_n$ be a sequence of real numbers and let $(v_n)_n$ be a sequence of non negative real numbers such that:

1. the sequence $(v_n)_n$ is decreasing and converges to 0.
2. the sequence $\left(S_n = \displaystyle\sum_{k=1}^{n} u_k\right)_n$ is bounded.

Then the series $\displaystyle\sum_{n\geq 1} u_n v_n$ is convergent.

**Proof** .
We use the Cauchy criterion (1) for the existence of the limit of sequences. Let $q > p \geq 1$,

$$\begin{aligned}\sum_{k=p+1}^{q} u_k v_k &= \sum_{k=p+1}^{q} (S_k - S_{k-1}) v_k = \sum_{k=p+1}^{q} S_k v_k - \sum_{k=p}^{q-1} S_k v_{k+1} \\ &= \sum_{k=p+1}^{q-1} (v_k - v_{k+1}) + S_q v_q - S_p v_{p+1}\end{aligned}$$

Since $|S_k| \leq M$, then $|\sum_{k=p+1}^{q} u_k v_k| \leq 2M v_{k+1} \xrightarrow[k\to+\infty]{} 0$.

**Remark 11 :**
The result holds also if we suppose that the sequence $(S_n)_n$ is bounded and the sequence $(b_n)_n$ converges to 0 and the series $\sum_{n=0}^{+\infty}(b_n - b_{n+1})$ is convergent.

**Examples 9 :**

1. Let $b_n = \dfrac{(-1)^{[\sqrt{n}]}}{n}$, for $n \geq 1$ and $a_n = e^{in\theta}$ for $0 < \theta < 2\pi$.
   $|\sum_{n=p}^{q} a_n| \leq \dfrac{1}{\sin\theta/2}$ and we can prove that $\sum_{n=2}^{+\infty}|b_n - b_{n-1}| \leq \sum_{n=2}^{+\infty}\dfrac{2}{(n-1)^2}$.
   It results that the series $\sum_{n\geq 1}\dfrac{(-1)^{[\sqrt{n}]}e^{in\theta}}{n}$ converges for all $0 < \theta < 2\pi$.

2. Let $s_n = \sum_{k=1}^{n}\dfrac{1}{k} - \ln n$, $n \geq 1$. We set $u_1 = S_1 = 1$ and for all $n \geq 2$; $u_n = S_n - S_{n-1} = \dfrac{1}{n} + \ln\dfrac{n-1}{n} = \dfrac{1}{n} + \ln(1 - \dfrac{1}{n}) = \dfrac{1}{n} + (-\dfrac{1}{n} - \dfrac{1}{2n^2} + o(\dfrac{1}{n^2}))$, then $u_n = \frac{-1}{2n^2} + o(\frac{1}{n^2})$, thus $(s_n)_n$ converges. We set $\gamma = \lim_{n\to+\infty} s_n$, $\gamma$ is called the "**Euler constant**.

## 2 Alternating Series

**Definition 2.1**

An alternating series is any series, $\sum_{n\geq 0} a_n$ such that $a_n a_{n+1} \leq 0$ for all $n \in \mathbb{N}$.

**Theorem 2.2: (Alternating series Test)**

Consider an alternating series $\sum_{n\geq 0}(-1)^n a_n$. If the sequence $(a_n)_n$ is decreasing and $\lim_{n\to 0} a_n = 0$, then the series $\sum_{n\geq 0}(-1)^n a_n$ is convergent.

Moreover for all $m \geq n \in \mathbb{N}$, $\left|\sum_{k=n}^{m}(-1)^k a_k\right| \leq a_n$.

**Proof** .
The convergence of the series results from the Abel theorem (1.5).

Consider the sequences $(S_n)_n$ defined by $S_n = \sum_{k=0}^{n}(-1)^k a_k$, $S_{2n}$ and $S_{2n+1}$. We have $S_{2n+1} - S_{2n} = -a_{2n+1} \leq 0$, $S_{2(n+1)} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0$ and $S_{2(n+1)+1} - S_{2n+1} = a_{2n+2} - a_{2n+3} \geq 0$. Hence the sequence $(S_{2n})_n$ is decreasing, the sequence $(S_{2n+1})_n$ is increasing and $0 \leq S_{2p+1} \leq S_{2n+1} \leq S_{2n} \leq S_{2q}$, for all $p \leq n \leq q$. We deduce that $S_{2p} - S_{2q+1} \geq 0$. Then for all $n \leq m$, $|S_m - S_n| = S_m - S_n$ if $n$ is even and $|S_m - S_n| = S_n - S_m$ if $n$ is odd. Also $|S_m - S_n| \leq |a_{n+1}$, for all $m > n$.

□

## 2.1 Exercises

**2-2-1** Consider a sequence $(u_n)_{n\geq 1}$ of real numbers such that the series $\sum\limits_{n\geq 1} n\, u_n$ is convergent. Prove that the series $\sum\limits_{n\geq 1} u_n$ is convergent.

**2-2-2** Let $(u_n)_{n\geq 1}$ be a decreasing sequence such that the series $\sum\limits_{n\geq 1} u_n$ is convergent.

(a) Prove that $\lim\limits_{n\to+\infty} nu_n = 0$.

(b) Prove that $\sum\limits_{n\geq 1} n(u_n - u_{n+1})$ converges and $\sum\limits_{n=1}^{+\infty} n(u_n - u_{n+1}) \sum\limits_{n=1}^{+\infty} u_n$.

(c) Compute for $0 \leq r < 1$ the following sums:
$\sum\limits_{n=1}^{+\infty} nr^n$ and $\sum\limits_{n=1}^{+\infty} n^2 r^n$.

**2-2-3** (a) Prove that the series $\sum\limits_{n\geq 0} \dfrac{(-1)^n}{n+1}$ is convergent.

(b) Show that $\left| \sum\limits_{k=0}^{n} \dfrac{(-1)^k}{k+1} - \displaystyle\int_0^1 \frac{dt}{1+t} \right| \leq \dfrac{1}{n+2}$.

(c) Deduce that $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} = \ln 2$.

**2-2-4** Find the following sums:

1) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$,

2) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)}$,

3) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{n!}$,

4) $\displaystyle\sum_{n=0}^{+\infty} \frac{2n^3+1}{n!}$,

5) $\displaystyle\sum_{n=2}^{+\infty} \ln\left(1 - \frac{1}{n^2}\right)$,

6) $\displaystyle\sum_{n=1}^{+\infty} \ln\left(\cos\frac{1}{2^n}\right)$,

7) $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x (\ln t)^n dt$.

**2-2-5** Study the convergence of the following series:

1) $\sum_{n\geq 1}\frac{2^n n!}{n^n}$,

2) $\sum_{n\geq 1}\frac{3^n n!}{n^n}$,

3) $\sum_{n\geq 1}\frac{n!}{n^n}$,

4) $\sum_{n\geq 2}\frac{(-1)^n\ln n}{n}$,

5) $\sum_{n\geq 2}\ln\left(1+\frac{(-1)^n}{n^\alpha}\right)$,

6) $\sum_{n\geq 1}\left(\frac{n}{n+1}\right)^{n^2}$,

7) $\sum_{n\geq 0}\frac{\cos n}{\sqrt{n}+\cos n}$,

8) $\sum_{n\geq 0}\frac{1}{C_{2n}^n}$,

9) $\sum_{n\geq 1}\frac{(2n)!}{n^n(n-1)!}$,

10) $\sum_{n\geq 1} n\sin(\frac{1}{n})$,

11) $\sum_{n\geq 1} e-\left(1+\frac{1}{n}\right)^n$,

12) $\sum_{n\geq 1}\cosh^\alpha n-\sinh^\alpha n$,

13) $\sum_{n\geq 1}\cos^{-1}\left(\frac{n^3+1}{n^3+2}\right)$,

14) $\sum_{n\geq 1}\ln\frac{(n^3+1)^2}{(n^2+1)^3}$,

15) $\sum_{n\geq 1}(\frac{1}{2})^{\sqrt{n}}$,

16) $\sum_{n\geq 1}\sqrt{1+\frac{(-1)^n}{\sqrt{n}}}-1$,

17) $\sum_{n\geq 1}\frac{(\ln n)^n}{n^{\ln n}}$,

18) $\sum_{n\geq 1}\frac{1}{(\ln n)^{\ln n}}$,

19) $\sum_{n\geq 1}\sin\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)$,

20) $\sum_{n\geq 1}\frac{(-1)^n}{n^\alpha+(-1)^n}$,

21) $\sum_{n\geq 3}\frac{1}{n\ln n\big(\ln(\ln n)\big)^\alpha}$,

22) $\sum_{n\geq 1}\left(\cos\frac{1}{\sqrt{n}}\right)^n-\frac{1}{\sqrt{e}}$,

23) $\sum_{n\geq 1}\ln\frac{1}{\sqrt{n}}-\ln\left(\sin\frac{1}{\sqrt{n}}\right)$,

2-2-6 Let $a,b$ and $c$ three real numbers. Consider the sequence $(u_n)_n$ defined by:

$$u_n=a\ln n+b\ln(n+1)+c\ln(n-1),\ n\geq 2.$$

(a) Express in term of $a,b$ and $c$ the necessary condition of the convergent of the series $\sum_{n\geq 2}u_n$.

(b) If this condition is satisfied, prove that the series $\sum_{n\geq 2} u_n$ is absolutely convergent.

(c) Chooses $a=-2$, $b=c=1$, prove that the series $\sum_{n\geq 2} u_n$ is convergent and compute its sum.

$\boxed{\text{2-2-7}}$ Consider $f(n)=\dfrac{n!}{n^n e^{-n}\sqrt{n}}$ and $S_n=\ln f(n)$, for $(n\geq 1)$.

(a) Prove that the series $\sum_{n\geq 2} u_n$ is convergent, where $u_n=S_n-S_{n-1}$.

(b) Deduce the convergence of the sequence $(S_n)_n$.

(c) Set $\ell=\lim_{n\to+\infty} S_n$. Determine in term of $\ell$ an equivalent of $n!$ when $n\to+\infty$.

$\boxed{\text{2-2-8}}$ Define the sequence of real numbers $(u_n)_n$ by:
$u_0$ arbitrary and $u_{n+1}=1-e^{-u_n}$, $\forall n\geq 0$.

(a) Study the convergent of the sequence $(u_n)_n$.

(b) Assume $u_0>0$, compute $\lim_{n\to+\infty}\dfrac{u_{n+1}-u_n}{u_n^2}$ and study the convergence of the series $\sum_{n\geq 0} u_n^2$.

$\boxed{\text{2-2-9}}$ Verify that the series $\sum_{n\geq 1}\dfrac{(-1)^n}{\sqrt{n}}+\dfrac{1}{n}$ is alternate and divergent. Conclude.

$\boxed{\text{2-2-10}}$ (a) Consider the function $f(x)=|\sin(2\pi x)|$, for $x\geq 1$.
Prove that $\displaystyle\int_1^{+\infty} f(t)dt$ diverges and the series $\sum_{n\geq 1} f(n)$ converges.

(b) Consider the function

$$g(x)=\begin{cases} n^2x+1-n^3 & \text{for} \quad x\in\left[n-\frac{1}{n^2},n\right] \quad (n\geq 2)\\ -n^2x+1+n^3 & \text{for} \quad x\in\left[n,n+\frac{1}{n^2}\right] \quad (n\geq 2)\\ 0 & \text{for} \quad x \text{ does not in any of these intervals}\end{cases}.$$

Prove that $\displaystyle\int_0^{+\infty} g(t)dt$ converges and the series $\sum_{n\geq 1} g(n)$ diverges.
Conclude.

$\boxed{\text{2-2-11}}$ Let $f$ be a function of class $C^1$ such that the integral $\displaystyle\int_0^{+\infty} f(t)dt$ is convergent and the integral $\displaystyle\int_0^{+\infty} f'(t)dt$ is absolutely convergent.

(a) Prove that the series $\sum_{n\geq 0} f(n)$ converges. (Hint: We can use Taylor formula with integral remainder).

(b) Study the convergence of the following series $\sum_{n=1}^{+\infty} \frac{\sin(\pi\sqrt{n})}{n}$.

**2-2-12** (a) Prove that for any $\theta \in \left]0, \frac{\pi}{2}\right[$:

$$\sin(2m+1)\theta = (\sin^{2m+1}\theta)P_m(\cot^2\theta),$$

where $P_m$ the polynomial defined by: $P_m(x) = \sum_{k=0}^{m}(-1)^k C_{2m+1}^{2k+1} x^{m-k}$.

(One will be able to use the Moivre Formula).

(b) Deduce the roots of the polynomial $P_m$ and the following relation

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

(c) Prove that: $\quad \forall t \in \left]0, \frac{\pi}{2}\right[, \quad \cot^2 t \leq \frac{1}{t^2} \leq \cot^2 t + 1.$

(d) Apply this result for $t = \frac{k\pi}{2m+1}$, deduce that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

**2-2-13** Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two convergent series with non negative terms.

(a) Prove that the series $\sum_{n\geq 0} u_n^2$ and $\sum_{n\geq 0} \sqrt{u_n v_n}$ are convergent.

Let $\sum_{n\geq 0} w_n$ be a series with non negative terms and such that $\lim_{n\longrightarrow+\infty}(nw_n) = \ell$.

(b) Prove that if the series $\sum_{n\geq 0} w_n$ is convergent, then $\ell = 0$.

**2-2-14** Let $u_0$ be a number real of $]0,1[$ and define the sequence $(u_n)_n$ by: $u_{n+1} = u_n - u_n^2$.

(a) Prove that the sequence $(u_n)_n$ is a decreasing sequence.

(b) Prove that $\forall n \in \mathbb{N},\ u_n \in ]0,1[$.

(c) Deduce that the sequence $(u_n)_n$ is convergent and compute its limit.

(d) Prove that the series $\sum_{n\geq 0} u_n^2$ converges and give its sum.

(e) Prove that the series $\sum_{n\geq 0} \ln(\frac{u_{n+1}}{u_n})$ and $\sum_{n\geq 0} u_n$ are divergent.

(f) Define for $n \in \mathbb{N}$, $v_n = \dfrac{1}{u_n} - \dfrac{1}{u_{n-1}}$.

i. Prove that $\lim_{+\infty} v_n = 1$.

ii. Deduce that $u_n \approx \frac{1}{n}$.

iii. Study the convergence of the series $\sum_{n\geq 1} \sin(u_n^2)$ and $\sum_{n\geq 1} \dfrac{u_n}{\sqrt{n}}$.

**2-2-15** Let $(u_n)_n$ be a sequence of real numbers. Assume that $|u_n| < 1$, for any $n \in \mathbb{N}$.

(a) Prove that the series $\sum_{n\geq 0} \ln(1 + u_n)$ is absolutely convergent if and only if the series $\sum_{n\geq 0} u_n$ is absolutely convergent.

(b) What can we say about the convergence?

(c) Assume that the series $\sum_{n\geq 0} u_n$ is absolutely convergent.

(a) Prove that the series $\sum_{n\geq 0} u_n^2$, $\sum_{n\geq 0} \dfrac{u_n}{1 + u_n}$ are absolutely convergent.

(b) What can we say about the convergence?

**2-2-16** Let $(u_n)_n$ be a sequence of non negative numbers. Define $v_n = \dfrac{u_n}{1 + u_n}$.

Prove that the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ converge or diverge together.

**2-2-17** Let $(u_n)_n$, $(v_n)_n$ and $(w_n)_n$ be three reals sequences such that the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} w_n$ converge, and $u_n \leq v_n \leq w_n$ for any $n$.

Prove that the series $\sum_{n\geq 0} v_n$ is convergent.

**2-2-18** Consider the sequence $(u_n)_n$, with $u_n = \displaystyle\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(x^2)dx$.

(a) Prove that the series $\sum_{n\geq 1} u_n$ is an alternate series.

(b) Prove that $\forall n \in \mathbb{N}, \quad |u_n| = \displaystyle\int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{2\sqrt{t}} dt.$

Deduce that the series $\displaystyle\sum_{n\geq 1} u_n$ is convergent.

Prove that it is conditionally convergent.

**2-2-19** Study the convergence and the absolutely convergence of the following series $\displaystyle\sum_{n\geq 2} u_n$, where $u_n = \dfrac{(-1)^n}{n^{\frac{3}{4}} + \cos n}$.

**2-2-20** Let $(u_n)_{n\geq 0}$ be a sequence defined by : $u_0 > 0$, $\forall\, n \in \mathbb{N},\ u_{n+1} = u_n + u_n^2$.

(a) Prove that $\displaystyle\lim_{n\to+\infty} u_n = +\infty$.

(b) Set $v_n = 2^{-n} \ln u_n$.

Prove that the sequence $(v_n)_n$ is convergent. (Study the series $\displaystyle\sum_{n\geq 0} v_{n+1} - v_n)$

(c) Deduce that there exists $\alpha > 0$ such that $u_n \approx \alpha^{2^n}$.

**2-2-21** Let $f(x) = \dfrac{1}{1 + \cosh x \sin^2 x}$ defined on $[0, +\infty[$.

(a) Prove that $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent if and only if the series $\displaystyle\sum_{n\geq 0} u_n$ is convergent, with $u_n = \displaystyle\int_{n\pi}^{(n+1)\pi} f(x)dx$.

(b) Prove that for any $n \in \mathbb{N},\ 0 \leq u_n \leq \displaystyle\int_{n\pi}^{(n+1)\pi} \frac{1}{1 + \frac{e^{n\pi}}{2}\sin^2 x} dx.$

(c) Deduce that $\forall n \in \mathbb{N}$, $u_n \leq \dfrac{\pi^2}{\sqrt{2}} e^{\frac{-n\pi}{2}}$ and that the integral $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent.

**2-2-22** Let $(a_n)_n$ be a sequence of non negative numbers such that the series $\displaystyle\sum_{n\geq 0} a_n$ is convergent. Define the sequences $(R_n)_n$ and $(b_n)_n$ by: $R_n = \displaystyle\sum_{k=n+1}^{+\infty} a_k$ and $b_n = \dfrac{a_n}{R_{n-1}^{\alpha}}$, with $\alpha \in ]0, 1[$ fixed.

(a) Prove that for any $n \in \mathbb{N}^*$, $b_n \leq \dfrac{R_{n-1}^{1-\alpha} - R_n^{1-\alpha}}{1-\alpha}$. (We will be able to use the integral $\displaystyle\int_{R_n}^{R_{n-1}} \frac{dt}{t^{\alpha}}$).

Deduce that the series $\sum_{n\geq 1} b_n$ is convergent.

(b) Set for any $n \in \mathbb{N}^*$, $c_n = \frac{a_n}{R_n}$, $d_n = \frac{a_n}{R_{n-1}}$ and $e_n = \ln(\frac{R_{n-1}}{R_n})$. Prove that the series $\sum_{n\geq 1} c_n$ and $\sum_{n\geq 1} d_n$ are divergent. (Prove that the series $\sum_{n\geq 1} e_n$ diverges and $0 \leq e_n \leq c_n$ and $c_n = \frac{d_n}{1-d_n}$.)

(c) If $(u_n)$ is a given non negative sequence such that the series $\sum_{n\geq 0} u_n$ converges, is there exists a sequence $(v_n)_n$ such that the series $\sum_{n\geq 0} u_n v_n$ converges and $\lim_{n\to+\infty} v_n = +\infty$?

# 3 Series Product

**Definition 3.1**

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers. For $n \in \mathbb{N}$, we set

$$c_n = \sum_{k=1}^{n} u_k v_{n-k}. \tag{3.2}$$

The series $\sum_{n\geq 1} c_n$ is called the series product of the two given series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$.

In this definition we are not interested in whether the product of the series exists, because it depends on some conditions. Indeed we have the following example:
Consider $\sum_{n\geq 1} c_n$ the series product of the series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n+1}}$ with itself. The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent but the series $\sum_{n\geq 1} c_n$ is divergent. Indeed:

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}.$$

Then $|c_n| \geq 1$ and the series $\sum_{n\geq 1} c_n$ is divergent.

The following theorem affirms the existence of the series product under some conditions.

**Theorem 3.2**

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers.

1. Assume that the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ are absolutely convergent. Then the series $\sum_{n\geq 1} c_n$ is absolutely convergent and we have
$$\sum_{n=1}^{+\infty} c_n = (\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \tag{3.3}$$
2. Assume that the series $\sum_{n\geq 1} u_n$ is absolutely convergent and the series $\sum_{n\geq 1} v_n$ is convergent. Then the series $\sum_{n\geq 1} c_n$ is convergent and we have:
$$\sum_{n=1}^{+\infty} c_n = (\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \tag{3.4}$$

**Proof** .

It suffices to proves 2). We set

$$A_n = \sum_{k=1}^{n} u_k, \qquad B_n = \sum_{k=1}^{n} v_k, \qquad C_n = \sum_{k=1}^{n} c_k,$$
$$A = \sum_{n=1}^{+\infty} u_n, \qquad \alpha = \sum_{n=1}^{+\infty} |u_n| \quad \text{and} \quad B = \sum_{n=1}^{+\infty} v_n.$$

Then
$$C_n = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} u_j B_{n-j} = \sum_{j=1}^{n} u_j (B_{n-j} - B) + BA_n.$$

Since $\lim\limits_{n\to+\infty} B.A_n = A.B$, then to show that $\lim\limits_{n\longrightarrow+\infty} C_n = A.B$, it suffices to show that the sequence $(\Delta_n)_n$ converges to 0, where $\Delta_n = \sum\limits_{j=1}^{n} a_j(B_{n-j} - B)$.

Let $\varepsilon > 0$: $\exists N \in \mathbb{N}$ such that $|B_n - B| < \frac{\varepsilon}{2\alpha}$ and $\sum\limits_{j=N}^{+\infty} |a_j| \leq \dfrac{\varepsilon}{2M}$, $\forall\, n \geq N$. Thus for every $n \geq 2N$,

$$|\Delta_n| \leq \sum_{j=1}^{N} |a_j||B_{n-j} - B| + \sum_{j=N+1}^{n} |a_j||B_{n-j} - B| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It results that $\lim\limits_{n\longrightarrow+\infty} |\Delta_n| = 0$. $\square$

## 3.1 Exercises

$\boxed{\text{2-3-1}}$

# CHAPTER III

# INTEGRALS DEPENDING ON PARAMETERS

We recall in this chapter, that a piecewise continuous function $f$ is called integrable on $I$ if the integral $\displaystyle\int_I |f(x)|dx$ is convergent.

## 1 Convergence Theorem

**Theorem 1.1**

[Monotone Convergence Theorem]
Let $\Big(f_n : I \longrightarrow \mathbb{R}\Big)_n$ be a sequence of integrable piecewise continuous functions on $I$. Assume that
i) the sequence $(f_n)_n$ is increasing, (i.e. $f_n \leq f_{n+1}$)
ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function $f$ on $I$.
Then $f$ is integrable on $I$ if and only if the sequence $\Big(\int_I f_n(x)dx\Big)_n$ is bounded above. Moreover with these assumptions

$$\int_I f(x)dx = \sup_{n\in\mathbb{N}} \int_I f_n(x)dx = \lim_{n\to+\infty} \int_I f_n(x)dx.$$

**Remark 12 :**
Let $\Big(f_n : I \longrightarrow \mathbb{R}\Big)_n$ be a sequence of integrable piecewise continuous functions on $I$. We assume that

i) the sequence $(f_n)_n$ is decreasing, (i.e. $f_n \geq f_{n+1}$)

ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function $f$ on $I$. Then $f$ is integrable on $I$ if and only if the sequence $\left(\int_I f_n(x)dx\right)_n$ is lower bounded. Moreover with these assumptions

$$\int_I f(x)dx = \inf_{n\in\mathbb{N}} \int_I f_n(x)dx = \lim_{n\to+\infty} \int_I f_n(x)dx.$$

**Theorem 1.2**

[Dominated Convergence Theorem]
Let $\left(f_n : I \longrightarrow \mathbb{R}\right)_n$ be a sequence of integrable piecewise continuous functions on $I$. We assume that
i) the sequence $(f_n)_n$ is increasing, (i.e. $f_n \leq f_{n+1}$),
ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function $f$ on $I$,
iii) there exists an integrable function $\varphi : I \longrightarrow \mathbb{R}^+$ such that $|f_n| \leq \varphi$, for any $n \in \mathbb{N}$. (This assumption is called the domination assumption).
Then for any $n \in \mathbb{N}$, $f_n$ is integrable on $I$ and $f$ is integrable on $I$. Moreover

$$\lim_{n\to+\infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

## 1.1 Continuity

**Theorem 1.3**

Let $\Omega$ be a subset of $\mathbb{R}^m$ and $f : \Omega \times I \longrightarrow \mathbb{C}$ a continuous function on $\Omega \times I$ and fulfills the domination assumption, (i.e. there exists an integrable function $\varphi : I \longrightarrow \mathbb{R}^+$ such that $|f(x,t)| \leq \varphi(t)$, for all $x \in \Omega$.) Then the function $x \longmapsto F(x) = \displaystyle\int_I f(x,t)dt$ is continuous on $\Omega$.

**Theorem 1.4**

Let $\Omega$ a subset of $\mathbb{R}^m$ and $f : \Omega \times I \longrightarrow \mathbb{C}$ a continuous function on $\Omega \times I$ and fulfills the local domination assumption, (i.e. for any compact $K \subset \Omega$, there exists an integrable function $\varphi : I \longrightarrow \mathbb{R}^+$ such that $|f(x,t)| \leq \varphi(t)$, for all $x \in K$.) Then the function $x \longmapsto F(x) =$

$\displaystyle\int_I f(x,t)dt$ is continuous on $\Omega$.

### 1.2 Differentiability

**Theorem 1.5**

Let $J$ be an interval and $f\colon J\times I \longrightarrow \mathbb{R}$ a continuous function on $J\times I$. We assume that
i) For any $x\in J$, the function $t\longmapsto f(x,t)$ is integrable on $I$
ii) $\dfrac{\partial f}{\partial x}$ exists, continuous on $J\times I$ and fulfills the domination assumption, (i.e. there exists an integrable function $\varphi\colon I\longrightarrow \mathbb{R}^+$ such that $\left|\dfrac{\partial f}{\partial x}\right|\leq \varphi(t)$, for all $x\in J$.)
Then the function $x\longmapsto F(x)=\displaystyle\int_I f(x,t)dt$ is of class $\mathcal{C}^1$ on $J$.

## 2 Generalized Integral Depending on Parameter

### 2.1 Convergence Theorem of Generalized Integral

Let $f(t,x)$ be a function defined on $]a,b[\times]\alpha,\beta[$; with $-\infty\leq a<b\leq+\infty$ and $-\infty\leq\alpha<\beta\leq+\infty$. We intend to study the continuity and the differentiability of the function

$$F(x)=\int_a^b f(t,x)dt.$$

To study this problem it suffices to study the case $a\in\mathbb{R}$. In which follows we consider the case $a\in\mathbb{R}$. To study the function $F$, we consider a sequence $(u_n)_n$ of $[a,b[$ which converges to $b$ and we study the sequence

$$F_n(x)=\int_a^{u_n} f(t,x)dt$$

and we apply for each function $F_n$ the previous results and deduce the regularity of the function $F=\lim\limits_{n\to+\infty}F_n$.

**Definition 2.1**

Let $X$ be a subset of $\mathbb{R}$ and $f$ a function defined on $[a, b[\times X$ such that the integral $\displaystyle\int_a^b f(t,x)dt$ converges for any $x \in X$.

We say that the integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on $X$ if, $\forall \varepsilon > 0$, $\exists c$ independent of $x$ such that $|\int_s^b f(t,x)dt| \leq \varepsilon$; for any $c \leq s < b$.

We remark that if the integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on $X$, then for any sequence $(u_n)_n$ of $[a, b[$ convergent to $b$, the sequence $F_n(x) = \displaystyle\int_a^{u_n} f(t,x)dt$ converges unoformly on $X$.

**Theorem 2.2**

[The Cauchy Criterion]
Let $X$ be a subset of $\mathbb{R}$ and $f$ a function defined on $[a, b[\times X$ such that the integral $\displaystyle\int_a^b f(t,x)dt$ converges for any $x \in X$.

The integral $\displaystyle\int_a^b f(t,x)dt$ converges uniformly on $X$ if and only if $\forall \varepsilon > 0$, $\exists c$ independent of $x$ such that $|\int_u^v f(t,x)dt| \leq \varepsilon$, for any $c \leq u \leq v < b$.

**Theorem 2.3**

Let $X$ be a subset of $\mathbb{R}$ and $f$ a function defined on $[a, b[\times X$. We assume that there exists an integrable function defined on on $[a, b[$ such that $|f(t,x)| \leq \varphi(t)$, for any $x \in X$. Then

i) The integral $\displaystyle\int_a^b f(t,x)dt$ converges absolutely for any $x \in X$.

ii) The integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on $X$.

**Example 10 :**

1. Consider the integral $\int_0^{+\infty} e^{-t^2} e^{itx} dt$, for $x \in \mathbb{R}$. As $|e^{-tx} e^{itx}| \leq e^{-t^2}$ which is integrable, thus $\int_0^{+\infty} e^{-t^2} e^{itx} dt$ converges unoformly on $\mathbb{R}$.

2. Vonsider the integral $\int_0^{+\infty} e^{-tx} \frac{\sin t}{t} dt$. This integrable converges unoformly on any interval $[a, +\infty[$; for any $a > 0$.

**Theorem 2.4**

[Abel Rule for the Uniform Convergence]
Let $X$ be subset of $\mathbb{R}$ and $f, g$ two functions defined one $[a, +\infty[\times X$ such that
i) There exists a real $M$ independent of $x$ such that $|\int_a^u f(t,x)dt| \leq M$, for any $u \in [a, +\infty$.
ii) The function $t \longmapsto g(t,x)$ is decreasing for any $x \in X$ and there exists a non negative decreasing function $\varphi$ on $[a, +\infty[$ such that $|f(t,x)| \leq \varphi(t)$ and $\lim_{t \to +\infty} \varphi(t) = 0$. Then the integral $\int_a^{+\infty} f(t,x)dt$ converges uniformly on $X$.

Therefore the integral $\int_0^{+\infty} e^{-tx} \frac{\sin t}{t} dt$ converges uniformly on $]0, +\infty[$. It suffices to take $f(t) = \sin t$ and $g(t,x) = \frac{e^{-tx}}{t} \leq \frac{1}{t}$.

## 2.2 Continuity

**Theorem 2.5**

Let $f$ be a continuous function on $[a, b[\times]\alpha, \beta[$ such that the integral $\int_a^b f(t,x)dt$ converges uniformly on any compact $[\zeta, \xi] \subset ]\alpha, \beta[$. Then the function

$$F(x) = \int_a^b f(t,x)dt$$

is continuous on $]\alpha, \beta[$.

## 2.3 Differentiability

**Theorem 2.6**

Let $f$ be a continuous function on $[a,b[\times]\alpha,\beta[$ such that $\dfrac{\partial f}{\partial x}$ exists and is continuous on $[a,b]\times]\alpha,\beta[$, for any $x\in]\alpha,\beta[$, the integral $\displaystyle\int_a^b f(t,x)dt$ converges and the integral $\displaystyle\int_a^b \frac{\partial f}{\partial x}(t,x)dt$ converges uniformly on any compact $[\zeta,\xi]\subset]\alpha,\beta[$. Then the function

$$F(x)=\int_a^b f(t,x)dt$$

is differentiable on $]\alpha,\beta[$ and

$$F'(x)=\int_a^b \frac{\partial f}{\partial x}(t,x)dt$$

.

**Example 11 :**

1. Let $F_n(x)=\displaystyle\int_0^1 t^x\ln^n t dt$, for $x\in]-1,0]$. $F_n$ is well defined. Moreover the functions $f_n(t,x)=t^x\ln^n t$ and $\dfrac{\partial f_n}{x}(t,x)=f_{n+1}(t,x)$ are continuous on $]0,1]\times]-1,0]$ and for $x\in[a,0]$, with $-1<a<0$, one has $t^x|\ln^n t|\leq t^a|\ln^n t|$. Thus the integral $\displaystyle\int_0^1 t^x\ln^n t dt$ converges uniformly on $[a,0]$ and $F_n$ is continuous and of class $C^\infty$ on $]-1,0]$. $F_n(x)=\dfrac{(-1)^n n!}{(x+1)^{n+1}}$.

2. Consider the function $G$ defined for $x\geq 0$ by:

$$G(x)=\int_0^{+\infty}\frac{e^{-xt^2}}{1+t^2}dt.$$

The function $g(t,x)=\frac{e^{-xt^2}}{1+t^2}$ is continuous on $[0,+\infty[\times[0,+\infty[$. $g(t,x)\leq \dfrac{1}{1+t^2}$, thus $G$ is continuous on $[0,+\infty[$.
$\dfrac{\partial g}{\partial x}(t,x)=-t^2e^{\frac{-xt^2}{1+t^2}}$ which is continuous on $[0,+\infty[\times[0,+\infty[$ and $\displaystyle\int_0^{+\infty}\frac{\partial g}{\partial x}(t,x)dt$ converges uniformly on any interval $[a,+\infty[$, for any $a>0$, because

$|\frac{\partial g}{\partial x}(t,x)| \leq e^{-at^2}$ which is integrable, for $x \geq a$. Therefore the function $G$ is differentiable on $]0, +\infty[$ and

$$G'(x) = \int_0^{+\infty} \frac{\partial g}{\partial x}(t,x)dt.$$

## 2.4 Exercises

**3-2-1** Let $E$ be the vector space of continuous functions on $[0,1]$, and let $K$ be the function of two variables defined by:

$$K(x,y)=\begin{cases}(x-1)y & \text{si } y\leq x\\ x(y-1) & \text{si } x\leq y\end{cases}$$

To any function $f$ of $E$ we associate the function

$$\tilde{f}(x)=\int_0^1 K(x,y)f(y)dy.$$

(a) Prove that for any $f\in E$, $\tilde{f}$ is of class $\mathcal{C}^2$, $\tilde{f}(1)=\tilde{f}(0)=0$ and $\tilde{f}''=f$.

(b) Prove that for any $f,g$ of $E$ :

$$\int_0^1 \tilde{g}(x)f(x)dx=\int_0^1 \tilde{f}(x)g(x)dx.$$

**3-2-2** (a) Study the convergence of the following integral with respect to the parameter $x\in\mathbb{R}$.

$$\int_1^{+\infty}\frac{t^{-(x+1)}}{\sqrt{t^2-1}}dt.$$

Let $I$ be the set of $x$ for which the integral is convergent.

(b) For $x\in I$, define

$$F(x)=\int_1^{+\infty}\frac{t^{-(x+1)}}{\sqrt{t^2-1}}dt.$$

Prove that $F$ is of class $C^\infty$ on $I$.

**3-2-3** We claim to compute the following integral

$$F(x)=\int_0^{+\infty}\frac{1-\cos(tx)}{t^2}.e^{-t}dt;\quad x>0$$

(a) Verify the existence of this integral.

(b) Prove that $F''(x)=\dfrac{1}{1+x^2}$

(c) Deduce the expression of $F$.

**3-2-4** For $x>0$ define the functions $F(x)=\displaystyle\int_0^{+\infty}\frac{\sin t}{t+x}dt$ and $G(x)=\displaystyle\int_0^{+\infty}\frac{e^{-tx}}{1+t^2}dt.$

(a) Prove that $F$ and $G$ fulfills the same differential equation $y''+y=\frac{1}{x}$.

(b) Prove that $F=G$.

(c) Deduce the value of the Dirichlet integral $\int_0^{+\infty}\frac{\sin t}{t}dt$.

3-2-5 Let $f$ be a continuous function and bounded on $\mathbb{R}_+$. We define for $x>0$ the function $F(x)=\int_0^{+\infty} f(t)e^{-xt}dt$ and $G(x)=\int_0^{+\infty} tf(t)e^{-xt}dt$.

(a) Verify that $F$ and $G$ are well defined for $x>0$.

(b) Determine the limit of $F$ at $+\infty$.

(c) Prove that $F$ is differentiable and compute $F'(x)$.

3-2-6 Let $\psi(t)=\frac{1}{\pi(1+t^2)}$ and $f$ a continuous function on $\mathbb{R}$ such that $\int_{-\infty}^{+\infty}|f(t)|dt<+\infty$.

Define

$$\varphi(x)=\int_{-\infty}^{+\infty} f(x-t)\psi(t)dt.$$

(a) Prove that $\varphi$ is continuous on $\mathbb{R}$.

(b) Prove that $\varphi$ is of class $C^\infty$ on $\mathbb{R}$.

(c) Prove that

$$\int_{-\infty}^{+\infty}\varphi(x)dx=\int_{-\infty}^{+\infty} f(t)dt.\int_{-\infty}^{+\infty}\psi(t)dt.$$

(d) Let $\tilde{\varphi}(x)=\int_{-\infty}^{+\infty}\frac{\cos(x-t)}{\pi(1+t^2)}dt$.

a) Prove that $\tilde{\varphi}$ is of class $C^\infty$ and fulfills a differential equation of second order.

b) Compute $\tilde{\varphi}(0)$ and deduce the expression of $\tilde{\varphi}$.

3-2-7 (a) Let $I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx$.

a) Compute $I_{2p}$ and $I_{2p+1}$, for any $p\in\mathbb{N}$.

b) Prove that for any $n\in\mathbb{N}$, $I_nI_{n+1}\le I_n^2\le I_nI_{n-1}$ and deduce the Wallis formula:

$$I_n\sim_{+\infty}\sqrt{\frac{\pi}{2n}}.$$

(b) a) Prove that $f\colon x \mapsto \displaystyle\int_0^{\frac{\pi}{2}} \sin^x t dt$ is $\mathcal{C}^\infty$ on $]-1,+\infty[$.

b) Give a simple equivalent of $f$ at $+\infty$.

c) Give an asymptotic rxpansion of three terms of $f$ at $-1$.

**3-2-8** Let $F(x) = \displaystyle\int_0^{+\infty} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$.

(a) Prove that $F$ is of class $C^1$ on $]0,+\infty[$.

(b) Find a relation between $F(x)$ and $F(\frac{1}{x})$.

(c) Determine the limit of $F(x)$ when $x \longrightarrow +\infty$.

(d) Remark that $F(x) > \displaystyle\int_0^1 \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$ and determine $\lim\limits_{x\to 0} F(x)$.

(e) a) Prove that $F(x) = 2\displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$.

b) Prove that $F(x) \sim_0 2\displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{x^2+t^2}}$.

c) Deduce a simple equivalent of $F$ in a neighborhood of $0$ and $+\infty$.

**3-2-9** Define $f(x) = \displaystyle\int_0^1 \frac{t^x(1-t)}{\ln t} dt$.

(a) Determine the domain of definition of $f$.

(b) Prove that $f$ is differentiable on $]-1,+\infty[$ and determine $f'(x)$ for any $x > -1$.

(c) Give $\lim\limits_{x \longrightarrow +\infty} f(x)$ and deduce the value of $f(x)$ for any $x > -1$.

**3-2-10** Let $f$ be a continuous function on $[0,+\infty[$ and

$$\mathcal{D} = \{(u,t) \in \mathbb{R}^2;\ 0 < u < x,\ 0 < t < u\}.$$

Define the function

$$g(x) = \iint_{\mathcal{D}} \frac{f(t)}{\sqrt{(x-u)(u-t)}} dudt.$$

(a) Prove that $g$ is well defined.

(b) Compute

$$\int_t^x \frac{du}{\sqrt{(x-u)(u-t).}}$$

(We will be able make the change of variables $u = t\cos^2\varphi + x\sin^2\varphi$.)

(c) Prove that $g(x) = \pi \displaystyle\int_0^x f(t)dt$ and deduce the expression of $f$ in term of $g$.

**3-2-11** Define the function $F$ by:

$$F(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin^2 t}} dt.$$

(a) Prove that the domain of definition of $F$ is $]-1, 1[$.

(b) Prove that $F$ is of class $\mathcal{C}^2$ on $]-1, 1[$, and give the expression in integral form of $F'$ and $F''$.

(c) a) Use the change of variables $u = x \sin t$ to prove that

$$F(x) \geq \int_0^x \frac{du}{1 - u^2}, \quad 0 < x < 1.$$

b) Deduce $\lim_{x \to 1^-} F(x)$.

# CHAPTER IV

## SEQUENCES AND SERIES OF FUNCTIONS

## 1 Sequences of Functions

**Definition 1.1**

Let $(f_n)_n$ be a sequence of functions defined on a subset $A$ of $\mathbb{R}$.

1. The sequence $(f_n)_n$ is called pointwise convergent on $A$ if for every $x \in A$, the sequence $(f_n(x))_n$ is convergent.
2. The sequence $(f_n)_n$ is called uniformly convergent to $f$ on $A$ if

$$\lim_{n \to +\infty} \sup_{x \in A} \|f_n(x) - f(x)\| = 0.$$

**Remark 13 :**

1. The sequence $(f_n)_n$ converges to $f$ on $A$ if and only if

$$\forall x \in A, \ \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N.$$

2. The sequence $(f_n)_n$ converges uniformly to $f$ on $A$ if and only if

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N \text{ and } \forall x \in A.$$

**Examples 12 :**

1. Let $(f_n)_n$ the sequence of functions defined on $I=[0,1]$ by: $f_n(x)=x^n$, for all $x\in I$ and $n\in\mathbb{N}$. The sequence $(f_n)_n$ converges to the function $f$ defined by:

$$f(x)=\begin{cases}0 & \text{if} \quad 0\leq x<1\\ 1 & \text{if} \quad x=1\end{cases}$$

$\sup\limits_{x\in[0,1]}|f_n(x)-f(x)|=\sup\limits_{x\in[0,1[}x^n=1$, then the sequence $(f_n)_n$ is not uniformly convergent on $[0,1]$ and also on $[0,1[$. Moreover, the sequence $(f_n)_n$ converges uniformly on any interval $[0,a]$, $\forall a\in[0,1[$. Indeed, $\lim\limits_{n\to+\infty}\left(\sup\limits_{x\in[0,a]}x^n\right)=\lim\limits_{n\to+\infty}a^n=0$.

2. Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}$ by: $f_n(x)=\dfrac{\sin(nx)}{n}$. The sequence $(f_n)_n$ converges uniformly to 0 on $\mathbb{R}$. ($|f_n(x)|\leq\frac{1}{n}$).

3. Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}^+=[0,+\infty[$ by: $f_n(x)=\dfrac{x}{n+x}$. The sequence $(f_n)_n$ converges to 0 on $\mathbb{R}^+$ and not uniformly since $\sup\limits_{x\in\mathbb{R}^+}f_n(x)=1$. Moreover the sequence $(f_n)_n$ converges uniformly on any closed interval $[a,b]\subset\mathbb{R}^+$.

4. Let $f_n(x)=xe^{-nx}$ for $x\in\mathbb{R}^+$. We have $\sup_{x\in\mathbb{R}^+}f_n(x)=\frac{1}{n}$. Then the sequence $(f_n)_n$ converges uniformly to 0 on $\mathbb{R}^+$.

5. Let $f_n(x)=\dfrac{x\sqrt{n}}{1+nx^2}$ for $x\in\mathbb{R}$. The sequence $(f_n)_n$ converges to 0, but $\sup_{x\in\mathbb{R}}f_n(x)=\frac{1}{2e}$. Then the sequence $(f_n)_n$ is not uniformly convergent on $\mathbb{R}$. Moreover for all $a>0$, the sequence $(f_n)_n$ converges uniformly on $[a,+\infty[$. Indeed for $n$ large enough $\sup_{x\in[a,+\infty[}f_n(x)=f_n(a)$.

## 1.1 Cauchy Criterion for the Convergence

**Theorem 1.2**

(Cauchy Criterion for the uniform convergence)
Let $(f_n)_n$ be a sequence of functions defined on an open subset $\Omega$ of $\mathbb{R}$. The sequence $(f_n)_n$ converges uniformly on a $A\subset\Omega$ if and only if

$$\lim_{p,q\to+\infty}\sup_{x\in A}|f_p(x)-f_q(x)|=0.$$

This is still equivalent to:

$$\forall\, \varepsilon > 0, \exists N,\ \sup_{x\in A} |f_{n+p}(x) - f_n(x)| \leq \varepsilon, \quad \forall\, n \geq N,\ \forall\, p \in \mathbb{N}.$$

**Remark 14 :**
If the sequence $(f_n)_n$ converges uniformly to $f$ on $A \subset \Omega$, then for any sequence $(x_n)_n \in A$, the sequence $\big(u_n = |f_n(x_n) - f(x_n)|\big)_n$ converges to 0. This is because $u_n \leq \sup_{x\in A} |f_n(x) - f(x)|$.

## 1.2 Continuity and Uniform Convergence

**Theorem 1.3**

Let $(f_n)_n$ be a sequence of functions defined on an open subset $\Omega \subset \mathbb{R}$ which converges uniformly to $f$ on a subset $I \subset \Omega$. Let $a \in I$ and assume that $\lim_{x\to a} f_n(x) = \ell_n$ exists for any $n$, then the sequence $(\ell_n)_n$ converges and $\lim_{x\to a} f(x) = \lim_{n\to+\infty} \ell_n$. Otherwise

$$\lim_{\substack{x\to a,\\ x\in I}} \big(\lim_{n\to+\infty} f_n(x)\big) = \lim_{n\to+\infty} \big(\lim_{\substack{x\to a,\\ x\in I}} f_n(x)\big). \tag{1.1}$$

**Proof** .
To prove that the sequence $(\ell_n)_n$ is convergent, we prove that it is a Cauchy sequence.
For $\varepsilon > 0$, there exists $\exists N$ such that $|f_n(x) - f_m(x)| \leq \varepsilon$, $\forall n, m \geq N$ and $\forall x \in I$. The inequality is still true if $x$ tends to $a$. Then $\forall \varepsilon > 0$, $|\ell_n - \ell_m| \leq \varepsilon$, $\forall n, m \geq N$. The sequence $(\ell_n)_n$ is a Cauchy sequence in $\mathbb{R}$. Let $\ell = \lim_{n\to+\infty} \ell_n$.
For $n_0 \geq N$, we have:

$$|f(x) - \ell| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - \ell_{n_0}| + |\ell_{n_0} - \ell|.$$

Since the sequence $(f_n)_n$ converges uniformly to $f$, $|f(x) - f_{n_0}(x)| < \varepsilon$, $\forall x \in I$. (We take $m = n_0$ and we tends $n$ to $+\infty$). Since $\lim_{x\to a} f_{n_0}(x) = \ell_{n_0}$, there exists $\eta > 0$ such that $\forall x \in I$, with $0 < |x - a| < \eta$ we have: $|f_{n_0}(x) - \ell_{n_0}| < \varepsilon \Rightarrow |\ell_{n_0} - \ell| < \varepsilon$. We have: $\forall x \in I$ such that $|x - a| < \eta$, $|f(x) - \ell| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$, which proves the result.

□

**Example 13 :**

Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}^+$ by: $f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t} e^{-xt} dt.$

$f_n(x)-f_m(x) = \displaystyle\int_n^m \frac{\sin t}{t} e^{-xt} dt, \quad (m > n)$. The function $t \longrightarrow \frac{e^{-xt}}{t}$ is decreasing on $[n, m]$, by the second mean formula, [1] $|f_n(x) - f_m(x)| \leq \dfrac{e^{-xn}}{n}.2 \leq 2/n$, then $\sup\limits_{x\in\mathbb{R}+} |f_n(x)-f_m(x)| \leq 2/n$, which proves that the sequence $(f_n)_n$ converge uniformly on $\mathbb{R}^+$.

Moreover $\lim\limits_{x\to 0^+} f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t} dt$, because $\left|f_n(x) - \displaystyle\int_0^n \frac{\sin t}{t} dt\right| \leq xn \xrightarrow[x\to 0^+]{} 0$. Then

$$\lim_{x\to 0^+} \int_0^{+\infty} \frac{\sin t}{t} e^{-xt} dt = \int_0^{+\infty} \frac{\sin t}{t} dt.$$

**Theorem 1.5**

Let $(f_n)_n$ be a sequence of functions defined on an open subset $I \subset \mathbb{R}$. Assume that :

1. The sequence $(f_n)_n$ converges uniformly to $f$ on any closed interval $[a, b] \subset I$,
2. For any $n \in \mathbb{N}$, the function $f_n$ is continuous at $c \in I$.

Then $f$ is continuous at $c$.

**Proof** .
We consider a sequence $(x_n)_n \in \Omega$ which converges to $c$. By Theorem 1.2 $\lim\limits_{x\to c} f(x) = \lim\limits_{n\to+\infty} f_n(c) = f(c)$. □

**Examples 14 :**

1. Let $(f_n)_n$ be the sequences of functions defined on $\mathbb{R}^+$ by: $f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t+x} dt$. The function $f_n$ are continuous on $\mathbb{R}^+$.

---

[1]

**Theorem 1.4**

Let $f$ be non-negative decreasing continuous function on the interval $[a, b]$ and let $g$ be a Riemann integrable function on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a) \int_a^c g(x)\ dx.$$

$|f_n(x)-f_n(0)| \leq \left|\int_0^n \frac{\sin t}{t}(\frac{x}{t+x})dt\right| \leq x(\ln(n+x)-\ln x)$, then $\lim\limits_{x\to 0}|f_n(x)-f_n(0)| = 0$. It results that $f_n$ is continuous at 0.

For $x_0 > 0$, $|f_n(x) - f_n(x_0)| \leq M_n(x_0)|x - x_0|$, $\forall x > \frac{x_0}{2}$, with $M_n(x_0) = \displaystyle\int_0^n \frac{dt}{(t+x_0)(t+\frac{x_0}{2})}$.

In use the second mean formula, we get: $|f_n(x) - f_m(x)| \leq \frac{2}{n+x}$, for all $n < m$ and $x > 0$. Then $\sup\limits_{x\in\mathbb{R}^+} |f_n(x) - f_m(x)| \leq \dfrac{2}{n}$ and the sequence $(f_n)_n$ converges uniformly on $\mathbb{R}^+$. It results that the function $f$ defined by $f(x) = \displaystyle\int_0^{+\infty} \frac{\sin t}{t+x}dt$ is continuous on $\mathbb{R}^+$.

2. For $x > 0$, we set $f_n(x) = \dfrac{1}{x}\displaystyle\int_0^n \frac{\sin t}{t+x}dt$. The functions $f_n$ are continuous on $\mathbb{R}_+^*$. The sequence $(f_n)_n$ convergences uniformly on $[h, +\infty[$, $\forall h > 0$. It results that the function $g$ defined by $g(x) = \dfrac{1}{x}\displaystyle\int_0^{+\infty} \frac{\sin t}{t+x}dt$ is continuous on $\mathbb{R}_+^*$.

**Theorem 1.6**

Let $(f_n)_n$ be a sequence of continuous functions on an open set $\Omega \subset \mathbb{R}$ and converges uniformly on compact subsets of $I$ to a function $f$. Then $f$ is continuous on $I$.

## 1.3 Integrability and Uniform Convergence

Let $(f_n)_n$ be a sequence of Riemann integrable functions on an interval $[a, b]$. Assume that the sequence $(f_n)_n$ converges to the function $f$. Various problems arise, however

1. the function $f$ is it Riemann integrable?

2. if $f$ is Riemann integrable on $[a, b]$, can we have

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt?$$

The answer to the question a) is negative, it suffices to take the function $f$ defined on $[a, b]$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{if not} \end{cases}$$

This function is not Riemann integrable and it is a limit of Riemann integrable functions. ($\mathbb{Q}$ is countable).
The answer to the question b) is also negative. We can take $f_n(x) = nx(1-x^2)^n$ defined on $[0,1]$. The sequence $(f_n)_n$ converges to 0 and $\lim\limits_{n\to+\infty} \int_0^1 f_n(x)dx = \frac{1}{2}$.
We still have the following theorem:

**Theorem 1.7**

Let $(f_n)_n$ be a sequence of Riemann-integrable functions on an interval $[a,b]$. If the sequence $(f_n)_n$ converges uniformly to a function $f$ on $[a,b]$, then $f$ is Riemann-integrable on $[a,b]$ and we have:

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt.$$

Moreover the sequence $(F_n)_n$ defined by: $F_n(x) = \int_a^x f_n(t)dt$ converges uniformly to the function $F$ defined by: $F(x) = \int_a^x f(t)dt$ on $[a,b]$.

**Proof** .
As the sequence $(f_n)_n$ is uniformly convergent to $f$ on $[a,b]$, the function $f$ is bounded. Indeed, for $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $\sup\limits_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N_\varepsilon$. Then $\sup\limits_{x\in[a,b]} |f(x)| \leq \sup\limits_{x\in[a,b]} |f_{N_\varepsilon}(x)| + \varepsilon < +\infty$.
Let $\sigma = \{x_1, \ldots, x_p\}$ be a partition of $[a,b]$ and let $n \geq N_\varepsilon$. As $\forall x \in [a,b]$ $f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon$, we have:
$M_k^n - \varepsilon \leq M_k \leq M_k^n + \varepsilon$ and $m_k^n - \varepsilon \leq m_k \leq m_k^n + \varepsilon$, with $M_k = \sup\limits_{x\in[x_k,x_{k+1}]} f(x)$, $M_k^n = \sup\limits_{x\in[x_k,x_{k+1}]} f_n(x)$, $m_k = \inf\limits_{x\in[x_k,x_{k+1}]} f(x)$ and $m_k^n = \inf\limits_{x\in[x_k,x_{k+1}]} f_n(x)$.
It results that:

$$U(f_n,\sigma) - \varepsilon(b-a) \leq U(f,\sigma) \leq U(f_n,\sigma) + \varepsilon(b-a)$$

$$L(f_n,\sigma) - \varepsilon(b-a) \leq L(f,\sigma) \leq L(f_n,\sigma) + \varepsilon(b-a).$$

Then

$$L(f_n) - \varepsilon(b-a) \leq L(f) \leq U(f) \leq U(f_n) + \varepsilon(b-a). \tag{1.2}$$

Since the functions $f_n$ are Riemann integrable, we have $U(f_n) = L(f_n)$ for all $n \in \mathbb{N}$, and $0 \leq U(f) - L(f) \leq 2\varepsilon(b-a)$, for all $\varepsilon > 0$. It results that $f$ is Riemann integrable on $[a,b]$ and for all $n \in \mathbb{N}$: $|\displaystyle\int_a^b f(t)dt - \int_a^b f_n(t)dt| < \varepsilon(b-a)$. Moreover we also have

$$\forall x \in [a,b], |F_n(x) - F(x)| \leq (b-a) \sup_{t\in[a,b]} |f_n(t) - f(t)|.$$

□

**Corollary 1.8**

Let $\big(f_n \colon [a,b] \longrightarrow \mathbb{R}\big)_n$ be a sequence of piecewise continuous functions on $[a,b]$ and uniformly convergent to $f$ on $[a,b]$, then $f$ is Riemann-integrable on $[a,b]$ and we have:

$$\int_a^b f(t)dt = \lim_{n\to+\infty} \int_a^b f_n(t)dt.$$

## 1.4 Differentiability

**Theorem 1.9**

Let $(f_n)_n$ be a sequence of continuously differentiable functions (of class $\mathcal{C}^1$) on an interval $[a,b] \subset \mathbb{R}$. Assume that:

1. the sequence $(f_n)_n$ is pointwise convergent to $f$ on $[a,b]$.
2. the sequence $(f'_n)_n$ is uniformly convergent on $[a,b]$.

Then $f$ continuously differentiable on $[a,b]$ and: $\forall x \in [a,b]$, $f'(x) = \lim_{n\to+\infty} f'_n(x)$ and $(f_n)_n$ converges uniformly to $f$ on $[a,b]$. In particular $f$ is of class $\mathcal{C}^1$ on $[a,b]$.

**Proof** .

We have $\displaystyle\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$. Let $g$ be the limit of the sequence $(f_n')_n$. We have $\displaystyle\int_a^x g(t)dt = f(x) - f(a)$. Moreover $g$ is continuous, then $f$ is differentiable and $f'(x) = g(x), \forall x \in [a, b]$. $\square$

**Exercise 3 :**

Let $(f_n)_n$ be a sequence of differentiable functions on an interval $[a, b]$. Assume that the sequence $(f_n')_n$ is uniformly convergent on $[a, b]$ and there exists $x_0 \in [a, b]$ such that the sequence $(f_n(x_0))_n$ is convergent. Prove that the sequence $(f_n)_n$ is uniformly convergent on $[a, b]$ to a differentiable function $f$ and $f'(x) = \lim\limits_{n\to+\infty} f_n'(x)$.

(Hint: use the mean value theorem to the function $f_n - f_m$, for $n$ and $m$ large enough.)

## 1.5 Exercises

**4-1-1** Define the sequence of functions $(f_n)_n$ on $\mathbb{R}$ by: $f_n(x) = n^2x(1-x)^n$.

(a) Determine the domain of pointwise convergence of the sequence $(f_n)_n$.

(b) Compute $\displaystyle\int_0^1 f_n(x)dx$ and deduce that the sequence $(f_n)$ is not uniformly convergent on the interval $[0, 2[$.

(c) Compute the limit of $f_n(\frac{1}{n})$, when $n \longrightarrow +\infty$, and deduce an other time the previous result.

**4-1-2** Study the pointwise and the uniform convergence of the following sequences of functions $(f_n)_n$ defined by:

(a) $f_n(x) = \dfrac{nx}{1+n^2x^2}$ on $\mathbb{R}$,

(b) $f_n(x) = \begin{cases} 2n^2x & \text{if} \quad x \in [0, \frac{1}{2n}] \\ 0 & \text{if} \quad x \in [\frac{1}{n}, 1] \\ 2n - 2n^2x & \text{if} \quad x \in [\frac{1}{2n}, \frac{1}{n}] \end{cases}$ on $[0, 1]$,

(c) $f_n(x) = \begin{cases} x^2 \sin(\frac{1}{nx}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}$,

(d) $f_n(x) = \begin{cases} \frac{\sin(x)}{x} e^{-nx} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}_+$,

(e) $f_n(x) = n^\alpha x(1 - nx - |1 - nx|)$ on $\mathbb{R}_+$, $\alpha \in \mathbb{R}$,

(f) $f_n(x) = \begin{cases} n^\alpha x(1 - nx) & \text{if} \quad 0 \le x < \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{n} \le x \le 1 \end{cases}$, $\alpha \in \mathbb{R}$

(g) $f_n(x) = \begin{cases} nx - \frac{1}{n} & \text{if} \quad x \in [0, \frac{1}{n}[ \\ 1 - x & \text{if} \quad x \in [\frac{1}{n}, 1] \end{cases}$ on $[0, 1]$.

(h) $f_n(x) = \begin{cases} \dfrac{\sin nx}{n\sqrt{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

(i) $f_n(x) = \dfrac{x^n}{1+x^n}$ on each of the following intervals, with $0 < a < 1$.
$[0, 1-a]$, $[1-a, 1+a]$, $[1+a, +\infty[$.

(j) $f_n(x) = \begin{cases} \dfrac{\sin^2 nx}{nx} & \text{if} \quad x \notin \pi\mathbb{Z} \\ 0 & \text{if} \quad x \in \pi\mathbb{Z} \end{cases}$,

**4-1-3** (a) Consider the function $\varphi_n : ]0, n[ \longrightarrow \mathbb{R}$ defined for $n \geq 2$ by:

$$\varphi_n(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n .$$

i. Prove that $\varphi_n'$ has a unique zero on the interval $]0, n[$.

ii. Study the variations of $\varphi_n$ on $[0, n]$.

(b) Study the pointwise and uniform convergence of the sequence of functions $(f_n)_{n\geq 1}$ defined on $[0, +\infty[$ by:

$$f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{if} \quad x > n \end{cases}$$

**4-1-4** Study the pointwise and the uniform convergence of the following sequences of functions $(f_n)_n$ defined by:

(a) $f_n(x) = (\cos^n x) \sin x$ for $x \in [0, \frac{\pi}{2}]$.

(b) $g_n(x) = (1 + \dfrac{x}{n})^n$, if $x \geq -n$ and $g_n(x) = 0$ if $x < -n$.
Consider the case of the uniform convergence on $]-\infty, a]$, for $a \in \mathbb{R}$.

**4-1-5** Let $(f_n)_n$ be the sequence of functions defined by on $\mathbb{R}\backslash\{-2\}$ by: $f_n(x) = \dfrac{(x+1)^n - 1}{(x+1)^n + 1}$.

Study the pointwise and the uniform convergence of the sequence $(f_n)_n$ on $\mathbb{R}\backslash\{-2\}$ and on any closed interval which does not contain neither $-2$ and $0$.

**4-1-6** Let $u_n(x) = n^2 x e^{-nx}, \ x \in [0, 1]$.

(a) Find the pointwise limit of the sequence of functions $(u_n)_n$

(b) Find $\displaystyle\lim_{n \longrightarrow +\infty} \int_0^1 u_n(x) dx$.

(c) The convergence of the sequence $(u_n)_n$ on $[0, 1]$ is it uniform?

**4-1-7** Let $(f_n)_n$ be the sequence of functions defined on $[0, +\infty[$ by: $f_n(x) = \dfrac{nx}{1 + nx}$.

(a) Determine the pointwise limit $f$ of the sequence $(f_n)_n$.

(b) The convergence of $(f_n)_n$ to $f$ is it uniform on $[0, 1]$? on $[1, +\infty[$? and on $[0, +\infty[$?

(c) Let $F_n$ be the function defined on $[0, +\infty[$ by: $F_n(x) = \int_0^x f_n(t)dt$.

i. Determine the pointwise limit $F$ of the sequence $(F_n)_n$.

ii. The convergence of $(F_n)_n$ to $F$ on $[0, 1]$ is it uniform?

**4-1-8** Let $(f_n)_n$ be the sequence of functions defined by: $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$, for $x \in \mathbb{R}$.

(a) Prove that the sequence $(f_n)_n$ converges uniformly on $\mathbb{R}$.

(b) Prove that the functions $f_n$ are differentiable on $\mathbb{R}$ and the limit of the sequence $(f_n)_n$ is not differentiable.

**4-1-9** Define a sequence of functions $(f_n)_n$ on $\mathbb{R}_+^*$ by:

$$f_n(x) = n|\ln x|^n.$$

(a) Determine the domain $D$ of the pointwise convergence of the sequence $(f_n)_n$.

(b) Study the uniform convergence of the sequence $(f_n)_n$ to $f$ on $D$ and on the compacts of $D$.

**4-1-10** Define the sequence of functions $(f_n)_n$ on $\mathbb{R}_+$ by: $f_n(x) = \dfrac{ne^{-x}(x^3 + x)}{1 + nx}$.

(a) Determine the limit $f$ of the sequence $(f_n)_n$ and deduce that the sequence $(f_n)_n$ is not uniformly convergent on $\mathbb{R}_+$.

(b) Prove that the sequence $(f_n)_n$ converges uniformly on any closed and bounded interval of $]0, +\infty[$ to $f$.

(c) Prove that the sequence $(|f_n - f|)_n$ is bounded on $[0, 1]$.

(d) Deduce that $\lim\limits_{n\to+\infty} \int_0^1 f_n(t)dt = \int_0^1 f(t)dt$.

**4-1-11** Define the sequence $(f_n)_n$ of functions defined on $\mathbb{R}_+$ by: $f_n(x) = e^{-nx^n}$.

(a) Determine the domain $D$ of pointwise convergence of the sequence $(f_n)_n$.

(b) Prove that the sequence $(f_n)_n$ converges uniformly on $[1, +\infty[$.

(c) Prove that the sequence $(f_n)_n$ is not uniformly convergent on $[0, 1[$.

(d) Study the uniform convergence of the sequence $(f_n)_n$ on the compact subsets of $[0, 1[$?

Let $g_n = f_n'$.

(e) Determine the domain of pointwise convergence of the sequence $(g_n)_n$.

(f) Study the convergence of the sequence $\left(g_n(\frac{n-1}{n^2})^{\frac{1}{n}}\right)_n$.

(g) Study the uniform convergence of the sequence $(g_n)_n$ on the following intervals, $[0,+\infty[$, $[0,1[$ and $[1,+\infty[$.

**4-1-12** Define the sequence of functions $(f_n)_n$ on $\mathbb{R}_+^*$ by:

$$f_n(x) = (-1)^n x^{n^\beta} \ln(\frac{x^2+x+n}{n+x}).$$

(a) Prove that $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^{2+n^\beta}}{n}$.

(b) Determine, eventually according to the values of $\beta$ the domain $D_\beta$ of the pointwise convergence of the sequence $(f_n)_n$.

(c) Study the uniform convergence on $D_\beta$, and on the compacts of $D_\beta$.

**4-1-13** Let $f$ be a continuous function on $\mathbb{R}$. Assume that there exists a sequence $(P_n)_n$ of polynomials which converges uniformly on $\mathbb{R}$ to $f$.

(a) Prove that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ P_n - P_{n_0}$ is bounded on $\mathbb{R}$.

(b) Deduce that $f$ is a polynomial function.

**4-1-14** Study the pointwise and uniform convergence of the following sequence of functions $(f_n)_n$.

(a) $f_n(x) = \begin{cases} x^{2n}\ln x & \text{if } x \in ]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(b) $f_n(x) = \begin{cases} nx^n\ln x & \text{if } x \in ]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(c) $f_n(x) = \begin{cases} \dfrac{\sin^2 nx}{n\sin x} & \text{if } x \in ]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(d) $f_n(x) = 4^n\left(x^{2^{n+1}} - x^{2^n}\right)$.

(e) $f_n(x) = \dfrac{2^n x}{1+n2^n x^2}$ and compute $\lim\limits_{n\to+\infty} \displaystyle\int_0^1 f_n(t)dt$ and $\displaystyle\int_0^1 \lim_{n\to+\infty} f_n(t)dt$.

**4-1-15** Let $(f_n)_n$ be the sequence of functions defined by: $f_n(x) = x^2 \sin \dfrac{1}{nx}$ if $x \neq 0$ and $f_n(0) = 0$

(a) Prove that the sequence $(f_n)_n$ converges uniformly on any interval $[a, b] \subset \mathbb{R}$.

(b) The convergence is it uniform on $\mathbb{R}$?

(c) The sequence $(f'_n)_n$ is it uniformly convergent on $\mathbb{R}$.

**4-1-16** For $x \in [0, 1]$ and $n \in \mathbb{N}$, define $f_n(x) = \sum_{k=1}^{n} \dfrac{(-1)^{k-1} x^k}{k} - \ln(1 + x)$

(a) Prove that the sequence $(f_n)_n$ converges uniformly to 0 on $[0, 1]$. (We can compute $f'_n(x)$).

(b) Prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k} (\dfrac{n}{n+1})^k = \ln 2$.

**4-1-17** Let $(f_n)_n$ be the sequence defined on $[0, 1]$ by:

$$f_1(x) = \begin{cases} n^2 x & \text{if} \quad x \in [0, \frac{1}{n}] \\ -nx^2 + 2x & \text{if} \quad x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if} \quad x \in [\frac{2}{n}, 1] \end{cases}$$

(a) Study the pointwise and the uniform convergence of the sequence $(f_n)_n$.

(b) Compare $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ and $\int_0^1 \lim_{n\to\infty} f_n(x)dx$.

# 2 Series of Functions

**Definition 2.1**

Let $(f_n)_n$ be a sequence of functions defined on a subset $A$ of $\mathbb{R}$.

1. The series of functions $\sum_{n\geq 1} f_n$ is called pointwise convergent on $A$ if the sequence $\left(S_n = \sum_{k=1}^{n} f_k\right)_n$ is pointwise convergent on $A$.

2. The series $\sum_{n\geq 1} f_n$ is called uniformly convergent on $A$ if the se-

quence $\left(S_n = \sum_{k=1}^{n} f_k\right)_n$ converges uniformly on $A$.

**Remark 15 :**

1. If the series $\sum_{n\geq 0} f_n$ is pointwise convergent to a function $f$ on an interval $I$, then $\lim_{n\to+\infty} f_n(x) = 0$, for all $x \in I$.

2. A series $\sum_{n\geq 0} f_n$ is pointwise convergent on $J$, if and only if, the series $\sum_{n\geq 0} f_n(x)$ fulfills the Cauchy criterion, i.e.

$$\forall x \in I, \forall \varepsilon > 0, \exists N; |\sum_{k=n}^{n+p} f_k(x)| < \varepsilon, \quad \forall n \geq N, p \in \mathbb{N}.$$

**Examples 15 :**

1. Let $(f_n)_n$ be a sequence of functions defined by: $f_n(x) = x^n$, the series $\sum_{n\geq 0} f_n(x)$ is pointwise convergent on the interval $]0,1[$ to the function $f(x) = \dfrac{1}{1-x}$. If $|x| \geq 1$, $|f_n(x)| \geq 1$, then the series $\sum_{n\geq 0} f_n(z)$ is divergent on $\mathbb{R}\backslash]0,1[$.

2. For $x \geq 0$, we set $f_n(x) = \dfrac{\sin \frac{x}{n}}{n+x}$.

   For all fixed $x > 0$ we have: $\sin \dfrac{x}{n} = \dfrac{x}{n} - \dfrac{x^3}{6n^3} + O(\dfrac{1}{n^3})$, then

$$f_n(x) = \frac{x}{n(x+n)} - \frac{x^3}{6n^3(x+n)} + O(\frac{1}{n^3}).$$

   Then the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}^+$.

   Also, the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R} \setminus \mathbb{Z}_-$.

**Remark 16 :**

1. If the series $\sum_{n\geq 0} f_n$ is uniformly convergent to $f$ on $I$, then the series $\sum_{n\geq 0} f_n$ is pointwise convergent to $f$ on $I$.
2. A series $\sum_{n\geq 0} f_n$ is uniformly convergent on $I$, if and only if, it fulfills the Cauchy criterion for the uniform convergence i.e.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \sup_{x\in I} \|\sum_{k=n}^{n+p} U_k(x)\| < \varepsilon, \quad \forall n \geq N,\ p \in \mathbb{N}.$$

**Example 16 :**
The series $\sum_{n\geq 0} x^n$ is pointwise convergent on $]-1,1[$ to the function $f(x) = \frac{1}{1-x}$, but the convergence is not uniform because $\sup_{x\in]-1,1[} f_n(x) = 1$.

**Definition 2.2**

A series $\sum_{n\geq 0} f_n$ is called normally convergent on $I$, if the series $\sum_{n\geq 0} \sup_{x\in I} \|f_n(x)\|$ is convergent.

**Proposition 2.3**

If the series $\sum_{n\geq 0} f_n$ is normally convergent on $I$, then it is uniformly convergent on $I$.

For the proof we use the Cauchy criterion.

**Corollary 2.4**

If $\sup_{x\in I} |f_n(x)| \leq a_n$ and the series $\sum_{n=0}^{+\infty} a_n$ is convergent, then the series

$\sum_{n\geq 0} f_n$ is normally converge on $I$.

**Examples 17 :**

1. Let $f_n(x) = \frac{e^{inx}}{n^\alpha}$, $(\alpha > 1)$. $|f_n(x)| \leq \dfrac{1}{n^\alpha}$, then the series converges normally on $\mathbb{R}$.

2. For $x \in ]0,+\infty[$, we have: $xe^{-x} \leq 1$, then $f_n(x) = \dfrac{e^{-nx}}{n} \leq \dfrac{1}{x.n^2} \leq \dfrac{1}{h.n^2}$ for all $x \in [h,+\infty[$. It results that the series $\sum_{n\geq 1} f_n$ converges uniformly on $[h,+\infty[, \forall h > 0$.

3. Let $f_n(z) = \frac{1}{n(x+n)}$ defined on $\mathbb{R}\setminus\mathbb{Z}_-^*$. $|f_n(x)| \leq \frac{1}{n|x+n|} \leq \frac{1}{n|n-|x||}$. Let $K$ be any compact of $\mathbb{R}\setminus\mathbb{Z}_-^*$, there exists $R > 0$ such that $K \subset ]-R,R[$. Let $n_0 \in \mathbb{N}$ such that $R < n_0$, we have: $|f_n(x)| \leq \frac{1}{n(n-R)}$, $\forall n \geq n_0$, $\forall x \in K$. Then the series $\sum_{n\geq 1} f_n$ converges uniformly on $K$.

## 2.1 Abel's Criterion for the Uniform Convergence

**Theorem 2.5**

Let $(f_n)_n$ be a sequence of functions defined on a subset $X \subset \mathbb{R}$ and let $(g_n)_n$ be a sequence of functions defined on a subset $Y \subset \mathbb{R}$. The series $\sum_{n\geq 1} f_n(x)g_n(y)$ is uniformly convergent on $X \times Y$ under any one of the following conditions.

1. The series $\sum_{n\geq 1} f_n$ is uniformly convergent on $X$ and the sequence $(g_n)_n$ is bounded and monotone on $Y$.
2. The partial sums of the series $\sum_{n\geq 1} f_n$ are uniformly bounded on $X$ and the sequence $(g_n)_n$ is monotone and uniformly convergent to 0 on $Y$.
3. The series $\sum_{n\geq 1} f_n$ is uniformly convergent on $X$ and the series $|g_0| + \sum_{n\geq 1} |g_n - g_{n+1}|$ is bounded on $Y$.

**Proof** .

1. We set $S_n(x) = \sum_{p=1}^{n} f_p(x)$ and $S(x) = \sum_{n=1}^{+\infty} f_n(x)$. Assume that the sequence $(S_n)_n$ is uniformly convergent to $S$ on $X$ and the sequence $(g_n)_n$ is decreasing and bounded on $Y$. Then

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\quad \forall\, n \geq N \quad \sup_{x\in X} |S_n(x) - S(x)| \leq \varepsilon.$$

Let $M > 0$ such that $|g_n(y)| \leq M$ for every $n \in \mathbb{N}$ and every $y \in Y$. If $p \geq N + 1$ and $q > p$, then

$$\begin{aligned}\sum_{n=p}^{q} f_n(x)g_n(y) \quad &= \quad \sum_{n=p}^{q-1} \left(S_n(x) - S(x)\right)\left(g_n(y) - g_{n+1}(y)\right) \\ &\quad + \left(S_q(x) - S(x)\right) g_q(y) - \left(S_{p-1}(x) - S(x)\right) g_p(y).\end{aligned}$$

Then

$$\begin{aligned}\sup_{x\in X, y\in Y}|\sum_{n=p}^{q} f_n(x)g_n(y)| &\leq \varepsilon \sup_{y\in Y}\sum_{n=p}^{q-1}[g_n(y)-g_{n+1}(y)] \\ &+\varepsilon\sup_{y\in Y}(|g_q(y)|+|g_p(y)|)\leq 2\varepsilon M.\end{aligned}$$

It follows that the series $\sum_{n\geq 1} f_n(x)f_n(y)$ converges uniformly on $X\times Y$.

2. Let $M>0$ such that $|S_n(x)|\leq M,\ \ \forall\ x\in X$ and $\forall n\in\mathbb{N}$. Assume that the sequence $(g_n)_n$ is decreasing:

$$\sum_{n=p}^{q} f_n(x)g_n(y) = \sum_{n=p}^{q-1} S_n(x)(g_n(y)-g_{n+1}(y))+S_q(x)g_q(y)-S_{p-1}(x)g_p(y)$$

We have: $\ \forall\ \varepsilon>0,\ \exists\ N\in\mathbb{N}$ such that $\forall\ \ n\geq N,\ \sup_{y\in Y}|g_n(y)|\leq\varepsilon.$

For $p\geq N+1$ and $q>p$

$$\sup_{x\in X, y\in Y}|\sum_{n=p}^{q} f_n(x)g_n(y)|\leq \sup_{y\in Y}\Big[M(g_p(y)-g_q(y))+Mg_q(y)+Mg_p(y)\Big]\leq 2M\varepsilon.$$

3. Let $M>0$ be such that

$|g_0(y)|+\sum_{n=0}^{+\infty}|g_n(y)-g_{n+1}(y)|\leq M,\ \ \forall\ y\in Y.$

Let $n\geq N$,

$$g_n(y)=\sum_{p=1}^{n-1}(g_{p+1}(y)-g_p(y))+g_0(y).$$

It follows that $|g_n(y)|\leq M,\ \ \forall\ n\in\mathbb{N}$ and $\ \forall\ y\in Y$.

$$\begin{aligned}\sum_{n=p}^{q} f_n(x)g_n(y) &= \sum_{n=p}^{q-1}(S_n(x)-S(x))(g_n(y)-g_{n+1}(y)) \\ &+(S_q(x)-S(x))g_q(y)-\big(S_{p-1}(x)-S(x)\big)g_p(y).\end{aligned}$$

Thus

$$\sup_{x\in X, y\in Y}|\sum_{n=p}^{q} f_n(x)g_n(y)|\leq\varepsilon\sup_{y\in Y}(\sum_{n=p}^{q}|g_n(y)-g_{n+1}(y)|+2M)\leq 3\varepsilon M.$$

$\square$

**Examples 18 :**

1. Let $(a_n)_n$ be a sequence of non negative decreasing real numbers and convergent to 0. The series $\sum\limits_{n\geq 0} a_n e^{inx}$ is uniformly convergent on any compact subset of $\mathbb{R}\setminus 2\pi\mathbb{Z}$.

2. Consider the series $\sum\limits_{n\geq 0} \dfrac{e^{inx}}{n+x}$ and $K$ a compact of $\mathbb{R}\setminus \mathbb{Z}_-^*$, $\exists R>0$ such that $K\subset[-R,R]$. The sequence $g_n(x)=\frac{1}{n+x}$ is decreasing positive $\forall n\geq n_0$, $(n_0>R)$. The series is pointwise convergent on $\mathbb{R}\setminus\mathbb{Z}_-^*$ and it is uniformly convergent on any compact subset $K\subset \mathbb{R}\setminus(\mathbb{Z}_-\cup 2\pi\mathbb{Z})$. In particular this series converges uniformly on any interval $[\delta, 2\pi-\delta]$; $\forall\delta>0$.

**Proposition 2.6**

Let $\left(f_n\colon I\longrightarrow\mathbb{R}\right)_n$ be a sequence on continuous functions at a point $a\in I$. Assume that the series $\sum_{n\geq 0} f_n$ is uniformly convergent on $I$ to a function $f$. Then $f$ is continuous at $a$.

**Proof** .
We apply the theorem (1.2) of the previous section. $\square$

**Proposition 2.7**

Let $I$ be an open set in $\mathbb{R}$ and $\left(f_n\colon I\longrightarrow\mathbb{R}\right)_n$ a sequence of continuous functions. Assume that the series $\sum\limits_{n\geq 0} f_n$ is uniformly convergent on any compact of $I$ to a function $f$. Then $f$ is continuous on $I$.

**Theorem 2.8**

Let $\left(f_n\colon [a,b]\longrightarrow\mathbb{R}\right)_n$ be a sequence of Riemann integrable functions. Assume that the series $\sum\limits_{n\geq 0} f_n$ is uniformly convergent on $[a,b]$ to a function $f$. Then $f$ is Riemann integrable and we have:

$$\sum_{n=0}^{+\infty} \int_a^b f_n(x)dx = \int_a^b \sum_{n=0}^{+\infty} f_n(x)dx.$$

**Proposition 2.9**

Let $\left(f_n : [a,b] \longrightarrow \mathbb{R}\right)_n$ be a sequence of continuously differentiable functions ($\mathcal{C}^1$ functions). Assume that

1. the series $\sum\limits_{n\geq 0} f_n$ is pointwite convergent on $[a,b]$ to a function $f$.
2. the series $\sum\limits_{n\geq 0} f_n'$ converge uniformly on $[a,b]$.

Then $f$ is continuously differentiable on $[a,b]$ and we have:

$$f'(x) = \sum_{n=0}^{+\infty} f_n'(x), \quad \forall x \in [a,b].$$

Moreover the series $\sum\limits_{n\geq 0} f_n$ converges uniformly on $[a,b]$ to $f$.

**Corollary 2.10**

Let $I$ be an interval of $\mathbb{R}$ and let $\left(f_n : I \longrightarrow \mathbb{R}\right)_n$ be a sequence of continuously differentiable functions. Assume

1. the series $\sum_{n\geq 0} f_n$ is pointwise convergent on $I$ to $f$,
2. the series $\sum_{n\geq 0} f_n'$ converges uniformly on any compact of $I$.

Then $f$ is continuously differentiable and we have:

$$f'(x) = \sum_{n=0}^{+\infty} f_n'(x), \quad \forall x \in I.$$

## 2.2 Exercises

**4-2-1** Study the pointwise, absolute, normally and uniform convergence of the following series of general term:

1) $\sum_{n\geq 1} \frac{\sin(n^2x)}{n^2}$, $x \in \mathbb{R}$,

2) $\sum_{n\geq 1} \frac{1}{n}\tan^{-1}\frac{x}{n}$, $x \in \mathbb{R}$,

3) $\sum_{n\geq 1} x^{n^2}\sin(n\pi x)$, $x \in [0,a]$, $0 < a < 1$.

4) $\sum_{n\geq 1} \frac{x}{(1+x^2)^n}$, $x \in \mathbb{R}$.

5) $\sum_{n\geq 1} xe^{-nx^2}$, $x \in \mathbb{R}$,

6) $\sum_{n\geq 1} x^2e^{-x\sqrt{n}}$, $x \in \mathbb{R}_+$,

7) $\sum_{n\geq 1} \frac{nx^2}{1+n^3x}$, $x \in \mathbb{R}_+$,

8) $\sum_{n\geq 1} \frac{(-1)^n}{n^x}$, $x \in \mathbb{R}$,

9) $\sum_{n\geq 1} \frac{x^{2n}}{1+x^{2n}}$, $x \in \mathbb{R}$,

10) $\sum_{n\geq 1} \frac{(-1)^n}{x^2+n}$, $x \in \mathbb{R}$,

11) $\sum_{n\geq 1} \frac{x}{(1+nx^2)^n}$, $x \in \mathbb{R}$,

12) $\sum_{n\geq 1} \frac{(-1)^n x}{(1+x^2)^n}$, $x \in \mathbb{R}$,

13) $\sum_{n\geq 1} \frac{x}{n^\alpha(1+nx^2)}$, $\alpha > 0$.

**4-2-2** (a) Study the pointwise convergence of the series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ on $\mathbb{R}_+$.

(b) Study the uniform and normal convergence of this series on any closed bounded interval in $\mathbb{R}_+$.

**4-2-3** Find the domain of definition and the domain of continuity of the function: $f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n e^{-nx}}{n+1}$.

**4-2-4** (a) Find the domain of definition $\mathcal{D}$ of the function $g(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n e^{-nx}}{n^2+1}$.

(b) Prove that $g$ is of class $C^1$ on $\mathcal{D}$.

**4-2-5** (a) Prove that the series $\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1}$ is uniformly convergent on $[-1,1]$.

Let $f(x) = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, for $x \in [-1,1]$.

(b) Prove that $f$ is differentiable on $]-1,1[$ and compute $f'$.

(c) Deduce the expression of $f(x)$, for $-1 \leq x \leq 1$.

(d) Compute $\displaystyle\int_0^1 \tan^{-1} x dx$ and deduce the value the of the following sum $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)}$.

**4-2-6** Consider the series of functions $\displaystyle\sum_{n\geq 1} f_n$ defined on $\mathbb{R}_+$ by: $f_n(x) = x^n - x^{n-\frac{1}{2}}$.

(a) Prove that the series $\displaystyle\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,1]$.

Denote $f(x) = \displaystyle\sum_{n=1}^{+\infty} f_n(x)$, for $x \in [0,1]$.

(b) Prove that the remains $R_n(x) = \displaystyle\sum_{p=n+1}^{+\infty} u_p(x) = x^n f(x)$.

(c) Prove that the series $\displaystyle\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,1]$.

(d) Prove that there exists $M \in \mathbb{R}_+$ such that:

$$\left|\int_0^1 R_n(x)dx\right| \leq \frac{M}{n+1}$$

(e) Deduce that the series $\displaystyle\sum_{n\geq 1} g_n$, where $g_n = \displaystyle\int_0^1 f_n(x)dx$ is convergent and its sum is $\displaystyle\int_0^1 f(x)dx$.

(f) Compute $\displaystyle\int_0^1 f(x)dx$ and deduce the value of the following sum $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$.

**4-2-7** Define the series of functions $\displaystyle\sum_{n\geq 0} f_n$, where $f_n$ is defined by: $f_0(x) = 0$ and $f_n(x) = \dfrac{\sin n^2 x}{n^2}$, for $n \geq 1$.

(a) Prove that the series $\displaystyle\sum_{n\geq 0} f_n$ is uniformly convergent on $\mathbb{R}$.

(b) Study the convergence of the series $\sum_{n\geq 0} f_n'$.

**4-2-8** Let $f(x) = \sum_{n=1}^{+\infty} f_n(x)$; with $f_n(x) = \dfrac{(-1)^{n-1}}{\sqrt{n^2+x^2}}$.

(a) Prove that $f$ is continuous on $\mathbb{R}$.

(b) Study the uniform convergence of the series $\sum_{n\geq 1} f_n'$ and deduce that $f$ is of class $\mathcal{C}^1$.

**4-2-9** (a) Find the set of definition $D$ of the function $f(x) = \sum_{n=1}^{+\infty} \dfrac{(-1)^{n+1}}{n^x}$ and prove that $f$ is of class $\mathcal{C}^\infty$ on $D$.

(b) For $x > 1$; express $f(x)$ in term of $\sum_{n=1}^{\infty} \dfrac{1}{n^x}$.

**4-2-10** (a) Prove that the series $\sum_{n\geq 1} \dfrac{x \sin nx}{2\sqrt{n} + \cos x}$ is pointwise convergent on $]0, 2\pi[$

(b) Prove that the convergence of the series is uniform on any interval of the form: $[\alpha, 2\pi - \alpha] \quad \forall 0 < \alpha < 2\pi$.

**4-2-11** Let $\alpha \in \mathbb{R}$ and $f_n(x) = \dfrac{1}{n^\alpha} \ln(1 + n^\alpha x^2)$, for $n \geq 1$ and $x \in \mathbb{R}$.

(a) Prove that the series $\sum_{n\geq 1} f_n(x)$ is pointwise convergent on $\mathbb{R}$ if and only if $\alpha > 1$.

(b) Assume that $\alpha > 1$.

i. The series $\sum_{n\geq 1} f_n(x)$ is it uniformly convergent on $\mathbb{R}$?

ii. Prove that the function $f(x) = \sum_{n=1}^{+\infty} f_n(x)$ is continuous on $\mathbb{R}$.

(c) Prove that if $\alpha > 2$; $f$ is differentiable on $\mathbb{R}^*$.

(d) Assume $1 < \alpha \leq 2$.

i. Prove that $f$ is differentiable on $\mathbb{R}^*$.

ii. Prove that $\forall n \geq 1;\ f(n^{\frac{-\alpha}{2}}) \geq \ln 2 . \sum_{k=n}^{\infty} \dfrac{1}{k^\alpha}$.

Deduce that $\forall n \geq 1$;

$$n^{\frac{\alpha}{2}} f(n^{\frac{-\alpha}{2}}) > \frac{\ln 2}{\alpha - 1}.$$

$f$ is it differentiable at 0?

**4-2-12** Define $f_n(x) = \dfrac{x}{(1+x^2)^n}$, for $x \in \mathbb{R}$.

Prove that

(a) The series $\sum_{n\geq 0} f_n$ and $\sum_{n\geq 0} (-1)^n f_n$ converge and compute their sum.

(b) $\forall a > 0$, the series $\sum_{n\geq 0} f_n$ converges uniformly on $[a, +\infty[$;

(c) The series $\sum_{n\geq 0} (-1)^n f_n$ converges uniformly on $\mathbb{R}$.

**4-2-13** Let $f_n(x) = \dfrac{\ln(1+nx)}{nx^n}$, for $x > 0$. Prove that

(a) the domain of the pointwise convergence of the series $\sum_{n\geq 1} f_n(x)$ is $]1, +\infty[$. Let $f = \sum_{n=1}^{+\infty} f_n$ on $]1, +\infty[$.

(b) the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $]1, +\infty[$ and normally convergent on $[a, +\infty[$, for all $a > 1$.

(c) $f$ is continuous on $]1, +\infty[$ and $\lim_{x\to 1^+} f(x) = +\infty$.

**4-2-14** Define the sequence $(f_n)_n$ by: $f_n(x) = \dfrac{e^{-x\sqrt{n}}}{1+\sqrt{n^3}}$.

(a) Determine the domain of convergence of the series $\sum_{n\geq 0} f_n$.

Denote $f = \sum_{n=0}^{+\infty} f_n$.

(b) Prove that $f$ is continuous on $\mathbb{R}_+$.

(c) Prove that $f$ is differentiable on $\mathbb{R}_+^*$.

**4-2-15** Let $f\colon ]-1,+\infty[$ defined by:

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x}.$$

Prove that $f$ is continuous on $]-1,+\infty[$ and compute $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to(-1)^+} f(x)$.

**4-2-16** Study the pointwise and uniform convergence of the series of functions $\sum\limits_{n\geq 0} \frac{e^{-nx}}{1+n^2}$. Let $f(x) = \sum\limits_{n=0}^{+\infty} \frac{e^{-nx}}{1+n^2}$.

Prove that $f$ is of class $\mathcal{C}^1$ on $\mathbb{R}_+^*$.

**4-2-17** Define the series of functions $\sum\limits_{n\geq 1} f_n$, where $f_n(x) = \dfrac{\sin nx \sin n^2x}{n}$.

Recall that $2\sin kx \sin k^2x = \cos k(k-1)x - \cos k(k+1)x$.

Prove that the series $\sum\limits_{n\geq 1} f_n(x)$ converges uniformly on $\mathbb{R}$.

**4-2-18** Let $f_n(x) = \dfrac{\ln(1+n^\beta x^2)}{n^\alpha}$; with $\alpha$ and $\beta$ two positive numbers.

Under what conditions the series $\sum\limits_{n\geq 1} f_n(x)$ and $\sum\limits_{n\geq 1} f_n'(x)$ are pointwise convergent on $\mathbb{R}$?

**4-2-19** Define by induction the sequence of functions $(f_n(x))_n$ on the interval $[0,1]$ by:

$$f_0(x) = 1 \quad \text{and} \quad f_n(x) = 1 + \int_0^x f_{n-1}(t-t^2)dt.$$

(a) Prove that for each $n \in \mathbb{N}$, the function $f_n$ is a polynomial and that $f_n(x) + f_n(1-x)$ is constant.

(b) Prove that for any $n \in \mathbb{N}$ and any $x \in [0,1]$

$$0 \leq f_n(x) - f_{n-1}(x) \leq \frac{x^n}{n!}.$$

(c) Deduce that the sequence $(f_n)_n$ converges uniformly on $[0,1]$ to a function $f$ of class $\mathcal{C}^1$ on $[0,1]$ and fulfills $f'(x) = f(x-x^2)$.

**4-2-20** Consider the sequence of functions $(f_n)_n$ defined on $]0,+\infty[$ by: $f_n(x) = \dfrac{1}{(nx+1)^2}$.

(a) Prove that the series $\sum\limits_{n\geq 0} f_n$, $\sum\limits_{n\geq 0} f'_n$ and $\sum\limits_{n\geq 0} f''_n$ are uniformly convergent on $[a,+\infty[$, with $a>0$.

(b) Let $F(x) = \sum\limits_{n=1}^{+\infty} f_n(x)$. Recall that $\sum\limits_{n=1}^{+\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. Compute $F(\frac{1}{2})$, $F(1)$ and $F(2)$.

(c) Prove that $F$ is $\mathcal{C}^2$ on $]0,+\infty[$ and give the sign of $F'$ and $F''$ on $]0,+\infty[$.

(d) Determine $\lim\limits_{x\to+\infty} F(x)$ and $\lim\limits_{x\to 0^+} F(x)$.

**4-2-21** Consider the series $\sum\limits_{n\geq 0} e^{-n^2x}$ and $\sum\limits_{n\geq 0} xe^{-n^2x}$ and denote $f(x) = \sum\limits_{n=0}^{+\infty} e^{-n^2x}$ and $g(x) = \sum\limits_{n=0}^{+\infty} xe^{-n^2x}$ in the domains of convergence respective $D_f$ and$D_g$.

(a) Determine $D_f$ and $D_g$

(b) i. Prove that $f$ is decreasing on $D_f$.

ii. Give $\lim\limits_{x\to 0^+} f(x)$.

(c) The function $f$ is it continuous on $D_f$?

(d) i. Compute $\sup\limits_{x\geq 0} xe^{-n^2x}$.

ii. The series $\sum\limits_{n\geq 0}^{+\infty} xe^{-n^2x}$ is it uniformly convergent on $D_g$?

**4-2-22** (a) Prove that the series $\sum\limits_{n\geq 0} (-1)^n \dfrac{e^{-nx}}{n+1}$ defines a continuous function on its domain of definition $D$.

(b) Prove that the series $\sum\limits_{n\geq 0} (-1)^n \dfrac{e^{-nx}}{n^2+1}$ defines a function $g$ of class $C^\infty$ on its domain of convergence.

**4-2-23** Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}$ by: $f_n(x) = nxe^{-nx^2}$.

(a) Study the pointwise convergence of the series $\sum_{n\geq 0} f_n$.

(b) Prove that the series $\sum_{n\geq 0} f_n$ is not normally convergent on $\mathbb{R}$.

(c) Prove that it is normally convergent on $[a, +\infty[$, for all $a > 0$.

(d) Let $f(x) = \sum_{n=0}^{+\infty} f_n(x)$. Prove that $f$ is the derivative of a well known function. Deduce the expression of $f$.

**4-2-24** Let $f_n(x) = \dfrac{x}{1+n^2x^2}$.

(a) Prove that the series $\sum_{n\geq 0} f_n$ is pointwise convergent on $\mathbb{R}$.

(b) For $a > 0$, prove that the series $\sum_{n\geq 0} f'_n$ converges normally on $]-\infty, -a] \cup [a, +\infty[$.

(c) The series $\sum_{n\geq 0} f'_n$ is it uniformly convergent on $\mathbb{R}$?

(d) Determine the set where the function $F(x) = \sum_{n=0}^{+\infty} f_n(x)$ is differentiable.

**4-2-25** (a) Prove that $\forall x, y \in \mathbb{R}_+^*$, $x^{\ln(y)} = y^{\ln(x)}$.

(b) Let $x \in \mathbb{R}_+^*$, we set: $f_n(x) = x^{\ln(n)}$.

Prove that the series $\sum_{n\geq 1} f_n(x)$ is convergent if and only if $x < \dfrac{1}{e}$.

(c) i. Let $a, b$ such that $0 < a < b < \dfrac{1}{e}$.

Prove that the series $\sum_{n\geq 1} f_n$ is normally convergent on $[a, b]$.

ii. Let $f(x)$ be the sum of the series $\sum_{n\geq 1} f_n(x)$. $\left(f(x) = \sum_{n=1}^{+\infty} f_n(x)\right)$

Deduce that $f$ is continuous on $]0, \dfrac{1}{e}[$.

(d) Compare the function $f$ the sum of the series $\sum_{n\geq 1} f_n(x)$ with an integral and prove that:

$$\forall x \in ]0, \frac{1}{e}[, \qquad \frac{-1}{1+\ln(x)} \leq f(x) \leq \frac{\ln(x)}{1+\ln(x)}$$

The function $f$ is it bounded on $]0, \frac{1}{e}[$?

**4-2-26** Consider the series of functions $\sum_{n\geq 0} f_n$, with $f_n(x) = \frac{(-1)^n}{n!}\frac{1}{x+n}$ for $x \in \mathbb{R}$.

(a) Give the domain of definition of $f_n$.

(b) Give the set $D$ where the series $\sum_{n\geq 0} f_n$ is convergent.

(c) Denote for $x \in D$, $\quad f(x) = \sum_{n=0}^{+\infty} f_n(x)$.

i. Compute $f(1)$ in term of $e = \sum_{n=0}^{+\infty} \frac{1}{n!}$.

ii. Prove that for any $x \in D$, the function $xf(x) - f(x+1)$ is constant. Give its value.

(d) Study the uniform convergence of the series $\sum_{n\geq 0} f_n'$, and $\sum_{n\geq 0} f_n''$ and deduce that $f$ is two times differentiable on $D$.

**4-2-27** Define the sequence $(f_n)_n$ with $f_n \colon ]0, +\infty[ \longrightarrow \mathbb{R}$ defined by: $f_n(x) = \frac{(-1)^n \ln n}{n^x}$ and set

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n \ln n}{n^x}$$

(a) i. Prove that the series $\sum_{n\geq 1} f_n'(x)$ converges normally on any closed interval $[a, b] \subset ]1, +\infty[$.

ii. Deduce that $f$ is of class $\mathcal{C}^1$ on $]1, +\infty[$.

(b) i. Prove that the series $\sum_{n\geq 1} f_n'(x)$ converges uniformly on any interval $[\alpha, +\infty[$, with $\alpha > 0$.

ii. Deduce that $f$ is of class $\mathcal{C}^1$ on $]0, +\infty[$.

(c) Prove by the same method that the function $f$ is of class $C^\infty$ on $]0, +\infty[$.

**4-2-28** (a) i. Prove that the series $\sum_{n\geq 1} \frac{e^{-2nx}}{4n^2-1}$ converges uniformly on $[0,+\infty[$.

We set $f(x) = \sum_{n=1}^{+\infty} \frac{e^{-2nx}}{4n^2-1}$.

ii. Prove that $\forall x \in ]0,+\infty[; |3e^{2x}f(x) - 1| \leq 3e^{-2x} \sum_{n=2}^{+\infty} \frac{1}{4n^2-1}$.

iii. Deduce that $f(x) \underset{+\infty}{\approx} \frac{1}{3}e^{-2x}$.

(b) Let $g(x) = \sum_{n=1}^{+\infty} \frac{e^{-(2n+1)x}}{2n-1}$.

i. Prove that the series $\sum_{n\geq 2} \frac{e^{-(2n+1)x}}{2n-1}$ is pointwise convergent on $]0,+\infty[$ and uniformly convergent on $[a,+\infty[$ for any $a > 0$.

ii. Let $a > 0$. Prove that:

$\forall x \in [a,+\infty[ \quad |e^{3x}g(x) - 1| \leq e^{-x} \sum_{n=2}^{+\infty} \frac{e^{-(2n-3)a}}{2n-1}$.

iii. Deduce that $g(x) \underset{+\infty}{\approx} e^{-3x}$.

(c) Let $u(x) = \sum_{n=1}^{+\infty} e^{-(2n-1)x}$.

i. Prove that the series $\sum_{n\geq 1} e^{-(2n-1)x}$ is pointwise convergent on $]0,+\infty[$ and uniformly convergent on $[a,+\infty[$, $\forall a > 0$.

ii. Prove that $\forall x \in ]0,+\infty[, u(x) = \frac{1}{2\sinh x}$.

(d) Let $F(x) = e^{-x}f(x)$ and $G(x) = e^{2x}g(x)$.

i. Prove that $F$ and $G$ are differentiable on $]0,+\infty[$ and $F'(x) = -g(x)$ and $G'(x) = -u(x)$.

ii. Let $x \in ]0,+\infty[$. Compute the integral: $\int_x^{+\infty} \frac{dt}{\sinh t}$ and $\int_x^{+\infty} \frac{1}{t^3} \ln\left(\frac{e^t-1}{e^t+1}\right) dt$.

(e) Deduce the values of $g(x)$ and $f(x)$.

**4-2-29** Let $f$ be a continuous function on $[0,1]$. Define the sequence of polynomials $(B_n)_n$ called Bernstein polynomials associated to $f$,

$$B_n(x) = \sum_{k=0}^{n} C_n^k f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

(a) Let $\varphi_n(x,t) = \sum_{k=0}^{n} C_n^k e^{\frac{kt}{n}} x^k (1-x)^{n-k} = \sum_{k=0}^{n} C_n^k \left(e^{\frac{t}{n}} x\right)^k (1-x)^{n-k}$.

i. Compute $\frac{\partial \varphi_n}{\partial t}(x,t)$ and $\frac{\partial^2 \varphi_n}{\partial t^2}(x,t)$.

ii. Prove that

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1,$$

$$\sum_{k=0}^{n} k C_n^k x^k (1-x)^{n-k} = nx$$

and

$$\sum_{k=0}^{n} k^2 C_n^k x^k (1-x)^{n-k} = nx + n(n-1)x^2.$$

(b) Deduce that all $0 < \alpha < 1$,

$$\sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k (1-x)^{n-k} \leq \frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \leq \frac{1}{4n\alpha^2}.$$

(c) Using the uniform continuity of $f$, prove that the sequence $(B_n)_n$ converges uniformly to $f$.

(d) Deduce that any continuous function on a interval $[a,b]$ is uniform limit of a sequence of polynomials.

# 3 Approximation Theorems

In this section, we prove the Weierstrass theorem on the density of the space of polynomials on the space of continuous functions on the interval $[a,b]$.

**Definition 3.1**

A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is called a step function if there exist a partition $\sigma = (a_j)_{0\leq j\leq n}$ of $[a,b]$ such that $f$ is constant on any interval $]a_{j-1}, a_j[$, for all $1 \leq j \leq n$.
A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is called piecewise continuous function, if there exist a partition $\sigma = (a_j)_{0\leq j\leq n}$ of $[a,b]$ such that $f$ is continuous on any interval $]a_{j-1}, a_j[$, for all $1 \leq j \leq n$ and $f$ has a finite limit at the right on any point of $[a,b[$ and a finite limit at the left on any point

of $]a, b]$.

**Theorem 3.2**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a piecewise continuous function, then there exist a sequence of step functions on $[a,b]$ which converges uniformly to $f$. (A regulated function $f$ is a uniform limit of a sequence of step function.)

**Proof** .
If $f$ is continuous, it is uniformly continuous on $[a,b]$, then $\forall\ \varepsilon > 0$, $\exists \alpha > 0$ such that if $|x - x'| < \alpha$, $|f(x) - f(x')| \leq \varepsilon$. For all $n \in \mathbb{N}$, we consider the uniform partition $\sigma_n = (a_0, \ldots, a_n)$, with $a_k = a + k\frac{b-a}{n}$ for all $0 \leq k \leq n$ and we consider the step functions $f_n$ defined by: $f_n(x) = f(a_k)$, if $x \in [a_k, a_{k+1}[$ and $f(a_n) = f(b)$. If $n \geq \frac{b-a}{\alpha}$, we have:

$$||f_n - f||_\infty = \max_{0 \leq j \leq n-1} \Big( \sup_{x \in [a_j, a_{j+1}[} |f_n(x) - f(x)| \Big) \leq \varepsilon.$$

If $f$ is piecewise continuous and $\sigma = (a_0, \ldots, a_n)$ a partition associated to $f$, i.e. $f$ is continuous on $]a_j, a_{j+1}[$ for all $0 \leq j \leq n-1$. Let $f_j$ be a continuous function on $[a_j, a_{j+1}]$ such that $f_j = f$ on $]a_j, a_{j+1}[$. For every $f_j$ there exist a sequence of step functions $(f_{n,j})_n$ which converges uniformly to $f$ on $]a_j, a_{j+1}[$. Then the sequence $(f_n)_n$ defined by: $f_n(a_j) = f(a_j)$ and $f_n(x) = f_{n,j}(x)$ for $x \in ]a_j, a_{j+1}[$, converges uniformly to $f$ on $[a,b]$. $\square$

**Theorem 3.3**

[Weierstrass Theorem]
Let $f$ be a continuous function on an interval $[a,b]$. There exists a sequence of polynomials $(P_n)_n$ which converges uniformly to $f$ on $[a,b]$. (i.e. $\mathbb{R}[X]$ is dense in $\mathcal{C}([a,b])$ for the norm of uniform convergence.)

**Proof** .
Without loss of generality, we can assume that $[a,b] = [0,1]$.
Since $f$ is continuous on $[0,1]$, it is uniformly continuous. Then $\forall \varepsilon > 0$, $\exists \alpha > 0$; if $|x - y| \leq \alpha$, $|f(x) - f(y)| \leq \varepsilon$.
We consider the Bernstein polynomials sequence $(B_n)_n$ defined by:

$$B_n(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}.$$

$$\begin{aligned}
|f(x) - B_n(x) &= \Big|\sum_{k=0}^{n} C_n^k (f(x) - f(\frac{k}{n})) x^k (1-x)^{n-k}\Big| \\
&\leq \sum_{k=0}^{n} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k (1-x)^{n-k} \\
&= \sum_{|x-\frac{k}{n}|<\alpha} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k (1-x)^{n-k} \\
&\quad + \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k |f(x) - f(\frac{k}{n})| x^k (1-x)^{n-k} \\
&\leq \varepsilon + 2||f||_\infty \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k (1-x)^{n-k}
\end{aligned}$$

$$\sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k (1-x)^{n-k} \leq \frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2.$$

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 = x^2 - \frac{2x}{n} \sum_{k=0}^{n} C_n^k k x^k (1-x)^{n-k} + \frac{1}{n^2} \sum_{k=0}^{n} C_n^k k^2 x^k (1-x)^{n-k}.$$

Since $\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1$, then by derivative with respect to $x$ and if we set $h(x) = \sum_{k=0}^{n} C_n^k k x^k (1-x)^{n-k}$, we have: $h(x) = nx$. We iterate this process, we find:

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 = \frac{x(1-x)}{n}.$$

Then

$$\frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \leq \frac{1}{4n\alpha^2}.$$

The sequence $(B_n)_n$ converge uniformly to $f$ on $[0,1]$.
We give another proof of this theorem in the chapter of Fourier series. We give now another proof.

**Theorem 3.4**

Weierstrass Theorem
Let $f$ be a continuous function on an interval $I$, there exist a sequence $(f_n)_n$ of polynomials which converges uniformly on any interval compact of $I$ to $f$.

**Proof** .
Assume in the first case that $f$ is continuous on $\mathbb{R}$ and equal to 0 on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. We set

$$P_n(x) = c_n(1-x^2)^n,$$

with $c_n$ a constant such that $\displaystyle\int_{-1}^{1} P_n(x)dx = 1$. We define the sequence

$$f_n(x) = \int_{-\infty}^{+\infty} f(y)P_n(x-y)dy = \int_{-\infty}^{+\infty} f(x-y)P_n(y)dy. \tag{3.3}$$

**Lemma 3.5**

The functions $f_n$ are polynomials and the sequence $(f_n)_n$ converges uniformly to $f$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

**Proof** .
By the left side of (3.3), $f$ is a polynomial and by the right side of (3.3) we have for $|x| \leq \frac{1}{2}$:

$$f(x) - f_n(x) = \int_{-1}^{1} (f(x) - f(x-y))P_n(y)dy. \tag{3.4}$$

Let $\varepsilon > 0$, $M$ the maximum of $f$ on $\mathbb{R}$ and $\delta > 0$ such that $|f(x) - f(x-y)| < \varepsilon$ if $|y| < \delta$. It results from the formula (3.4) that

$$|f(x) - f_n(x)| \leq \int_{|y|<\delta} \varepsilon P_n(y)dy + \int_{\delta \leq |y| \leq 1} M \ P_n(y)dy.$$

We have to prove now that $\displaystyle\int_{\delta \leq |y| \leq 1} P_n(y)dy$ tends to 0 when $n$ tends to infinity.
Let $0 < r < 1$.

$$\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n dx \geq \int_{-r}^{r} (1-r^2)^n dx = 2r(1-r^2)^n.$$

Then $c_n \leq \dfrac{1}{2r(1-r^2)^n}$. Thus

$$\int_{\delta\leq|y|\leq 1} P_n(y)dy \leq \frac{1}{2r(1-r^2)^n}\int_{-1}^{1}(1-\delta^2)^n dy = \frac{(1-\delta^2)^n}{r(1-r^2)^n}.$$

The result is deduced if we take $r < \delta$ and we tends $n$ to infinity.

**Proof of theorem** (3) .

If $f$ is zeros on the complement of the interval $[-s, s]$, the function $F(x) = f(2sx)$ is zeros on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. By the previous lemma there exist a sequence $(f_n)_n$ of polynomials which converges uniformly to $F$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The sequence of polynomials $g_n(x) = f_n(\frac{x}{2s})$ converges uniformly to $f$ on the interval $[-s, s]$.

If now $f$ is continuous on the interval $I = (a, b)$. For all $n \in \mathbb{N}$ such that $n > \dfrac{2}{b-a}$, there exists a function $\varphi_n$ continuous on $I$ such that $\varphi_n = 1$ on $[a+\frac{1}{n}, b-\frac{1}{n}]$ and zeros on the complement of the interval $[a+\frac{1}{2n}, b-\frac{1}{2n}]$. There exists a polynomial $f_n$ such that $|f_n(x) - \varphi_n(x)f(x)| < \dfrac{1}{n}$ on $I$. The sequence $(f_n)_n$ is a solution to the problem.

□

**Corollary 3.6**

If $f$ is a continuous function on the interval $[a, b]$ such that $\displaystyle\int_a^b f(x)x^n dx = 0$, for all $n \in \mathbb{N}$, then $f = 0$.

**Proof** .

It results that for all polynomial $P$, $\displaystyle\int_a^b f(x)P(x)dx = 0$. Since $f$ is a uniform limit of sequence of polynomial $(P_n)_n$, then

$$\int_a^b f^2(x)dx = \lim_{n\to+\infty}\int_a^b f(x)P_n(x)dx = 0.$$

□

**Remark 17 :**

The previous result is wrong for the continuous functions on an unbounded interval. For example, let $f$ be the function defined by: $f(x) = e^{-x^{\frac{1}{4}}}\sin(x^{\frac{1}{4}})$, for $x \in [0, +\infty[$. Prove that $\displaystyle\int_0^{+\infty} x^n f(x)dx = 0$, for all $n \in \mathbb{N}$.

**Corollary 3.7**

Let $f\colon [a,b] \longrightarrow \mathbb{C}$ be a continuous function. There exist a sequence $(Q_n)_n \in \mathbb{R}[X]$ such that $(Q_n)_n$ converges uniformly to $f$ on $[a,b]$.

# CHAPTER V

# POWER SERIES

## 1 Power Series

### 1.1 Abel's Lemma

**Definition 1.1**

Let $(a_n)_n$ be a sequence of real or complex numbers. The series $\sum_{n\geq 0} a_n(x-x_0)^n$ is called a power series centered at $x_0$.

Let $\sum_{n\geq 0} a_n(x-x_0)^n$ be a power series, we look for its domain of convergence. The series converges at least for $x = x_0$. In which follows, we consider the series centered at 0.

**Proposition 1.2**

(Abel's lemma)

If the power series $\sum_{n\geq 0} a_n x_0^n$ is convergent for $x_0 \neq 0$, then

1. the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent on the interval $]-|x_0|, |x_0|[$,

2. for every $r < |x_0|$, the power series $\sum_{n\geq 0} a_n x^n$ is uniformly convergent on $[-r, r]$.

**Proof** .

1. Let $x \in ]-|x_0|, |x_0|[$, $\sum_{n=0}^{+\infty} |a_n x^n| \leq \sum_{n=0}^{+\infty} |a_n x_0^n| |\frac{x}{x_0}|^n$. Since the series $\sum_{n\geq 0} a_n x_0^n$ is convergent, the sequence $(a_n x_0^n)_n$ is bounded. Moreover the series $\sum_{n\geq 0} |\frac{x}{x_0}|^n$ is convergent, then the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent on $]-|x_0|, |x_0|[$.

2. Let $r < |x_0|$ and $x \in [-r, r]$, $|a_n x^n| \leq |a_n| r^n$ and $\sum_{n=0}^{+\infty} |a_n| r^n < +\infty$, thus the series $\sum_{n\geq 0} a_n x^n$ is uniformly convergent on $[-r, r]$.

□

**Corollary 1.3**

If the power series $\sum_{n\geq 0} a_n x_0^n$ is divergent then it is divergent for every $x$ such that $|x| > |x_0|$.

## 1.2 Radius of Convergence of Power Series

**Theorem 1.4**

For every power series $\sum_{n\geq 0} a_n x^n$, there exists a unique $R \in [0, +\infty]$ such that:

1. For every $|x| < R$, the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent.

2. For every $|x| > R$, the sequence $(a_n x^n)_n$ is not bounded and then

the series $\sum_{n\geq 0} a_n x^n$ is divergent.

The number $R$ is called the radius of convergence of the power series and $]-R, R[= \{x \in \mathbb{R};\ |x| < R\}$ is called the open interval of convergence of the power series.

**Proof** .
The uniqueness results from Abel's lemma. We set

$$R = \sup\{r \geq 1;\ \sum_{n=0}^{+\infty} |a_n| r^n < +\infty\}.$$

If $|x| < R$, the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent.
If there exists $|x| > R$ such that the series $\sum_{n\geq 0} |a_n| r^n$ is convergent. Then the series $\sum_{n\geq 0} |a_n| r^n$ is convergent for every $R < r < |x|$ which is absurd. □

**Remark 18 :**
From the proof of the theorem (1.2), we deduce that if $R$ is the radius of convergence of the series $\sum_{n\geq 0} a_n x^n$, then the series is uniformly convergent on any interval $[-r, r]$ with $0 < r < R$.

**Theorem 1.5**

(Cauchy 1821, used by Hadamard) (Cauchy-Hadamard Rule) Let $\sum_{n\geq 0} a_n x^n$ be a power series with $R$ its radius of convergence. Then

1. $R = \sup\{r \geq 0;\ \sum_{n=0}^{+\infty} |a_n| r^n < +\infty\} = \sup\{r \geq 0;$ the sequence $(a_n r^n)_n$is bounded $\}$.
2. If $\lim_{n\to+\infty} |\frac{a_n}{a_{n+1}}| = \beta \in [0, +\infty]$, then $R = \beta$.
3. $R = \frac{1}{\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|}}$. (With $R = +\infty$ if $\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|} = 0$ and $R = 0$ if $\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|} = +\infty$.)

**Theorem 1.6**

Let $\sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence $R > 0$. Define $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. Then the power series $\sum_{n\geq 1} n a_n x^{n-1}$ has $R$ as radius of convergence and the function $f$ is differentiable on $]-R, R[$ and $f'(x) = g(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.

For the proof, we need the following lemma:

**Lemma 1.7**

Let $x \in \mathbb{R}$ and $h \in \mathbb{R}$ such that $0 < |h| \leq r$, then for any $n \in \mathbb{N}$

$$|(x+h)^n - x^n - nhx^{n-1}| \leq \frac{|h|^2}{r^2}(|x|+r)^n \tag{1.1}$$

and

$$n|x|^{n-1} \leq \frac{1}{r}\big(2(|x|+r)^n + |x|^n\big). \tag{1.2}$$

**Proof** .
From the inequality (3.4)

$$\begin{aligned}
\left|(x+h)^n - x^n - nhx^{n-1}\right| &= \left|\sum_{k=0}^{n} C_n^k h^k x^{n-k} - x^n - nhx^{n-1}\right| = \left|\sum_{k=2}^{n} C_n^k h^k x^{n-k}\right| \\
&\leq |h|^2 \sum_{k=2}^{n} C_n^k |x|^{n-k} |h|^{k-2} \leq \frac{|h|^2}{r^2} \sum_{k=2}^{n} C_n^k |x|^{n-k} r^k \\
&\leq \frac{|h|^2}{r^2}(|x|+r)^n.
\end{aligned}$$

We have: $|(x+h)^n - x^n - nhx^{n-1}| \geq nr|x|^{n-1} - |x|^n - (|x|+r)^n$. From the relation (3.4), we deduce:

$$nr|x|^{n-1} \leq |x|^n + (|x|+r)^n + |(x+r)^n - x^n - nrx^{n-1}| \leq |x|^n + 2(|x|+r)^n.$$

$\square$

**Proof of the theorem** (1.2) .
We denote $R'$ the radius of convergence of the power series $\sum_{n\geq 1} na_n x^{n-1}$. It is obvious that $R' \leq R$. Let $r>0$ such that $|x|+r<R$. From the lemma (1.2); we have: $|na_n x^{n-1}| \leq \frac{1}{r}\big(2|a_n|(|x|+r)^n + |a_n||x|^n\big)$ and thus $\sum_{n\geq 1} na_n x^{n-1}$ is absolutely convergent on $]-R,R[$. Thus the radius of convergence of the series defining $g$ is greater than $R$. Thus $R=R'$.
From the inequality (3.4) we have:

$$|\frac{f(x+h)-f(x)}{h} - g(x)| \leq \frac{|h|}{r}\sum_{n=1}^{+\infty}|a_n|(|x|+r)^n.$$

This proves that when $h$ tends to 0; $f'(x)=g(x)$; for any $x \in ]-R,R[$. □

**Corollary 1.8**

If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then $f$ is infinitely continuously differentiable on $]-R,R[$ if $R>0$, $a_n = \frac{f^{(n)}(0)}{n!}$ and $f(x) = \sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n$. (This series is called the Taylor's series of $f$ at 0 or the Mac-Laurent series of $F$.)

**Example 19 :**

1. For $x \in \mathbb{R}$,

$$e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!} \qquad e^{-x} = \sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{n!},$$

$$\cosh x = \sum_{n=0}^{+\infty}\frac{x^{2n}}{(2n)!} \qquad \sinh x = \sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

$$\cos x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \qquad \sin x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

2. For $|x|<1$,

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty}x^n, \qquad \ln(1+x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{(n+1)},$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \text{ and } \tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)},$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)}.$$

3. Let $\alpha$ be a real number, $\alpha \notin \mathbb{N}$ and $f(x) = (1+x)^\alpha$ for $x \in ]-1,1[$. $f'(x) = \alpha(1+x)^{\alpha-1}$, then $f$ satisfies the following differential equation

$$(1+x)y' - \alpha y = 0. \tag{1.3}$$

We look for a power series $\sum_{n \geq 0} a_n x^n$ solution of the differential equation (1.3).

If $S = \sum_{n=0}^{+\infty} a_n x^n$ is a solution, we have:

$$(1+x) \sum_{n=0}^{+\infty} n a_n x^{n-1} - \alpha \sum_{n=0}^{+\infty} a_n x^n = 0,$$

then $(n+1)a_{n+1} + n a_n - \alpha a_n = 0 \iff a_{n+1} = \frac{\alpha - n}{n+1} a_n \ \forall n \geq 0$, which yields that

$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)} a_0.$$

Then

$$S(x) = a_0 (1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n).$$

By the uniqueness of the solution of the differential equation

$$(1-x)^\alpha = \sum_{n=0}^{+\infty} a_n x^n, \qquad \text{for } |x| < 1,$$

where $a_n = \dfrac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)}$.

For $\alpha = \frac{-1}{2}$, we have:

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^n, \qquad \sqrt{1+x} = 1 + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{n+1}}{n+1}.$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^{2n}, \qquad \sin^{-1} x = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}, \qquad \sinh^{-1} x = \sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

## 1.3 Exercises

**5-1-1** Find the sums of the following series and compute their radius of convergence:

1) $\sum_{n=0}^{+\infty} \frac{x^n}{2n-1}$,

2) $\sum_{n=1}^{+\infty} n^2 x^n$,

3) $\sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n$,

4) $\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)(n+3)}$,

5) $\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{4n^2-1}$,

6) $\sum_{n=1}^{+\infty} \frac{x^n}{n} \cosh(na), a > 0$

7) $\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n}$,

8) $\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n 2^n}$,

9) $\sum_{n=1}^{+\infty} \frac{n x^n \sin^2(n\theta)}{2^n}$,

10) $\sum_{n=0}^{+\infty} \frac{n^2+1}{n+1} x^n$,

11) $\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!}$,

12) $\sum_{n=0}^{+\infty} \frac{\sin^2(n\theta)}{n!} x^{2n}$,

13) $\sum_{n\geq 0} (2n+1) x^n$,

14) $\sum_{n=0}^{+\infty} \frac{x^{3n}}{(3n)!}$,

15) $\sum_{n=0}^{+\infty} (n^2+1) \frac{x^n}{n!}$,

16) $\sum_{n=0}^{+\infty} \frac{n x^n}{3^n (n+1)}$,

17) $\sum_{n=0}^{+\infty} (-1)^n \frac{(n^2+1) x^n}{n!}$,

18) $\sum_{n=0}^{+\infty} \frac{n x^n}{3^n (n+1)}$,

19) $\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{3n+1}$.

**5-1-2** (a) Define the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ by:
$\begin{cases} u_0 = 1 \\ v_0 = 0 \end{cases}$ and $\begin{cases} u_{n+1} = u_n + 2v_n \\ v_{n+1} = u_n + v_n. \end{cases}$
Determine the radius of convergence and the sum of the power series $\sum_{n\geq 0} u_n x^n$.

(b) Determine the radius of convergence of the power series:

$$\sum_{n\geq 0} a_n x^n; \text{ with } a_{2n} = 0 \text{ and } a_{2n+1} = \frac{(-1)^n}{(2n-1)(2n+1)}.$$

Let $f(x) = \sum_{n=1}^{+\infty} a_n x^n$, give a simple expression of the derivative $f'(x)$ in term of $x$ and $\tan^{-1} x$.
Deduce $f(x)$.

**5-1-3** Say if the following affirmations are true or false.

(a) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} (-1)^n a_n x^n$ have the same radius of convergence.

(b) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} |a_n| x^n$ have the same radius of convergence.

(c) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} (-1)^n a_n x^n$ have the same domain of convergence.

(d) If the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is infinite, then the series is uniformly convergent on $\mathbb{R}$.

(e) If the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is infinite and if $a_n$ are positives, then for any integer $p$, $\lim_{x\to+\infty} \frac{f(x)}{x^p} = +\infty$, with $f(x) = \sum_{n=0}^{+\infty} a_n x^n$.

**5-1-4** Give the expansion in power series in a neighborhood of 0 of the following functions

(a) $x \longmapsto \dfrac{\ln(1+x)}{1+x}$.

(b) $f(x) = (\sin^{-1} x)^2$. (We will be able to show that $f$ fulfills a differential equation of order 2.)

(c) $\dfrac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}$.

(d) $\ln(1 - 2x\cos\alpha + x^2)$.

(e) $e^{2x}\cos x$.

**5-1-5** Give the expansion in power series of the function $f(x) = \dfrac{x}{1-x-x^2}$.

**5-1-6** Give the expansion in power series of the following functions in a neighborhood of 0 and determine the corresponding radius of convergence:

1) $\dfrac{1}{(1-x)^2}$,

2) $\dfrac{1}{(x-2)(x-3)}$,

3) $\ln(1+x+x^2)$

4) $\sin^3 x$,

5) $\sinh^3 x$,

6) $(x-1)\ln(x^2-5x+6)$,

7) $x\ln(x+\sqrt{x^2+1})$,

8) $\dfrac{x-2}{x^3-x^2-x+1}$,

9) $\dfrac{1}{1+x-2x^3}$,

10) $\dfrac{1-x}{(1+2x-x^2)^2}$,

11) $\tan^{-1}(x+1)$,

12) $\tan^{-1}(x+\sqrt{3})$,

13) $\displaystyle\int_0^x \frac{\ln(t^2-\frac{5}{2}t+1)}{t}\,dt$,

14) $\left(\dfrac{(1+x)\sin x}{x}\right)^2$,

15) $\displaystyle\int_x^{2x} e^{-t^2}\,dt$,

16) $\displaystyle e^{-2x^2}\int_0^x e^{2t^2}\,dt$,

17) $\dfrac{e^x}{1-x}$,

18) $\dfrac{e^{x^2}}{1-x}$,

19) $\displaystyle\int_0^x \frac{\cos t-1}{t^2}dt$,

20) $\ln\left(\dfrac{1+x}{2-x}\right)$

21) $\ln\sqrt{1-2x\cosh a+x^2}$,

**5-1-7** Define $f(x)=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$.

(a) Prove that $f$ has an expansion in power series in a neighborhood of 0 and precise the radius of convergence.

(b) Prove that $f$ fulfills a differential equation.
Deduce the coefficients of the expansion in power series of $f$.

(c) Give the expansion in power series of $(\sin^{-1})^2(x)$.

**5-1-8** Give the expansion in power series the following functions at the corresponding point $x_0$.

(a) $f(x)=\cos x$, $(x_0=\dfrac{\pi}{4})$,

(b) $f(x)=(1-x^3)^{-\frac{1}{2}}$, $(x_0=0)$,

**5-1-9** Assume that the power series $\sum\limits_{n\geq 0} a_{2n}x^n$ and $\sum\limits_{n\geq 0} a_{2n+1}x^n$ have radius of convergence $R$ and $R'$ respectively.

Determine the radius of convergence of the power series $\sum\limits_{n\geq 0} a_n x^n$.

5-1-10 Let $(a_n)_n$ be a decreasing sequence and $\lim\limits_{n\to+\infty} a_n = 0$ and the series $\sum\limits_{n\geq 0} a_n$ diverges.

(a) Prove that the radius of convergence of the power series $\sum\limits_{n\geq 0} a_n x^n$ is 1.

(b) Study the convergence for $|x| = 1$.

5-1-11 (a) Let $(a_n)_n$ be a sequence of real numbers such that the series $\sum\limits_{n\geq 0} a_n$ is convergent.

We claim to prove that the power series $\sum\limits_{n\geq 0} a_n x^n$ is uniformly convergent on $[0,1]$.

Define $R_n = \sum\limits_{k=n+1}^{+\infty} a_k$ and $S_n = \sum\limits_{k=0}^{n} a_k x^k$.

i. Prove that for $p > n$; $S_p(x) - S_n(x) = R_n x^{n+1} - R_p x^p + \sum\limits_{k=n+1}^{p-1} (x^{k+1} - x^k) R_k$.

ii. Deduce that the series $\sum\limits_{n\geq 0} a_n x^n$ fulfills the Cauchy criterion for the uniform convergence on $[0,1]$.

(b) Let $\sum\limits_{n\geq 0} b_n x^n$ be a power series of radius of convergence $R$ and let $f(x)$ its sum. Let $x_0 \in \mathbb{R}$ such that $|x_0| = R \neq 0$. Assume that the series $\sum\limits_{n\geq 0} b_n x_0^n$ is convergent.

i. Prove that $\lim\limits_{\substack{x\longmapsto x_0\\ x\in[0,x_0]}} f(x) = \sum\limits_{n=0}^{+\infty} b_n x_0^n$ $([0, x_0] = \{t x_0,\ t \in [0,1]\})$.

ii. Deduce the value of the following sum $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n}$.

5-1-12 For each of the following power series, determine the interval of convergence of this series and prove that its sum is a solution of the suitable differential equation.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}, \quad y^{(4)} = y$$

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2},\quad xy^{''}+y'-y=0$$

$$f(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!},\quad y^{''}+4y=0$$

**5-1-13** (a) Prove that there exists a solution as power series of the following differential equation

$$x(x-1)y^{''}+3xy'+y=0.$$

(b) Determine the radius of convergence of the obtained series.

**5-1-14** For any $\lambda\in\mathbb{R}$, consider the following differential equation

$$y^{''}(x)-2xy'(x)+2\lambda y(x)=0 \tag{1.4}$$

(a) Prove that the equation (1.6) has a unique even solution $P_\lambda$ as a power series on $\mathbb{R}$ and fulfills $P_\lambda(0)=1$.

(b) Prove that (1.6) has a unique odd solution $Q_\lambda$ as a power series on $\mathbb{R}$ and fulfills $Q'_\lambda(0)=1$.

(c) Determine all the values of $\lambda$ for that the equation (1.6) has a non vanishing polynomial solution.

**5-1-15** (a) Find the solutions as power series of the following differential equations:

i. $y'-2xy=0; y(0)=1$

ii. $y^{''}+xy'+y=0$

iii. $4xy^{''}+2y'-y=0,\ x>0$

(b) Give the expansion in power series the function $f(x)=e^{\frac{-x^2}{2}}\displaystyle\int_0^x e^{\frac{t^2}{2}}\,dt$.

**5-1-16** Define $u_n(x)=(-1)^n\dfrac{x^n}{n(n-1)}$, for $n\geq 2$.

(a) Determine the interval of convergence of the series $\displaystyle\sum_{n\geq 2}(-1)^n\frac{x^n}{n(n-1)}$ and study this series to the endpoints of this interval.

(b) Study the series $\displaystyle\sum_{n\geq 2}u'_n(x)$ and the series $\displaystyle\sum_{n\geq 2}u^{''}_n(x)$.

(c) Deduce the sum of the series $\displaystyle\sum_{n\geq 2}u_n(x)$.

**5-1-17** (a) Consider the sequence $(a_n)$ defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = 0$.

i. Compute $F(x) = \sum_{n=0}^{+\infty} a_n x^n$.

ii. Deduce the expression of $a_n$.

(b) Consider the sequence $(a_n)$ defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = n$.
Compute the expression of $a_n$.

(c) Consider the sequence $(a_n)_n$ defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 8a_{n+1} + 16a_n = 0$.
Find the expression of $a_n$.

**5-1-18** Let $(a_n)_n \in \mathbb{R}^*$ be a convergent sequence of real numbers and let $a = \lim_{n\to+\infty} a_n$.

(a) Find the radius of convergence of the power series $\sum_{n\geq 0} \frac{a_n x^n}{n!}$.

Define $f(t) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} t^n$, for $t \in \mathbb{R}$.

(b) Compute $\lim_{t\longmapsto+\infty} e^{-t} f(t)$.

**5-1-19** Prove that the equation $3xy' + (2 - 5x)y = x$ has a solution as a power series in a neighborhood of 0 and give its radius of convergence.

**5-1-20** Consider the following differential equation

$$x^2 y'' + xy' - (x^2 + x + 1)y = 0. \tag{1.5}$$

(a) Find a solution of the equation (1.1) $\varphi(x) = \sum_{n=0}^{+\infty} a_n x^n$ with $a_1 = 1$.

(b) Prove that, for $n \geq 1$, $|a_n| \leq \frac{1}{(n-1)!}$ and deduce the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$.

(c) Solve the equation (1.1) in putting $y = \frac{e^{-x}}{x} z$.

**5-1-21** We claim to prove that the following differential equation

$$x^2 y'(x) = y(x) - x^2 \tag{1.6}$$

has no solution as sum of a power series.

Assume that this equation has a solution $y = \sum_{n=0}^{+\infty} a_n x^n$.

(a) Give the values of $a_0, a_1$ and $a_2$?

(b) Give the relation between $a_{n+1}$ and $a_n$ for $n \geq 2$.

(c) Prove that the relations stated in 1) and 2) give the uniqueness of the power series $\sum_{n \geq 0} a_n x^n$. Compute its coefficients and prove that it diverges.

# CHAPTER VI

## FOURIER SERIES

In this chapter, we consider the locally Riemann integrable functions. The reader can always take the piecewise continuous functions.
The aim of this chapter is the study the expansion of function (in physics we said a signal) of one real variable then of the synthase or reconstitution of this function has from of the its composite elements.

# 1 Fourier Series Expansion

## 1.1 Preliminary

1. Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a locally Riemann-integrable function and $T-$periodic with $T > 0$, then

$$\int_a^{a+T} f(t)dt = \int_0^T f(t)dt \quad \forall a \in \mathbb{R}.$$

Indeed, $\displaystyle\int_a^{a+T} f(t)dt = \int_a^0 f(t)dt + \int_0^T f(t)dt + \int_T^{a+T} f(t)dt$. Taking the change of variable $u = t - T$ in the last integral, we get the result. This means that the integral of a $T-$periodic function on an interval of length $T$ does not depends of the chosen interval.

2. For $n, m \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{int}\, dt = \begin{cases} 0 & \text{if} \quad n \neq 0 \\ 1 & \text{if} \quad n = 0 \end{cases}\text{'}$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin(mt)\cos(nt)\ dt = 0,$$

$$\frac{1}{\pi}\int_0^{2\pi} \cos(mt)\cos(nt)\ dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \neq 0 \end{cases}\text{'}$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin(mt)\sin(nt)\ dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \neq 0 \\ 0 & \text{if} \quad n = m = 0 \end{cases}$$

**Definition 1.1**

We consider the space $\mathscr{E}$ of continuous functions $2\pi$-periodic defined on $\mathbb{R}$ with complex values. The map defined on $E \times E$ by:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{a-\pi}^{a+\pi} f(t)\overline{g(t)}dt$$

is a inner product. It defines a norm called the Euclidean norm denoted by $\| \ \|_2$.

**Remark 19 :**
The system $\{1, \cos(nt), \sin(nt), \quad n \in \mathbb{N}\}$ is an orthogonal system. Also the system $\{e^{int}, \quad n \in \mathbb{Z}\}$ is orthogonal.

## 1.2 Bessel Inequality

**Definition 1.2**

1. A trigonometric polynomial of degree $\leq N$ is a complex linear combination of $\{1, \cos(kx), \sin(kx), \quad 1 \leq k \leq N\}$, i.e. a trigonometric polynomial $P$ of degree $\leq N$ has the form

$$P(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)), \tag{1.1}$$

with $a_n, b_n \in \mathbb{C}$. In particular a trigonometric polynomial is a function of class $C^\infty$ and $2\pi$-periodic.

2. A trigonometric series is a series of functions in the form

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos(nx) + b_n \sin(nx)),$$

with $a_n$ and $b_n \in \mathbb{C}$.

**Remark 20 :**

Let $P(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx))$ a trigonometric polynomial of degree $\leq N$, then

$$P(x) = \frac{a_0}{2} + \sum_{n=1}^{N} e^{inx}(\frac{a_n}{2} - \mathrm{i}\frac{b_n}{2}) + \sum_{n=1}^{N} e^{-inx}(\frac{a_n}{2} + \mathrm{i}\frac{b_n}{2}) = \sum_{n=-N}^{N} C_n e^{inx}, \quad (1.2)$$

with

$$C_n = (\frac{a_n}{2} - \mathrm{i}\frac{b_n}{2}), \quad C_{-n} = (\frac{a_n}{2} + \mathrm{i}\frac{b_n}{2})$$

for $n \geq 1$ and $C_0 = \frac{a_0}{2}$. This form is called the exponential form of $P$, and the form (1.1) is called trigonometric form of $P$.
If $P$ is a trigonometric polynomial of degree $\leq N$ in the form (1.1) or (1.1), then

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} P(t)e^{-int}\,dt, \ \forall n \in \mathbb{Z},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} P(t)\cos(nt)\,dt, \ \forall n \in \mathbb{N}\cup\{0\},$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} P(t)\sin(nt)\,dt, \ \forall n \in \mathbb{N}.$$

**Theorem 1.3**

Let $f\colon [0, 2\pi] \longrightarrow \mathbb{C}$ be a Riemann-integrable function. define

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt, \quad n \in \mathbb{Z}$$

$$S_N(x) = \sum_{n=-N}^{N} C_n e^{inx}, \qquad N \in \mathbb{N} \cup \{0\}.$$

Then :

1. For any trigonometric polynomial $P$ of degree $\leq N$,

$$\int_0^{2\pi} |f(t) - S_N(t)|^2 \, dt \leq \int_0^{2\pi} |f(t) - P(t)|^2 \, dt. \tag{1.3}$$

2. The series $\sum_{n\in\mathbb{Z}} |C_n|^2$ is convergent and

$$\sum_{n=-\infty}^{+\infty} |C_n|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt \quad \text{(Bessel Inequality)}. \tag{1.4}$$

The property (1.3) shows that $S_N$ realized the best approximation in quadratic mean of $f$ by a trigonometric polynomial of degree $\leq N$.

**Proof** .

1. Let $P(x) = \sum_{n=-N}^{N} d_n e^{inx}$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(t) - P(t)|^2 \, dt &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \, dt - \frac{1}{2\pi}\int_0^{2\pi} f(t)\bar{P}(t)dt \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} P(t)\bar{f}(t)dt + \frac{1}{2\pi}\int_0^{2\pi} |P(t)|^2 dt.\end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{P(t)}dt = \sum_{n=-N}^{N} \bar{d}_n \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int} \, dt = \sum_{n=-N}^{N} \bar{d}_n C_n.$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \bar{f}(t)P(t) \, dt = \sum_{n=-N}^{N} d_n \bar{C}_n, \qquad \frac{1}{2\pi}\int_0^{2\pi} |P(t)|^2 \, dt = \sum_{n=-N}^{N} |d_n|^2.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(t)-P(t)|^2\,dt=\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt-\sum_{n=-N}^{N}|C_n|^2+\sum_{n=-N}^{N}|d_n-C_n|^2.$$

If the polynomial $P$ is the polynomial $S_N$, we have:

$$\frac{1}{2\pi}\int_0^{2\pi}|f(t)-S_N(t)|^2\,dt=\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt-\sum_{n=-N}^{N}|C_n|^2,$$

this yields the result.

2. $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}|f(t)-P(t)|^2\,dt=\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt-\sum_{n=-N}^{N}|C_n|^2$, thus $\displaystyle\sum_{n=-N}^{N}|C_n|^2\leq\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt$ and we take the limit when $N\longrightarrow+\infty$.

□

**Corollary 1.4**

If $f\colon[0,2\pi]\longrightarrow\mathbb{C}$ is a Riemann-integrable function, then

$$\lim_{n\to+\infty}\int_0^{2\pi}f(t)\cos(nt)\,dt=0\quad\text{and}\quad\lim_{n\to+\infty}\int_0^{2\pi}f(t)\sin(nt)\,dt=0.$$

**Proof** .
As the series $\displaystyle\sum_{n\in\mathbb{Z}}|C_n|^2$ converges, then $\displaystyle\lim_{n\to\infty}|C_n|^2=0$. If we set $\displaystyle a_n=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos(nt)\,dt$ and $\displaystyle b_n=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin(nt)\,dt$, for $n\in\mathbb{N}$ we have: $a_n=C_n+C_{-n}$ and $b_n=\mathrm{i}(C_n-C_{-n})$, and we have the result.

□

**Theorem 1.5**

(Riemam-Lebesgue Lemma)
Let $f\colon[a,b]\longrightarrow\mathbb{C}$ be a Riemann-integrable function, then

$$\lim_{\lambda\to+\infty}\int_a^b f(t)\cos(\lambda t)\ dt = 0 \text{ and } \lim_{\lambda\to+\infty}\int_a^b f(t)\sin(\lambda t)\ dt = 0.$$

**Proof** .
As $\int_a^b f(t)\cos(\lambda t)\quad dt \;=\; \int_a^b \operatorname{Re} f(t)\cos(\lambda t)\quad dt + \int_a^b \operatorname{Im} f(t)\cos(\lambda t)\quad dt$, it suffices to prove the theorem for $f$ real .

• If $f = \chi_{[\alpha,\beta]}$ is the characteristic function of an interval $[\alpha,\beta]$, we have:

$$\int_a^b f(t)\cos(\lambda t)\quad dt = \int_\alpha^\beta \cos(\lambda t)\quad dt = \frac{\sin(\lambda\alpha)}{\lambda} - \frac{\sin(\lambda\beta)}{\lambda} \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

• If $f$ is a step function on $[a,b]$, there exists a partition $\sigma = \{x_0 = a < x_1 < \ldots < x_n = b\}$ of $[a,b]$ such that $f = c_j$ on $]x_j, x_{j+1}[$. In this case

$$\int_a^b f(t)\cos(\lambda t)dt = \sum_{j=0}^{n-1} c_j \int_{x_j}^{x_{j+1}} \cos(\lambda t)dt.$$

Thus

$$\Big|\int_a^b f(t)\cos(\lambda t)dt\Big| \le \frac{2}{\lambda}\sum_{j=0}^{n-1}|c_j| \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

In the general case: as $f$ is Riemann-integrable on $[a,b]$, for $\varepsilon > 0$, there exists a step function $f_\varepsilon$ such that $f_\varepsilon \le f$ and $\int_a^b (f(t) - f_\varepsilon(t))\ dt < \varepsilon$. Then

$$\int_a^b f(t)\cos(\lambda t)\ dt = \int_a^b (f(t) - f_\varepsilon(t))\cos(\lambda t)\ dt + \int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt.$$

We deduce that

$$\Big|\int_a^b f(t)\cos(\lambda t)\ dt\Big| \le \Big|\int_a^b (f(t) - f_\varepsilon(t))\ dt\Big| + \Big|\int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt\Big|.$$

As $f_\varepsilon$ is a step function, $\lim_{\lambda\to+\infty}|\int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt| = 0$ and the result is deduced. □

## 1.3 Fourier Series

1. Let $f$ be a complex $2\pi-$periodic function, Riemann-integrable on $[0, 2\pi]$. We set

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\mathrm{i}nt}\,dt, \quad n \in \mathbb{Z},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt, \quad n \in \mathbb{N}_0,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt, \quad n \in \mathbb{N}$$

The coefficients $(C_n)_n$ will be called the exponential Fourier coefficients of $f$ and $a_n$ and $b_n$ will be called the trigonometric Fourier coefficients of $f$. We recall that:

$$a_0 = 2C_0, \quad a_n = C_n + C_{-n}, \quad b_n = \mathrm{i}(C_n - C_{-n}), \qquad \forall n \geq 1.$$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}.$$

$$\lim_{n\to+\infty} S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n=-\infty}^{+\infty} C_n e^{\mathrm{i}nx}$$

The series $\displaystyle\sum_{n\in\mathbb{Z}} C_n e^{\mathrm{i}nx} = \frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos(nx) + b_n\sin(nx))$ will be called the Fourier series of $f$. We will denote formally $\tilde{f}(x)$ the sum of this series.

We say that the Fourier series of $f$ converges at $x_0 \in \mathbb{R}$ if the sequence $(S_N)_N$, $\displaystyle S_N(x) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}$ converges at $x_0$.

2. If $f$ is $T-$periodic, the function $g(x) = f(\frac{Tx}{2\pi})$ is $2\pi$-periodic on $\mathbb{R}$. Moreover the function $f$ is locally Riemann integrable on $\mathbb{R}$, we associate to $f$ the Fourier coefficients defined from the Fourier coefficients of $g$ by:

$$C_n = \frac{1}{T}\int_0^{T} f(t)e^{-\mathrm{i}n\frac{2\pi}{T}t}\,dt, \qquad \forall n \in \mathbb{Z},$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi}{T}nt\ dt, \qquad \forall n \in \mathbb{N}_0,$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi}{T}nt\ dt,\ n \in \mathbb{N}.$$

The exponential Fourier series of $f$ is

$$\sum_{n\in\mathbb{Z}} C_n e^{-\mathrm{i}n\frac{2\pi}{T}t}$$

and the trigonometric Fourier series is

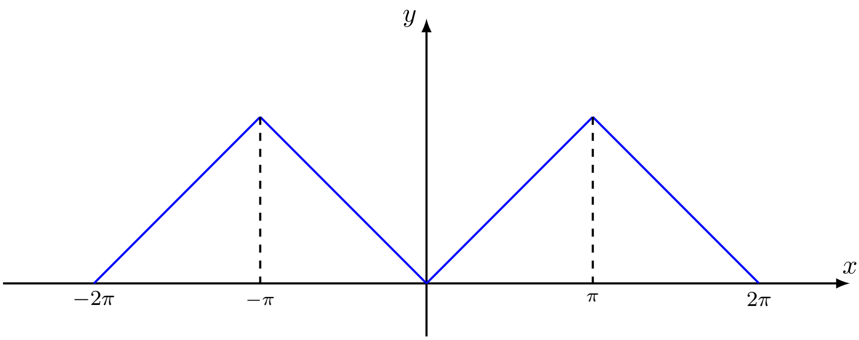
$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos\frac{2\pi}{T}nx + b_n\sin\frac{2\pi}{T}nx).$$

**Definition 1.6**

Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a $2\pi-$periodic function and Riemann-integrable on $[0, 2\pi]$. Develop $f$ in Fourier series, means that find Fourier trigonometric or exponential series of $f$, study the convergence of the series $\tilde{f}$ of $f$ and give its value.

**Examples 20 :**

1. $f(x) = |x|$ if $|x| \leq \pi$ and $f$ $2\pi-$periodic. The curve of $f$ on $[-2\pi, 2\pi]$ has the following form:




We have:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\ dt = \pi, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\cos(nt)\ dt = \frac{2}{n^2\pi}((-1)^n - 1), \quad n \geq 1.$$

As $f$ is even $b_n = 0$. The Fourier series of $f$ converges uniformly on $\mathbb{R}$.

2. Let $f(x) = \sin x$, for $x \in [0, \pi]$ even and $2\pi$-periodic. Thus $b_n = 0$ and $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx)\, dx$. $a_{2n+1} = 0$ and $a_{2n} = \frac{-4}{\pi(4n^2-1)}$. The Fourier series of $f$ converges uniformly on $\mathbb{R}$.

3. Let $\alpha \in \mathbb{C} \setminus (\mathrm{i}\mathbb{Z})$, $f(x) = e^{\alpha . x}$ on $]-\pi, \pi[$ and $2\pi-$periodic.

$$C_n = \frac{(-1)^n \sinh \alpha\pi}{\pi(\alpha - in)}, \qquad \tilde{f}(x) = \frac{\sinh \pi\alpha}{\pi}\sum_{-\infty}^{+\infty}(-1)^n \frac{e^{inx}}{\alpha - in}.$$

4. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power series ($z \in \mathbb{C}$) of radius of convergence $R > 0$. For $r \in [0, R[$, the map $\theta \overset{F}{\longmapsto} f(re^{\mathrm{i}\theta})$ is $2\pi-$periodic and we have:

$$f(re^{\mathrm{i}\theta}) = \sum_{n=0}^{+\infty}(a_n r^n)e^{in\theta} \tag{1.5}$$

and the trigonometric series converges uniformly on $\mathbb{R}$.

Thus $\frac{1}{2\pi}\int_0^{2\pi} f(re^{\mathrm{i}\theta})e^{-\mathrm{i}p\theta}d\theta = \sum_{n=0}^{+\infty}(a_n r^n)\int_0^{2\pi} e^{\mathrm{i}(n-p)\theta}d\theta.$

The series (1.5) is the Fourier series of $f$. Moreover $a_p = \frac{1}{2\pi}\int_0^{2\pi} f(re^{\mathrm{i}\theta})e^{-\mathrm{i}p\theta}d\theta$, then $|a_p| \leq \frac{M(r)}{r^p}$, with $M(r) = \sup_{|z|=r}|f(z)|$.

If we take the function $f(z) = \frac{1}{1-z}$, we know that for $|z| < 1$, $\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$. Thus for any $\theta \in \mathbb{R}$ and any $r \in [0, 1[$, $\frac{1}{1-re^{\mathrm{i}\theta}} = \sum_{n=0}^{+\infty} r^n e^{in\theta}$ and in taking the real part of each member we get:

$$\frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta} = \sum_{n=0}^{+\infty} r^n \cos(n\theta).$$

## 1.4 The Dirichlet Theorem

The natural question in Fourier analysis is: "In what condition the Fourier series of a function $f$ is convergent and the relation between the limit and the function $f$.

**Definition 1.7**

[Dirichlet Kernel]
The Dirichlet kernel of degree $N \in \mathbb{N}_0$ is the trigonometric polynomial $D_N$ defined by:

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

The function $D_N$ is even and $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} D_N(t)dt = 1$.

**Theorem 1.8**

(Dirichlet Theorem)
Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a $2\pi$-periodic function and Riemann-integrable on $[0, 2\pi]$. Let $x \in \mathbb{R}$ such that $f(x+) = \lim\limits_{t\to x,\ t>x} f(t)$ and $f(x-) = \lim\limits_{t\to x,\ t<x} f(t)$ exist in $\mathbb{C}$. We assume also that there exists $\delta_x > 0$ (depends of $x$) and $M_x \geq 0$ (depends of $x$) such that: $\forall t,\ 0 < |t| < \delta_x,$:

$$\frac{|f(x+t) + f(x-t) - f(x+) - f(x-)|}{|t|} \leq M_x \tag{1.6}$$

then the Fourier series of $f$ at $x$ converges to $\dfrac{f(x+) + f(x-)}{2}$, i.e.

$$\lim_{N\to+\infty} \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)) = \frac{f(x+) + f(x-)}{2}. \tag{1.7}$$

The condition (1.6) is called the Dirichlet condition at $x$.

**Proof** .
Let $C_n$ be the Fourier exponential coefficients of $f$, with $n \in \mathbb{Z}$.

$$\begin{aligned} S_N(x) &= \sum_{n=-N}^{N} C_n e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}e^{inx}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)D_N(t-x)\ dt. \\ &\overset{u=t-x}{=} \frac{1}{2\pi}\int_0^{2\pi} f(u+x)D_N(u)\ du. \end{aligned}$$

If we denote $y = \dfrac{f(x+)+f(x-)}{2}$ we have:

$$\begin{aligned} S_N(x) - y &= \frac{1}{2\pi}\int_0^{\pi}(f(x-u)+f(x+u)-2y)D_N(u)du \\ &= \frac{1}{2\pi}\int_0^{\pi}\frac{f(x-u)+f(x+u)-f(x+)-f(x-)}{u}\frac{u}{\sin\frac{u}{2}}\sin(\frac{(2N+1)u}{2})du. \end{aligned}$$

The function $\varphi$ defined on $]0,\pi[$ by:

$$\varphi(u) = \frac{f(x-u)+f(x+u)-f(x+)-f(x-)}{u}\frac{u}{\sin\frac{u}{2}}$$

is Riemann-integrable on $]0,\pi]$. Moreover $\forall\ u \in ]0,\delta_x[$, we have: $\varphi(u)| \leq M_x\pi$. $(\frac{2}{\pi} \leq \frac{\sin t}{t} \leq 1,\ \forall t \in [0,\frac{\pi}{2}])$ and by the Riemman-Lebesgue lemma (1.2), $\lim\limits_{N\to+\infty}\int_0^{\pi}\varphi(u)\sin(N+\frac{1}{2})u\ du = 0$. Thus $\lim\limits_{N\to+\infty} S_N(x) = \dfrac{f(x+)+f(x-)}{2}$.

□

**Theorem 1.9**

1. Let $x \in \mathbb{R}$ such that $f(x+)$, $f(x-)$, $f'(x+) = \lim\limits_{t\to0,\ t>0}\dfrac{f(x+t)-f(x+)}{t}$ and $f'(x-) = \lim\limits_{t\to0,\ t>0}\dfrac{f(x-t)-f(x-)}{t}$, exist in $\mathbb{C}$. Then the Dirichlet condition is realized at $x$ and the Fourier series of $f$ at $x$ converges to $\frac{f(x+)+f(x-)}{2}$.

2. If $f$ is also continuous at $x$, then the Fourier series of $f$ at $x$ converges to $f(x)$.

3. If $f$ is $2\pi-$periodic and of class piecewise continuously differentiable $[0,2\pi]$, then $\forall x \in \mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos(nx)+b_n\sin(nx)) = \sum_{n=-\infty}^{+\infty} C_n e^{inx}.$$

**Examples 21 :**

1. Let $f$ be the function defined by: $f(x) = |x|$ if $x \in [-\pi, \pi]$ and $2\pi-$periodic. $f$ is continuous at the left of $\pi$ and at the right of $-\pi$, by parity and periodicity, $f$ is continuous at $\pi$ and at $-\pi$. $f$ is continuously differentiable on $[-\pi, \pi]$, thus by Dirichlet theorem, the Fourier series of $f$ coincides with $f$ at any point $x \in \mathbb{R}$. Thus for $|x| \leq \pi$, we have:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{+\infty} \frac{\cos(2k+1)x}{(2k+1)^2}.$$

For $x = 0$, we have:

$$\frac{\pi^2}{8} = \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2}.$$

The Fourier series of $f$ converges uniformly to $f$ on $\mathbb{R}$.

2. Let $f$ be the function defined by: $f(x) = x$ on $]-\pi, \pi[$ and $2\pi-$periodic. (we associate an arbitrary value at $\pi$). $f$ is continuously differentiable on $]-\pi, \pi[$ and has a derivative at the left and at the right at any point on $\mathbb{R}$. By Dirichlet theorem, we have for any $x \in \mathbb{R} \setminus \{(2k+1)\pi, k \in \mathbb{Z}\}$,

$$f(x) = 2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

In particular for $x = \frac{\pi}{2}$

$$\frac{\pi}{4} = \sum_{p=0}^{+\infty} \frac{(-1)^p}{2p+1}.$$

## 1.5 The Parseval Theorem

**Definition 1.10**

(The Cesaro Summation)
Let $(U_n)_n$ be a sequence of complex numbers. We define the sequence $S_N = \sum_{k=0}^{N} U_k$. We say that the series $\sum_{n \geq 0} U_n$ is Cesaro summable if the sequence $T_N = \dfrac{S_0 + \ldots S_N}{N+1}$ converges in $\mathbb{C}$.

**Examples 22 :**

1. If $U_n = (-1)^n$, $S_{2p} = 1$ and $S_{2p+1} = 0$, $T_{2n} = \frac{n}{2n+1}$ and $T_{2n+1} = \frac{n}{2n+2}$, thus the series $\sum_{n\geq 0} U_n$ is Cesaro summable and has $\frac{1}{2}$ as sum, but the series $\sum_{n\geq 0} U_n$ diverges.
2. If the series $\sum_{n\geq 0} U_n$ converges to $\ell$, then it is Cesaro summable and has $\ell$ as sum.

**Definition 1.11**

[Fejer Kernel]

For $N \in \mathbb{N}_0$, we set $F_N(x) = \sum_{n=0}^{N} D_N(x)$, $x \in \mathbb{R}$, with $D_N$ the Dirichlet kernel. $F_N$ is a polynomial trigonometric called the Fejer kernel of degree $N$.
$F_N$ is even function and $\frac{1}{2\pi}\int_0^{2\pi} F_N(t)dt = N + 1$.

**Notations**
Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Riemann-integrable function on $[0, 2\pi]$ and $2\pi-$periodic. Let $(a_n)_n$ and $(b_n)_n$ its trigonometric Fourier coefficients. We define for all $N \in \mathbb{N}_0$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)),$$

and

$$\Lambda_N(f, x) = \frac{S_0(x) + \ldots S_N(x)}{N + 1},$$

then as in the proof of Dirichlet theorem, we have:

$$S_N(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+u)D_N(u)\, du = \frac{1}{2\pi}\int_0^{2\pi} f(x-u)D_N(u)\, du.$$

$$\Lambda_N(f, x) = \frac{1}{2\pi(N+1)}\int_0^{2\pi} f(x+u)F_N(u)\, du = \frac{1}{2\pi(N+1)}\int_0^{2\pi} f(x-u)F_N(u)\, du.$$

The real expression of $F_N$ is

$$F_N(x) = \frac{\sin^2 \frac{N+1}{2}x}{\sin^2 \frac{x}{2}}.$$

**Theorem 1.12**

Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a Riemann-integrable function on $[0, 2\pi]$ and $2\pi-$periodic.

1. Let $x \in \mathbb{R}$ such that $f(x+)$ and $f(x-)$ exist, then
$$\lim_{N\to+\infty} \Lambda_N(f, x) = \frac{f(x+) + f(x-)}{2}.$$
2. The sequence $(\Lambda_N)_N$ converges uniformly on any compact $K$ on which $f$ is continuous.

**Proof** .

1. We know that $\Lambda_N(f, x) = \dfrac{1}{2\pi(N+1)} \displaystyle\int_0^{2\pi} f(x+u)F_N(u)\, du$.

   Let $y$ be a constant, as $\dfrac{1}{2\pi(N+1)} \displaystyle\int_0^{2\pi} F_N(u)\, du = 1$, we have:
$$\begin{aligned}
\Lambda_N(f, x) - y &= \frac{1}{2\pi(N+1)} \int_0^{2\pi} (f(x+u) - y)F_N(u)\, du \\
&= \frac{1}{2\pi(N+1)} \int_0^{\pi} (f(x+u) + f(x-u) - 2y)F_N(u)\, du.
\end{aligned}$$

   We take $y = \frac{f(x+)+f(x-)}{2}$. Let $\varepsilon > 0$, $\exists \delta_x > 0$ such that $\forall u \in ]0, \delta_x[$, $|f(x+u) - f(x+)| < \dfrac{\varepsilon}{2}$ and $|f(x-u) - f(x-)| < \dfrac{\varepsilon}{2}$. There it results that
$$\begin{aligned}
|\Lambda_N(f, x) - y| &\leq (\frac{\varepsilon}{2} + \frac{\varepsilon}{2})\frac{1}{2\pi(N+1)} \int_0^{\delta_x} F_N(u)\, du \\
&+ \frac{1}{2\pi(N+1)} \int_{\delta_x}^{\pi} |f(x+u) + f(x-u) - 2y| F_N(u)\, du \\
&\leq \varepsilon + \frac{1}{2\pi(N+1)\sin^2 \delta_x/2} \int_0^{\pi} |f(x+u) + f(x-u) - 2y|\, du.
\end{aligned}$$

$f$ is bounded on $\mathbb{R}$, then there exists $N_0 \in \mathbb{N}_0$ such that for any $N \geq N_0$

$$\frac{1}{2\pi(N+1)\sin^2 \delta_x/2} \int_0^\pi |f(x+u)+f(x-u)-2y|\, du \leq \varepsilon.$$

2. We take $\delta > 0$ which does not depends on $x \in K$. (This is possibly, because $f$ is uniformly continuous on $K$.)

□

**Corollary 1.13**

Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function and $2\pi-$periodic. If the sequence $(S_N)_N$ converges, then its limit is $f$.

**Proof** .
Let $g = \lim\limits_{N\to+\infty} S_N$. The sequence $(\Lambda_N)_N$ converges uniformly to $f$, then $g = f$.
□

**Corollary 1.14**

Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function and $2\pi-$periodic, then $\forall \varepsilon > 0$, there exists a trigonometric polynomial $P_\varepsilon$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - P_\varepsilon(x)| < \varepsilon.$$

Otherwise a continuous function $2\pi-$periodic is limit uniform of trigonometric polynomials.

## 1.6 The Parseval Identity

Let $f$ be a $2\pi-$periodic function, Riemann-integrable on $[0, 2\pi]$. If

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt, \quad \text{for } n \in \mathbb{Z}.$$

For $N \in \mathbb{N}_0$,we posed

$$S_N(x) = \sum_{n=-N}^{N} C_n e^{inx}, \quad \text{and} \quad \Lambda_N(f,x) = \frac{S_0 + \ldots + S_N(x)}{N+1} = \sum_{k=-N}^{N} \gamma_k e^{ikx}.$$

$\gamma_0 = C_0$, $\gamma_1 = \frac{N}{N+1}C_1$, $\gamma_{-1} = \frac{N}{N+1}C_{-1}$, $\gamma_p = \frac{N-p+1}{N+1}C_p$ and $\gamma_{-p} = \frac{N-p+1}{N+1}C_{-p}$, $\forall\, p \geq 2$. Then

$$\Lambda_N(f,x) = \sum_{k=-N}^{N}(1 - \frac{|k|}{N+1})C_k e^{\mathrm{i}kx}.$$

**Theorem 1.15**

(Parseval Identity)
Let $f$ be a $2\pi-$periodic function and piecewise continuous on $[0, 2\pi]$, then:

$$\sum_{-\infty}^{+\infty}|C_n|^2 = \frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\, dt = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{+\infty}(|a_n|^2 + |b_n|^2).$$

**Lemma 1.16**

With the same notations

$$\lim_{N\to+\infty}\int_0^{2\pi}|\Lambda_N(f,x)|^2\, dx = \int_0^{2\pi}|f(t)|^2\, dt.$$

**Proof** .
Let $x_0 = 0 < x_1 < \ldots < x_s = 2\pi$ such that $f$ is continuous on $]x_i, x_{\mathrm{i}+1}[$ $\forall i \in \{0, \ldots, s-1\}$, thus $(\Lambda_N(f))_N$ converges uniformly to $f$ on $I_\eta = [x_i+\eta, x_{\mathrm{i}+1}-\eta]$, for any $\eta > 0$, $\eta \in ]0, (x_{\mathrm{i}+1} - x_i)/2[$.
For any $x \in \mathbb{R}$, $|\Lambda_N(f,x)| \leq \dfrac{1}{2\pi(N+1)}\displaystyle\int_{-\pi}^{\pi}|f(x+u)|F_N(u)\, du \leq M$, with $M = \displaystyle\sup_{x\in[0,2\pi]}|f(x)|$, where $(|\Lambda_N(f,.)|^2)_N$ converges uniformly to $|f|^2$ on $I_\eta$ and then
$\displaystyle\lim_{N\to+\infty}\int_{x_i}^{x_{\mathrm{i}+1}}|\Lambda_N(f,x)|^2\, dx = \int_{x_i}^{x_{\mathrm{i}+1}}|f(x)|^2\, dx$ because for $\varepsilon > 0$,

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx &= \int_{x_i}^{x_i+\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&\quad + \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&+ \int_{x_{i+1}-\varepsilon}^{x_{i+1}} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&\le 4\varepsilon M^2 + \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} |\ |\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx
\end{aligned}$$

$$\lim_{N\to+\infty} \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx = 0.$$

□

**Proof of the theoreme** (1.6) .

$$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\ dx = \sum_{k=-N}^{N} (1 - \frac{|k|}{N+1})|C_k|^2 \le \sum_{k=-N}^{N} |C_k|^2,$$

$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\ dx \le \sum_{k=-\infty}^{+\infty} |C_k|^2$ and the Bessel inequality yields that

$$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\ dx \le \sum_{k=-\infty}^{+\infty} |C_k|^2 \le \frac{1}{2\pi}\int_0 |f(x)|^2 dx.$$

□

**Corollary 1.17**

Let $f$ be a piecewise continuous function on $\mathbb{R}$ and $2\pi-$periodic. We assume that

$$\langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0,\ \forall n \in \mathbb{Z},$$

then $f$ is zero at all its points of continuity and $||f||_2 = 0$.

**Remark 21 :**
If $f$ and $g$ are two piecewise continuous functions and $2\pi-$periodic. Let $C_n$ (respectively $D_n$) be the Fourier coefficients of $f$ (respectively $g$). As the series

$\sum_{n\in\mathbb{Z}} |C_n|^2$ and $\sum_{n\in\mathbb{Z}} |D_n|^2$ converge, then the series $\sum_{n\in\mathbb{Z}} C_n\overline{D_n}$ converges absolutely. We consider the map $h(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t+x)}dt$. In using the Fubini formula we prove $\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-inx}dx = C_n\overline{D_n}$.
It results that the Fourier series of $h$ converges uniformly and at any point $x$ of continuity of $h$, $h(x) = \sum_{-\infty}^{+\infty} C_n\overline{D_n}e^{inx}$.

## 1.7 Weierstrass Theorem

**Proposition 1.18**

Let $f\colon [a,b] \longrightarrow \mathbb{C}$ be a continuous function. There exists a sequence of polynomials $(Q_n)_n \in \mathbb{C}[X]$ such that $(Q_n)_n$ converges uniformly to $f$ on $[a,b]$.

**Proof** .
**First case**: We assume that $a = 0$, $b = 2\pi$ and $f(0) = f(2\pi)$. In this case $f$ can be extended to a continuous function on $\mathbb{R}$ and $2\pi-$periodic. From the corollary (1.5) $\forall\varepsilon > 0$, there exists $P_\varepsilon$ a trigonometric polynomial such that

$$\sup_{x\in[0,2\pi]} |f(x) - P_\varepsilon(x)| < \varepsilon.$$

$$P_\varepsilon(x) = \sum_{n=-N}^{N} \alpha_n e^{inx}.$$

Moreover, we know that the series $\sum_{n\geq 0} \frac{z^n}{n!}$ converges uniformly on any compact to the function $e^z$. Thus for any $-N \leq n \leq N$, there exists $d_n \geq 0$ such that

$$\sup_{x\in[0,2\pi]} |e^{inx} - \sum_{p=0}^{d_n} \frac{(\mathrm{i}n)^p x^p}{p!}| < \frac{\varepsilon}{\sum_{n=-N}^{N} |\alpha_n|}.$$

We set $R_n(x) = \sum_{p=0}^{d_n} \frac{(\mathrm{i}n)^p x^p}{p!}$ and $H_N = \sum_{-N}^{N} \alpha_n R_n(x)$. $H_N$ is a polynomial.

$$\sup_{x\in[0,2\pi]} |f(x) - H_N(x)| \leq \sup_{x\in[0,2\pi]} |f(x) - P_\varepsilon(x)| + \sup_{x\in[0,2\pi]} |P_\varepsilon(x) - H_N(x)|$$

$$\sup_{x\in[0,2\pi]} |P_\varepsilon(x) - H_N(x)| \leq \sum_{n=-N}^{N} \sup_{x\in[0,2\pi]} |\alpha_n e^{\mathrm{i}nx} - \alpha_n R_n(x)| < \varepsilon.$$

Thus $\sup\limits_{x\in[0,2\pi]} |f(x) - H_N(x)| \leq 2\varepsilon$ and the corollary is proved in this case.

**General Case**: A from of $f$ one constructed a function which verifies the conditions of the first case.

Define the continuous function $g$ on $[a, b]$ by: $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$, $g(a) = f(a) = g(b)$ and let $h$ be the function defined on $[0, 2\pi]$ by: $h(x) = g(x.\frac{b-a}{2\pi} + a)$. $h$ is continuous on $[0, 2\pi]$ and $h(0) = h(2\pi)$. Let $\varepsilon > 0$, by the first case, there exists $K_\varepsilon \in \mathbb{C}[x]$ such that $\sup\limits_{x\in[0,2\pi]} |h(x) - K_\varepsilon(x)| < \varepsilon$, thus $\sup\limits_{x\in[a,b]} |g(y) - K_\varepsilon(\dfrac{2\pi}{b - a}(y - a))| < \varepsilon$.

We set $Q_\varepsilon(y) = K_\varepsilon(\frac{2\pi}{b-a}(y - a))$. This is a polynomial and gives an answer to the corollary.

### Other Proof

**Theorem 1.19**

(Weierstrass Theorem)
Let $f$ be a continuous function on an interval $I$, there exists a sequence $(f_n)_n$ of polynomials which converges uniformly on any closed and bounded interval $I$ to $f$.

**Proof** .
We assume in the first case that $f$ is continuous on $\mathbb{R}$ and identically zero on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. In this case we set

$$P_n(x) = c_n(1 - x^2)^n$$

where $c_n$ is chosen such that $\displaystyle\int_{-1}^{1} P_n(x)dx = 1$. We define the sequence

$$f_n(x) = \int_{-\infty}^{+\infty} f(y)P_n(x - y)dy = \int_{-\infty}^{+\infty} f(x - y)P_n(y)dy. \tag{1.8}$$

$\square$

**Lemma 1.20**

The functions $f_n$ are polynomials and converge uniformly to $f$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

**Proof** .
From the left side of the formula (1.8), $f$ is a polynomial. From the right side of the formula (1.8), we have for $|x| \leq \frac{1}{2}$

$$f(x) - f_n(x) = \int_{-1}^{1} f(x-y)P_n(y)dy \tag{1.9}$$

Let $\varepsilon > 0$, $M$ the maximum of $f$ on $\mathbb{R}$ and $\delta > 0$ such that $|f(x) - f(x-y)| < \varepsilon$ if $|y| < \delta$. It results from the formula (1.9) that

$$|f(x) - f_n(x)| \leq \int_{|y|<\delta} \varepsilon P_n(y)dy + \int_{\delta \leq |y| \leq 1} M\ P_n(y)dy.$$

We intend to prove that $\displaystyle\int_{\delta \leq |y| \leq 1} P_n(y)dy$ tends to 0 when $n$ tends to infinity.
Let $0 < r < 1$.

$$\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n dx \geq \int_{-r}^{r} (1-r^2)^n dx = 2r(1-r^2)^n.$$

Thus $c_n \leq \dfrac{1}{2r(1-r^2)^n}$ and

$$\int_{\delta \leq |y| \leq 1} P_n(y)dy \leq \frac{1}{2r(1-r^2)^n} \int_{-1}^{1} (1-\delta^2)^n dy = \frac{(1-\delta^2)^n}{r(1-r^2)^n}.$$

The result is deduced if we take $r < \delta$ and tends $n$ to infinity.
**Proof of the theorem**
If $f$ is zero outside the interval $[-s, s]$, the function $F(x) = f(2sx)$ is zero outside the interval $[-\frac{1}{2}, \frac{1}{2}]$. From the previous lemma there exists a sequence $(f_n)_n$ of polynomials which converges uniformly to $F$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The sequence of polynomials $g_n(x) = f_n(\frac{x}{2s})$ converges uniformly to $f$ on the interval $[-s, s]$.
If $f$ is continuous on the interval $I = (a, b)$. For any $n \in \mathbb{N}_0$ and $n > \dfrac{2}{b-a}$, there exists a continuous function $\varphi_n$ on $I$ such that $\varphi_n = 1$ on $[a + \frac{1}{n}, b - \frac{1}{n}]$ and zero outside $[a + \frac{1}{2n}, b - \frac{1}{2n}]$. There exists a polynomial $f_n$ such that $|f_n(x) - \varphi_n(x)f(x)| < \dfrac{1}{n}$ on $I$. The sequence $(f_n)_n$ is a solution of the problem.

□

## 1.8 Exercises

**6-1-1** Let $t \in \mathbb{R} \setminus \mathbb{Z}$ and $f(x) = \cos tx$, for $-\pi \le x \le \pi$ and $2\pi-$periodic.

(a) Give the Fourier series of $f$.

(b) Deduce that $\cos tx = \dfrac{\sin t\pi}{\pi}\left[\dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n 2t}{t^2 - n^2}\cos(nx)\right]$, for $x \in [-\pi, \pi]$.

(c) Show that

i. $$\frac{\pi}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{(-1)^n 2t}{t^2 - n^2}, \text{ for } t \notin \mathbb{Z}.$$

ii. $$\pi \text{cotan} \pi t = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}.$$

iii. $$\frac{\pi^2}{\sin^2 \pi t} = \sum_{-\infty}^{+\infty} \frac{1}{(t+n)^2}.$$

**6-1-2** Let $\delta \in ]0, \frac{\pi}{2}]$ and let $f$ be the even function $2\pi$-periodic defined by:

$$f(x) = \begin{cases} \frac{2\pi}{\delta}\left(1 - \frac{x}{2\delta}\right) & \text{if } 0 \le x \le 2\delta \\ 0 & \text{if } 2\delta \le x \le \pi \end{cases}$$

(a) Give the Fourier series the function $f$ and prove that this series converges uniformly to $f$ on $\mathbb{R}$.

(b) Compute $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin^2 n\delta}{n^2}$ and $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin^4 n\delta}{n^4}$.

**6-1-3** Let $f$ be a continuous function on $\mathbb{R}$ and $2\pi-$periodic.

Prove that if the Fourier series of $f$ is convergent, then $f$ is the sum of its Fourier series.

**6-1-4** (a) Prove the following formulas which gives an expansion in trigonometric series of the function $f(x) = x$ in divers intervals, in looking in each case, the periodic function $\varphi(x)$ whose expansion in Fourier series yields the given result.

$$x = \pi - 2\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n} \quad \text{pour} \quad 0 < x < 2\pi.$$

$$x = -2\sum_{n=1}^{+\infty} \frac{(-1)^n \sin(nx)}{n} \quad \text{pour} \quad -\pi < x < \pi.$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} \quad \text{pour} \quad 0 \leq x \leq \pi.$$

$$x = \frac{4}{\pi}\sum_{n=0}^{+\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)^2} \quad \text{for} \quad \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}.$$

$$x = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} + \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}\sin(nx)}{n} \quad \text{for} \quad 0 \leq x < \pi.$$

(b) Deduce

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad ; \quad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad ; \quad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

(c) i. In use of the formulas of the question 1) to compute the sum $g(x)$ of the trigonometric series $\displaystyle\sum_{n\geq 0} \frac{\sin(2n+1)\pi x}{(2n+1)^3}$.

ii. Verify the result in compute the Fourier coefficients of $g$.

**6-1-5** Let $f$ be the even function, $2\pi$ periodic defined by: $f(x) = \begin{cases} 1 & \text{if} \quad x \in \left[0, \frac{\pi}{2}\right[ \\ -1 & \text{if} \quad x \in \left[\frac{\pi}{2}, \pi\right[ \end{cases}$

(a) Determine the Fourier coefficients of $f$.

(b) Deduce the value of the sum $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$.

**6-1-6** (a) Does there exists a locally Riemann integrable function $f$ such that its Fourier series is $\displaystyle\sum_{n\geq 1} \frac{\sin(nx)}{\sqrt{n}}$?

(b) Same question for the series $\displaystyle\sum_{n\geq 1} \frac{\sin(nx)}{n^3}$.

**6-1-7** (a) Determine, for $a > 0$ the expansion in Fourier series of the function $f(x) = \dfrac{1}{\cosh(a) - \cos(x)}$.

(b) Deduce the value of $\displaystyle\int_0^{2\pi} \frac{dx}{\cosh(a) - \cos(x)}$.

**6-1-8** (a) Compute the Fourier series of the following $2\pi$-periodic functions on $\mathbb{R}$ given by:

i. $f(x) = \pi - x$ if $0 \leq x < 2\pi$.

ii. $g(x) = \pi - x$ if $0 \leq x < \pi$, $g$ even.

(b) Deduce that the Fourier series of the $2\pi-$periodic odd function $h$ defined by: $h(x) = x(\pi - \dfrac{x}{2})$ for $0 \leq x \leq \pi$.

**6-1-9** Let $\varphi$ be the $2\pi$-periodic function on $\mathbb{R}$ defined on $]-\pi, \pi]$ by $\varphi(x) = e^x$.

(a) Compute its Fourier coefficients.

(b) Prove that:

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{\pi \cosh \pi + \sinh \pi}{2 \sinh \pi}.$$

**6-1-10** (a) Find the Fourier series of the $2\pi$-periodic function

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ x^2 & \text{if} \quad 0 \leq x < \pi \end{cases}$$

(b) Use the first question to compute the following sums:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\,, \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$$

**6-1-11** Let $g$ be the odd $2\pi$-periodic function such that:

$$g(x) = x(\pi - x), \text{ for } 0 \leq x \leq \pi.$$

(a) Give the Fourier series of $g$.

(b) Use the Parseval identity to compute

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^6}.$$

**6-1-12** (a) Compute the sum of the following series $\sum_{n \geq 1} r^n \cos n\theta$, for $0 < r < 1$.

(b) Deduce the following equality:

$$Q_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = 1 + 2\sum_{n=1}^{+\infty} r^n \cos n\theta = \sum_{-\infty}^{+\infty} r^{|n|} e^{in\theta}.$$

(c) Using the theory of Fourier series, deduce the following value of the integral:

$$I_n(r) = \int_0^{2\pi} \frac{\cos n\theta}{1 - 2r\cos\theta + r^2} d\theta.$$

**6-1-13** Let $h$ be the function defined by:

$$h(x) = \frac{x^2 - 1}{x^2 - 4x + 1}.$$

(a) i. Give the power series of $h$ in a neighborhood of 0.

ii. Compute the radius of convergence of the obtained series.

Let $a$ and $z$ be two complex numbers, such that $|a| \neq |z|$ and $az \neq 0$. Recall that:

$$\frac{1}{z-a} = \begin{cases} \frac{1}{z}\sum_{n=0}^{+\infty}(\frac{a}{z})^n & if \quad |a| < |z|. \\ \frac{-1}{a}\sum_{n=0}^{+\infty}(\frac{z}{a})^n & \text{if} \quad |a| > |z| \end{cases}$$

(b) Prove that there exists a sequence of real numbers $(\lambda_n)_{n\leq 1}$, such that $\forall z \in \mathbb{C}$ such that $(|z| \in ]2-\sqrt{3}, 2+\sqrt{3}[)$:

$$h(z) = \sum_{n=1}^{+\infty} \frac{\lambda_n}{z^n} - \sum_{n=1}^{+\infty} \lambda_n z^n$$

Let $f$ be the $2\pi$-periodic function on $\mathbb{R}$ defined by: $f(t) = \dfrac{\sin(t)}{2-\cos(t)}$.

(c) Prove that $h(e^{\mathrm{i}t}) = -\mathrm{i}f(t)$ , $\forall t \in \mathbb{R}$.

(d) Deduce the expansion of $f$ in Fourier series.

(e) Deduce the value of the following integral $\displaystyle\int_0^{2\pi} \frac{\sin^2 x}{2-\cos x} dx$.

Let $F$ be the $2\pi$-periodic function defined by: $F(t) = \ln(2-\cos t)$.

(f) Say why $F$ can has an expansion in Fourier series.

(g) Compute $F'(t)$ and deduce, without compute the Fourier coefficients of $F$ that the Fourier series of $F$ converges normally to $F$.

(h) Deduce the value of the integral $\displaystyle\int_0^{\pi} \ln(2-\cos x)dx$.

**6-1-14** Define the sequence $(f_n)_n$ by: $f_n(x) = \dfrac{1}{a^2 + (x + 2n\pi)^2}$ and $a > 0$.

(a) Prove that the series $\sum_{n\geq 1} f_n$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(b) i. Prove that for any $n \geq 1$ and any $t \in \mathbb{R}$, $|f_n'(t)| \leq \dfrac{f_n(t)}{a}$.

ii. Deduce that the series $\sum_{n=1}^{+\infty} f_n'$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(c) Deduce that the function

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{a^2 + (x + 2n\pi)^2}$$

is even, $2\pi$-periodic and equal in each point to its Fourier series on $\mathbb{R}$.

(d) i. For any $k \in \mathbb{Z}$ compute the integral

$$I_k(a) = \int_{-\infty}^{+\infty} \frac{\cos kx}{a^2 + x^2} dx.$$

ii. Prove that $\int_0^{2\pi} f(x) \cos kx dx = I_k(a)$.

iii. Give the expression of $f$.

# CHAPTER VII

# LEBESGUE INTEGRAL

In this chapter, we present the Lebesgue measure theory and compare it with the Riemann integral.

## 1 Classes of Subsets of $\mathbb{R}$

### 1.1 Algebra and $\sigma-$Algebra

**Definition 1.1**

1. A non empty collection of subsets $\mathcal{A}$ of $\mathbb{R}$ is called an algebra or a field if:
   (a) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
   (b) If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.
2. An algebra $\mathscr{A}$ in $\mathscr{P}(\mathbb{R})$ is called a $\sigma-$algebra if every countable intersection of a collection of elements of $\mathscr{A}$ is again in $\mathscr{A}$. That is if $(A_j)_j$ is a sequence in $\mathscr{A}$ then $\bigcap_{j=1}^{+\infty} A_j \in \mathscr{A}$.
   If $\mathscr{A}$ is a $\sigma-$algebra. The pair $(\mathbb{R}, \mathscr{A})$ is called a **measurable space**, and the elements of $\mathscr{A}$ are called measurable subsets.

## Properties 1.2

Let $\mathcal{A}$ be an algebra, then

1. $\emptyset, \mathbb{R} \in \mathcal{A}$;
2. $\mathcal{A}$ is closed under finite union and finite intersection. (i.e. if $A_1, \ldots, A_n \in \mathscr{A}$, then $\bigcap_{j=1}^{n} A_j \in \mathscr{A}$ and $\bigcup_{j=1}^{n} A_j \in \mathscr{A}$).
3. Let $\mathscr{A}$ be a $\sigma-$algebra then: if $(A_j)_j$ is a sequence in $\mathscr{A}$, then $\bigcup\limits_{j=1}^{+\infty} A_j \in \mathscr{A}$.

**Proof** .

1. Since $\mathcal{A}$ is non empty there exists $A \in \mathcal{A}$. So $A^c \in \mathcal{A}$, hence $\emptyset = A \cap A^c \in \mathcal{A}$ and $\mathbb{R} = \emptyset^c \in \mathcal{A}$.
2. Let $A, B \in \mathcal{A}$, then $A^c, B^c \in \mathcal{A}$ and $A^c \cap B^c \in \mathcal{A}$. Since $(A \cup B)^c = A^c \cap B^c \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. By induction we prove that if $A_1, \ldots, A_n \in \mathcal{A}$ then $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{A}$ and $\bigcap_{j=1}^{n} A_j \in \mathcal{A}$.
3. We have $A_j^c \in \mathscr{A}$ and $\bigcap\limits_{j=1}^{+\infty} A_j^c \in \mathscr{A}$, hence $\left(\bigcap_{j=1}^{+\infty} A_j^c\right)^c = \bigcup_{j=1}^{+\infty} A_j \in \mathscr{A}$.

$\square$

**Example 23 :**

1. $\mathscr{A} = \{\emptyset, \mathbb{R}\}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.
2. The power set $\mathscr{P}(\mathbb{R})$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.
3. Let $\{A, B, C\}$ be a partition of $\mathbb{R}$. The set $\mathcal{A} = \{\emptyset, \mathbb{R}, A, B, C, A^c, B^c, C^c\}$ is an algebra. ($A \cup B = C^c, A \cup C = B^c, B \cup C = A^c$.)
4. Let $\mathcal{A}$ be the collection of subsets $A$ of $\mathbb{R}$ such that either $A$ or $A^c$ is finite. $\mathcal{A}$ is an algebra. but not a $\sigma$-algebra.
5. Let $\mathscr{A}$ be the collection of subsets $A$ of $\mathbb{R}$ such that either $A$ or $A^c$ is countable or $\emptyset$. $\mathscr{A}$ is a $\sigma-$algebra. Indeed: let $(A_j)_j$ be a sequence of elements of $\mathscr{A}$. If there exists $p$ such that $A_p$ is countable, then $\cap_{j=1}^{+\infty} A_j \subset A_p$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$. If the sets $A_j$ are all not countable, then the sets $A_j^c$ are countable. The set $\cup_{j=1}^{+\infty} A_j^c$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.

**Theorem 1.3**

Any intersection of algebras (resp $\sigma-$ algebra) is an algebra (resp $\sigma-$ algebra) i.e. if $(\mathcal{A}_j)_{j\in J}$ is a family of algebras (resp $\sigma-$ algebra) on $\mathbb{R}$, then $\bigcap\limits_{j\in J}\mathcal{A}_j$ is an algebra (resp $\sigma-$ algebra).

**Proof** .
Consider the case where $\mathcal{A}_j$ are algebra.
$\mathbb{R}\in\mathcal{A}_j$ for all $j\in J$, then $\mathbb{R}\in\bigcap\limits_{j\in J}\mathcal{A}_j$.
If $A\in\bigcap_{j\in J}\mathcal{A}_j$, as $A\in\mathcal{A}_j$ for all $j\in J$, then $A^c\in\bigcap_{j\in J}\mathcal{A}_j$.
Let $A_1,\ldots,A_n$ in $\bigcap\limits_{j\in J}\mathcal{A}_j$, then $A_1,\ldots,A_n$ are in $\mathcal{A}_j$ for all $j\in J$. Thus $\bigcap_{k=1}^n A_k\in\bigcap_{j\in J}\mathcal{A}_j$.
Now, if $\mathscr{A}_j$ are $\sigma-$ algebra.
If $(A_n)_n$ is a sequence in $\bigcap\limits_{j\in J}\mathscr{A}_j$, then $(A_n)_n\in\mathscr{A}_j$ for all $j\in J$. Thus $\bigcap_{n=1}^{+\infty}A_n\in\bigcap_{j\in J}\mathscr{A}_j$.

□

**Theorem 1.4**

Let $(\mathscr{A}_j)_{j\in J}$ be a family of $\sigma-$algebras on $\mathbb{R}$, then $\bigcap\limits_{j\in J}\mathscr{A}_j$ is a $\sigma-$ algebra.

**Proof** .
$\bigcap\limits_{j\in J}\mathscr{A}_j$ is an algebra. Let $(A_n)_n$ be a sequence in $\bigcap\limits_{j\in J}\mathscr{A}_j$. Since each $\mathscr{A}_j$ is a $\sigma-$ algebra then $\bigcap\limits_{n=1}^{+\infty}A_n\in\mathscr{A}_j$ for all $j\in J$. Thus $\bigcap\limits_{n=1}^{+\infty}A_n\in\bigcap\limits_{j\in J}\mathscr{A}_j$. □

**Definition 1.5**

Let $\mathcal{B}\subset\mathscr{P}(\mathbb{R})$. The intersection of the algebras (resp $\sigma-$ algebra) on $\mathbb{R}$ that contain $\mathcal{B}$ is the smallest algebra (resp $\sigma-$ algebra) denoted by $\mathcal{A}(\mathcal{B})$ (rep $\sigma(\mathcal{B})$) that contain $\mathcal{B}$. This algebra (resp $\sigma-$ algebra) is called the algebra (resp the $\sigma-$ algebra) generated by $\mathcal{B}$.

**Example 24 :**
Let $\mathscr{A}$ be the $\sigma-$ algebra of subsets $A \subset \mathbb{R}$ such that either $A$ or $A^c$ is countable. $\mathscr{A}$ is the $\sigma$-algebra generated by the singleton sets $S = \{\{x\} : x \in \mathbb{R}\}$.
It is evident that if $A$ or $A^c$ is countable then $A \in \sigma(S)$. Then $\mathscr{A} \subset \sigma(S)$. The other inclusion is evident.

**Exercise 4 :**
Let $\mathcal{A}$ and $\mathcal{B}$ two family of subsets of $\mathbb{R}$.
Prove that

$$\sigma(\mathcal{A}) = \sigma(\mathcal{B}) \iff \begin{cases} \forall A \in \mathcal{A}, & A \in \sigma(\mathcal{B}) \\ \quad \& & \\ \forall B \in \mathcal{B}, & B \in \sigma(\mathcal{A}) \end{cases}$$

**Solution**:
Il suffices to prove that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B}) \iff A \in \sigma(\mathcal{B}),\ \forall A \in \mathcal{A}$.
Assume that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. If $A \in \mathcal{A}$, then $A \in \mathcal{A} \subset \sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.
Assume that $A \in \sigma(\mathcal{B}),\ \forall A \in \mathcal{A}$. Then $\mathcal{A} \subset \sigma(\mathcal{B})$. Since $\sigma(\mathcal{A})$ is the smallest $\sigma-$ algebra that contain $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.

## 1.2 The Borelian $\sigma-$Algebra

**Definition 1.6: [The Borelian $\sigma-$Algebra on $\mathbb{R}$]**

Let $\mathscr{B}_{\mathbb{R}}$ be the $\sigma-$algebra generated by the family $\{[a, b[ : (a, b) \in \mathbb{R}^2\}$. This $\sigma-$algebra is called the Borel $\sigma-$algebra on $\mathbb{R}$. The elements of $\mathscr{B}_{\mathbb{R}}$ are called Borel subsets of $\mathbb{R}$.

We have the following theorem:

**Theorem 1.7**

1. The open and the closed subsets of $\mathbb{R}$ are Borel subsets;
2. $\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$;
3. $\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$;
4. $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a, +\infty[ : \ a \in \mathbb{R}\}$;
5. $\mathscr{B}_{\mathbb{R}}$ is generated by $\{] - \infty, a] : \ a \in \mathbb{R}\}$.

**Proof** .
For the proof we use the exercises (1.1).

1. As any open subset of $\mathbb{R}$ is countable union of open intervals. It suffices to prove that the open intervals are Borel sets. We have $]a,b[=\cup_{n=1}^{+\infty}[a+\frac{1}{n},b[$. Then $]a,b[\in \mathscr{B}_{\mathbb{R}}$.

2. Since $[a,b[=\cap_{n=1}^{+\infty}]a-\frac{1}{n},b[$, then $\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$;

3. Since $[a,b[=\cup_{n=1}^{+\infty}[a,b-\frac{1}{n}]$ and $[a,b]=\cap_{n=1}^{+\infty}[a,b+\frac{1}{n}[$, then $\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$;

4. The $\sigma-$Algebra generated by the family $\{]a,+\infty[:\ a\in\mathbb{R}\}$ is a subset of the $\sigma-$Algebra generated by open sets. To prove that $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a,+\infty[:\ a\in\mathbb{R}\}$, it suffices to prove that any open interval $]a,b[$ is in the $\sigma-$Algebra generated by the family $\{]a,+\infty[:\ a\in\mathbb{R}\}$.

   We have $]a,b]=]a,+\infty[\cap(]b,+\infty[)^c$ and $]a,b[=\cup_{n=1}^{+\infty}]a,b-\frac{1}{n}]$. Then $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a,+\infty[:\ a\in\mathbb{R}\}$.

5. With the same arguments as in the previous property, $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]-\infty,a]:\ a\in\mathbb{R}\}$.

$\square$

### 1.3 Exercises

**7-1-1** Find all $\sigma-$algebras that contain three elements in $\mathscr{P}(\mathbb{R})$.
Find all $\sigma-$algebras that contain four elements in $\mathscr{P}(\mathbb{R})$.

**7-1-2** Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Prove that the set $\mathscr{A} = \{A \subset \mathbb{R} : f^{-1}(f(A)) = A\}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.

**7-1-3** Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a bijective function.
Prove that the set

$$\mathscr{A} = \{A \subset X : \ f(A) \subset A \& \ f^{-1}(A) \subset A\}.$$

is a $\sigma-$algebra.

**7-1-4** Let $E$ be a non empty subset of $\mathbb{R}$.
Find all the $\sigma-$algebras generated by the set $\mathscr{C} = \{F : \mathcal{E} \subset F \subset \mathbb{R}\}$.

**7-1-5** Let $E$ be infinite subset of $\mathbb{R}$ and $S = \{\{x\} : x \in E\}$.
Find the $\sigma-$algebra generated by $S$. (Discuss the case of $E$ countable and not countable)

**7-1-6** Let $A$ be non-empty subset of $\mathbb{R}$.

(a) Find the $\sigma-$algebra generated by the set $\mathscr{C} = \{B \subset \mathbb{R} : A \subset B\}$.
(b) In which case this $\sigma-$algebra is equal to $\mathscr{P}(\mathbb{R})$?

# 2 The Lebesgue Measure on $\mathbb{R}$

## 2.1 Lebesgue Outer Measure

**Definition 2.1**

A set function $\mu^*\colon \mathscr{P}(\mathbb{R}) \longrightarrow [0, \infty]$ is called an outer measure or exterior measure on $\mathbb{R}$ if:

1. $\mu^*(\emptyset) = 0$;
2. $\mu^*$ is increasing (i.e. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$);
3. $\mu^*(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n)$, for any sequence $(A_n)_n$ of subsets of $\mathbb{R}$.

We give an example of an outer measure on $\mathbb{R}$ which helps us to construct the Lebesgue measure on $\mathbb{R}$.

**Proposition 2.2**

Let $\mathcal{A} \subset \mathscr{P}(\mathbb{R})$ be a family of subsets of $\mathbb{R}$ such that $\emptyset, \mathbb{R} \in \mathcal{A}$. Consider a function $\rho\colon \mathcal{A} \longrightarrow [0, +\infty]$ such that $\rho(\emptyset) = 0$. For all subset $A \subset \mathbb{R}$, define

$$\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \rho(A_n) : \ A_n \in \mathcal{A}, \ A \subset \cup_{n=1}^{+\infty} A_n\}. \tag{2.1}$$

The function $\mu^*$ is an outer measure on $\mathbb{R}$.

**Proof** .
For each subset $A \subset \mathbb{R}$, there exists a sequence $(A_n)_n \in \mathcal{A}$ such that $A \subset \cup_{n=1}^{+\infty} A_n$. (We can take $A_n = \mathbb{R}$). So the function $\mu^*$ is well-defined.
It is obvious that $\mu^*(\emptyset) = 0$ and that $\mu^*(A) \leq \mu^*(B)$ if it was $A \subset B$.
Let $(A_n)_n$ be a sequence in $\mathscr{P}(\mathbb{R})$ such that $A \subset \cup_{n=1}^{+\infty} A_n$.
If there exists $A_n$ such that $\rho(A_n) = +\infty$, then $\mu^*(A) \leq \sum_{k=1}^{+\infty} \mu^*(A_k) = +\infty$.
Now assume that $\rho(A_n) < +\infty$ for every $n \in \mathbb{N}$.
For $\varepsilon > 0$, and for each $n \in \mathbb{N}$, there is a sequence $(A_{n,k})_k$ in $\mathcal{A}$ such that $A_n \subset \cup_{k=1}^{+\infty} A_{n,k}$ and

$$\sum_{k=1}^{+\infty} \rho(A_{n,k}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

We have $A \subset \cup_{n,k=1}^{+\infty} A_{n,k}$ and $\displaystyle\sum_{n,k=1}^{+\infty} \rho(A_{n,k}) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon$. □

**Remark 22 :**
If we take $\mathcal{I}$ is the family of open intervals in $\mathbb{R}$ and the function $\rho(I) = \mathscr{L}(I)$, where $\mathscr{L}(I)$ is the length of $I$.
In this case, we denote the outer measure defined by this function by $\lambda^*$. It is called a the Lebesgue outer measure.

$$\lambda^*(A) = \inf\{\sum_{n=1}^{+\infty} \mathscr{L}(I_n) : \ I_n \in \mathcal{I}, \ A \subset \cup_{n=1}^{+\infty} I_n\}.$$

This outer measure fulfills the following properties:

**Lemma 2.3**

For any interval $I$ in $\mathbb{R}$, $\lambda^*(I) = \mathscr{L}(I)$.

**Proof** .
The result is obvious if the interval is not bounded, and if the interval is bounded $I$ and $a$ and $b$ are its limits, then for any $\varepsilon > 0$, $I \subset ]a-\varepsilon, b+\varepsilon[$. Then $\lambda^*(I) \leq \mathscr{L}(I) + 2\varepsilon$ and $\lambda^*(I) \leq \mathscr{L}(I)$.
Inversely if $(I_k)_k$ is open covering of $I$, then $[a+\varepsilon, b-\varepsilon] \subset \cup_{k=1}^{+\infty} I_k$. As the interval $[a+\varepsilon, b-\varepsilon]$ is compact, there is a finite covering $(I_k)_{1\leq k\leq n}$ of $[a+\varepsilon, b-\varepsilon]$.
Therefore $b-a-2\varepsilon \leq \displaystyle\sum_{k=1}^{n} \mathscr{L}(I_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(I_k)$. Then $b-a-2\varepsilon \leq \lambda^*(I)$ for every $\varepsilon > 0$. Therefore $\lambda^*(I) = \mathscr{L}(I)$.

□

**Lemma 2.4**

Let $\Omega$ be an open subset of $\mathbb{R}$ and let $(I_n)_n$ the connected components of $\Omega$. Then

$$\lambda^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n).$$

**Proof** .
Using the definition of the outer measure $\lambda^*$, we have $\lambda^*(\Omega) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$.
Inversely, let $(J_k)_k$ be a covering of $\Omega$ by open intervals. As $I_n = \cup_{k=1}^{+\infty} J_k \cap I_n$, then

$$\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \mathscr{L}(I_n \cap J_k) = \sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k).$$

On the other hand, since the intervals $(I_n)_n$ are disjoint, then $\displaystyle\bigcup_{n=1}^{m} (J_k \cap I_n) \subset J_k$ for every $m$. Therefore $\displaystyle\sum_{n=1}^{m} \mathscr{L}(J_k \cap I_n) \leq \mathscr{L}(J_k)$ and $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(J_k)$. Hence $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \lambda^*(\Omega)$ and therefore $\lambda^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. □

**Theorem 2.5**

For any subset $A \subset \mathbb{R}$, $\lambda^*(A) = \displaystyle\inf_{O \in \mathcal{O}_A} \lambda^*(O)$, where $\mathcal{O}_A$ the collection of open sets that contain the subset $A$.

**Proof** .

Let $(I_n)_n$ be any countable covering of $A \subset \mathbb{R}$ formed by open intervals. If $\omega = \bigcup_{n=1}^{+\infty} I_n$, then $\lambda^*(A) \le \lambda^*(\omega) \le \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. Then $\lambda^*(A) \le \inf\limits_{O \in \mathcal{O}_A} \lambda^*(O)$.
The converse inequality is evident if $\lambda^*(A) = +\infty$.
Assume that $\lambda^*(A) < +\infty$. For $\varepsilon > 0$, there exist a countable covering $(I_n)_n$ of $A$ by open intervals so that $\sum\limits_{n=1}^{+\infty} \mathscr{L}(I_n) \le \lambda^*(A) + \varepsilon$. The open interval $\Omega = \cup_{n=1}^{+\infty} I_n$ contains $A$ and $\lambda^*(\Omega) \le \sum\limits_{n=1}^{+\infty} \mathscr{L}(I_n) \le \lambda^*(A) + \varepsilon$. Then $\inf\limits_{O \in \mathcal{O}_A} \lambda^*(O) \le \lambda^*(A)$. □

**Corollary 2.6**

If $A$ is countable subset of $\mathbb{R}$, then $\lambda^*(A) = 0$.

As $\lambda^*\{a\} = \mathscr{L}([a,a]) = 0$, then if $A = \{a_n : \ n \in \mathbb{N}\}$, $\lambda^*(A) \le \sum_{n=1}^{+\infty} \lambda^*\{a_n\} = 0$.

**Corollary 2.7**

$\mathbb{R}$ and any interval $[a,b]$ are not countable, for $a \ne b$.

**Theorem 2.8**

Let $A \subset \mathbb{R}$ and $r \in \mathbb{R}$, then $\lambda^*(A + r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$.

**Proof** .
If $A = (a,b)$, then $A + r = (a+r, b+r)$ and if $r \ge 0$, $rA = (ra, rb)$ and if $r \le 0$, $rA = (rb, ra)$. Therefore $\lambda^*(A + r) = b - a = \lambda^*(A)$ and $\lambda^*(rA) = |r|(b-a) = |r|\lambda^*(A)$.
If $A$ is an open subset, then $A = \cup_{n=1}^{+\infty}(a_j, b_j)$ with $(a_j, b_j) \cap (a_k, b_k) = \emptyset$ for every $j \ne k$ and $\lambda^*(A) = \cup_{n=1}^{+\infty}(b_j - a_j)$. Therefore $\lambda^*(A + r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$.
In the general case since, for any subset $A \subset \mathbb{R}$, $\lambda^*(A) = \inf\limits_{O \in \mathcal{O}_A} \lambda^*(O)$, where $\mathcal{O}_A$ is the collection of open subsets that contain $A$, then $\lambda^*(A + r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$. □

## 2.2 The Lebesgue $\sigma-$algebra

**Definition 2.9**

Let $\mu^*$ be an outer measure on $\mathbb{R}$. We say that a subset $A$ of $\mathbb{R}$ is measurable with respect to the outer measure $\mu^*$ If

$$\forall X \subset \mathbb{R}: \quad \mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c).$$

**Theorem 2.10**

The set $\mathscr{B}$ of measurable subsets in $\mathbb{R}$ with respect to the outer measure $\mu^*$ is a $\sigma-$Algebra.

**Proof** .

1. As $\mu^*(X \cap \emptyset) + \mu^*(X \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(X) = \mu^*(X)$ for any subset $X$ in $\mathbb{R}$, then $\emptyset$ is measurable.

2. Let $A \in \mathscr{B}$, the for any subset $X$ in $\mathbb{R}$, $\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c)$. This definition is symmetric with respect to $A$ and $A^c$. Then $A^c$ is also measurable.

3. Let $A, B \in \mathscr{B}$ and $X$ a subset in $\mathbb{R}$. As $A$ is measurable

$$\begin{aligned} \mu^*(X \cap (A \cup B)) &= \mu^*(X \cap (A \cup B) \cap A) + \mu^*(X \cap (A \cup B) \cap A^c) \\ &= \mu^*(X \cap A) + \mu^*(X \cap B \cap A^c). \end{aligned}$$

   Then

$$\begin{aligned} \mu^*(X \cap (A \cup B)) + \mu^*(X \cap (A \cup B)^c) &= \mu^*(X \cap A) + \mu^*(X \cap B \cap A^c) \\ &\quad + \mu^*(X \cap A^c \cap B^c) \\ &= \mu^*(X \cap A) + \mu^*(X \cap A^c) \\ &= \mu^*(X). \end{aligned}$$

   We deduce that $A \cup B$ is measurable.

4. Let $A_1, A_2$ be two disjoint measurable sets and $X$ a subset in $\mathbb{R}$. Let $B = X \cap (A_1 \cup A_2)$. As $B \cap (A_1 \cup A_2)^c = \emptyset$, then

$$\begin{aligned} \mu^*(B) &= \mu^*(B \cap (A_1 \cup A_2)) + \mu^*(B \cap (A_1 \cup A_2)^c) \\ &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\ &= \mu^*(X \cap A_1) + \mu^*(X \cap A_2). \end{aligned}$$

Therefore $\mu^*(X \cap (A_1 \cup A_2)) = \mu^*(X \cap A_1) + \mu^*(X \cap A_2)$.
Let $(A_n)_n$ be disjoint sequence in $\mathscr{B}$ and $X \subset \mathbb{R}$.

$$\begin{aligned}
\mu^*(X) &= \mu^*(X \cap \bigcup_{j=1}^{n} A_j) + \mu^*(X \cap (\bigcup_{j=1}^{n} A_j)^c) \\
&\geq \mu^*(X \cap \bigcup_{j=1}^{n} A_j) + \mu^*(X \cap (\bigcup_{j=1}^{+\infty} A_j)^c) \\
&\geq \sum_{j=1}^{n} \mu^*(X \cap A_j) + \mu^*(X \cap (\bigcup_{j=1}^{+\infty} A_j)^c).
\end{aligned}$$

Then

$$\begin{aligned}
\mu^*(X) &\geq \sum_{n=1}^{+\infty} \mu^*(X \cap A_n) + \mu^*(X \cap (\bigcup_{n=1}^{+\infty} A_n)^c) \qquad (2.2) \\
&\geq \mu^*(X \cap \bigcup_{n=1}^{+\infty} A_n) + \mu^*(X \cap (\bigcup_{n=1}^{+\infty} A_n)^c).
\end{aligned}$$

The inverse inequality results from the outer measure property.
So that to complete the proof, consider a sequence $(B_n)_n$ in $\mathscr{B}$. We define the sequence $(A_n)_n$ as follows: $A_1 = B_1$, $A_n = B_n \setminus \bigcup_{j=1}^{n-1} B_j$. Hence $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$.
Since $\bigcup_{n=1}^{+\infty} A_n \in \mathscr{B}$ then $\bigcup_{n=1}^{+\infty} B_n \in \mathscr{B}$ Therefore $\mathscr{B}$ $\sigma-$algebra.

□

**Theorem 2.11**

The Borel sets are measurable with respect to the outer measure $\lambda^*$, i.e. $\mathscr{B}_{\mathbb{R}} \subset \mathscr{B}$.

**Proof** .
It suffice to prove that $]a, +\infty[ \in \mathscr{B}$ for any $a \in \mathbb{R}$.
Let $X$ be a subset in $\mathbb{R}$, We want to prove that:

$$\lambda^*(X) = \lambda^*(X\cap]a,+\infty[) + \lambda^*(X\cap]-\infty,a]).$$

As $\lambda^*$ is an outer measure

$$\lambda^*(X) \leq \lambda^*(X\cap]a,+\infty[) + \lambda^*(X\cap]-\infty,a]).$$

For the inverse inequality, the result is evident if $\lambda^*(X) = +\infty$.
Suppose that $\lambda^*(X) < +\infty$. So for any $\varepsilon > 0$, there exists an open set $\Omega_\varepsilon$ such that $X \subset \Omega_\varepsilon$ and $\lambda^*(\Omega_\varepsilon) \leq \lambda^*(X) + \varepsilon$.
Assume first that $a \notin \Omega_\varepsilon$.

$$\lambda^*(\Omega_\varepsilon) = \sum_{I\in\mathcal{C}} \mathscr{L}(I) = \sum_{I\in\mathcal{C}\cap]a,+\infty[} \mathscr{L}(I)) + \sum_{I\in\mathcal{C}\cap]-\infty,a[} \mathscr{L}(I),$$

where $\mathcal{C}$ is the set of component connected of $\Omega_\varepsilon$. Then

$$\begin{array}{rcl}\lambda^*(\Omega_\varepsilon) & = & \lambda^*(\Omega_\varepsilon \cap [a,+\infty[) + \lambda^*(\Omega_\varepsilon\cap]-\infty,a[) \\ & \geq & \lambda^*(X \cap [a,+\infty[) + \lambda^*(X\cap]-\infty,a[).\end{array}$$

Therefore $\lambda^*(X) \geq \lambda^*(X \cap [a,+\infty[) + \lambda^*(X\cap]-\infty,a])$.
If $a \in \Omega_\varepsilon$, we use the first case, by considering the open set $\Omega'_\varepsilon = \Omega_\varepsilon \setminus \{a\}$ instead of $\Omega_\varepsilon$. ($\lambda^*(\Omega'_\varepsilon) = \lambda^*(\Omega_\varepsilon)$.) □

**Exercise 1 :**
We say that a subset $A \subset \mathbb{R}$ is a zero set with respect to outer measure $\lambda^*$ if there exists a measurable subset $B$ so that $A \subset B$ and $\lambda^*(B) = 0$.
Prove that each zero set is measurable.
Solution
If $A$ is a zero set, there is $B \in \mathscr{B}$ such that $A \subset B$ and $\lambda^*(B) = 0$. If $X$ is a subset of $\mathbb{R}$, then $\lambda^*(X \cap A) = 0$ and

$$\lambda^*(X) \geq \lambda^*(X \cap A^c) = \lambda^*(X \cap A) + \lambda^*(X \cap A^c).$$

The inverse inequality results from the definition of the outer measure $\lambda^*$. So the set $A$ is measurable.

## 2.3 The Lebesgue Measure

### 2.3.1 Measure Theory

**Definition 2.12**

Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$. We say that a function $\mu\colon \mathscr{A} \to [0,\infty]$ is a measure (positive measure) on $\mathscr{A}$ if the following conditions are satisfied:

1. $\mu(\emptyset) = 0$,

2. For any disjoint sequence $(A_n)_n \in \mathscr{A}$, $\mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n)$

The set $(\mathbb{R}, \mathscr{A}, \mu)$ is called a measure space.

**Examples 25 :**

1. If $\mathscr{A} = \mathscr{P}(\mathbb{R})$ and $\mu(A) = \#A$ (number of elements of $A$ if $A$ is finite and $+\infty$ otherwise). The function $\mu$ is a measure on $\mathscr{A}$. This measure is called a the counting measure on $\mathbb{R}$.

2. Let $a \in \mathbb{R}$ and $\delta_a(A) = 1$ if $a \in A$ and $0$ if $a \notin A$.
$\delta_a$ is a measure called a point measure at $a$ or the Dirac measure at $a$.

3. Let $\mu$ be the function defined on $\mathscr{P}(\mathbb{R})$ as follows: $\mu(A) = 0$ if the set $A$ is finite and $\mu(A) = +\infty$ if the set $A$ is infinite.
The function $\mu$ is not a measure since $\mathbb{N} = \cup_{n=1}^{+\infty}\{n\}$, but $\mu(\mathbb{N}) = +\infty \neq \sum_{n=1}^{+\infty} \mu(\{n\}) = 0$.

**Theorem 2.13**

Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$ and $\mu$ a measure on $\mathscr{A}$. The measure $\mu$ satisfies the following properties:

1. If $A_1, \ldots, A_n \in \mathscr{A}$ are disjoint, then
$$\mu(\cup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j).$$

2. If $A, B \in \mathscr{A}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. ($\mu$ is increasing)

3. If $(A_n)_n \in \mathscr{A}$ and $A = \cup_{n=1}^{+\infty} A_n$, then
$$\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$

4. If $(A_n)_n$ is increasing sequence in $\mathscr{A}$ and $A = \cup_{n=1}^{+\infty} A_n$, then
$$\mu(A) = \lim_{n \to +\infty} \mu(A_n).$$

5. If $A, B \in \mathscr{A}$ and $A \subset B$ and $\mu(B) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$ (The result remains true if $\mu(A) < \infty$).

6. If $(A_n)_n$ is a decreasing sequence in $\mathscr{A}$ and $A = \cap_{n=1}^{+\infty} A_n = \lim\limits_{n \to +\infty} A_n$. If $\mu(A_1) < \infty$, then $\mu(A) = \lim\limits_{n \to +\infty} \mu(A_n)$.

**Proof** .

1. We prove this property by induction.

2. Since $B = A \cup (B \setminus A)$, then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

3. Let $B_1 = A_1$, and $B_n = A_n \setminus \cup_{j=1}^{n-1} B_j$, for every $n \geq 2$. The sets $(B_n)_n$ are disjoint and $A = \cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. Therefore

$$\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$

4. Let $(B_n)_n$ the sequence defined previously. As $\cup_{j=1}^{n} A_j = \cup_{j=1}^{n} B_j$, then

$$\begin{aligned}
\mu(A) &= \mu(\cup_{n=1}^{+\infty} A_n) = \mu(\cup_{n=1}^{+\infty} B_n) \\
&= \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu(B_j) \\
&= \lim_{n \to \infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n \to \infty} \mu(\cup_{j=1}^{n} A_j) = \lim_{n \to \infty} \mu(A_n).
\end{aligned}$$

5. $\mu(B \setminus A) + \mu(A) = \mu(B)$. If $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

6. We apply property (3) to the sequence $(A_1 \setminus A_n)_n$.

□

**Example 26 :**
Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$ and $\mu \colon \mathscr{A} \longrightarrow [0, +\infty]$ a function on $\mathscr{A}$. $\mu$ is a measure if and only if:

1. $\mu(\emptyset) = 0$

2. $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$.

3. If $(A_n)_n$ is an increasing sequence in $\mathscr{A}$, then $\mu(\cup_{n=1}^{+\infty} A_n) = \lim\limits_{n \to +\infty} \mu(A_n)$.

If $\mu$ is a measure, it fulfills the properties (1) and (2).
Let $(A_n)_n$ be an increasing sequence in $\mathscr{A}$. Define $B_1 = A_1$ and $B_n = A_n \setminus \cup_{j=1}^{n-1} A_j$ for every $n \in \mathbb{N}$. The sequence $(B_n)_n$ is disjoint and $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$. Then

$$\begin{aligned}\mu\left(\cup_{n=1}^{+\infty} A_n\right) &= \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n\to+\infty} \sum_{j=1}^{n} \mu(B_j) \\ &= \lim_{n\to+\infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n\to+\infty} \mu(A_n)\end{aligned}$$

Inversely, if $\mu$ is a function satisfying the properties (1), (2) and (3). If $(A_n)_n$ is a disjoint sequence of measurable sets. So the sequence $\left(B_n = \cup_{j=1}^{n} A_j\right)_n$ is increasing and $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$. Therefore

$$\mu(\cup_{n=1}^{+\infty} A_n) = \lim_{n\to+\infty} \mu(B_n) = \lim_{n\to+\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{n=1}^{+\infty} \mu(A_n).$$

### 2.3.2 The Uniqueness Theorem

**Theorem 2.14**

Let $\mu$ and $\nu$ two measure on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Assume that there exists a class $\mathscr{C} \subset \mathscr{B}_{\mathbb{R}}$ that satisfies the following properties:

1. $\mathbb{R} \in \mathscr{C}$ and if $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$
2. $\mathscr{C}$ generates the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$. $(\sigma(\mathscr{C}) = \mathscr{B}_{\mathbb{R}})$
3. $\mu(C) = \nu(C) < +\infty$ for every $C \in \mathscr{C}$.

Then $\mu = \nu$.

**Remarks 23 :**
Let $\mu$ and $\nu$ two measures that fulfill the hypotheses of the theorem (2.3.2). Define the family $\mathscr{F} = \{A \in \mathscr{B}_{\mathbb{R}} : \ \mu(A) = \nu(A)\}$. The class $\mathscr{F}$ verifies the following properties:

1. If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$.
   This is because $\mu(A^c) = \mu(\mathbb{R}) - \mu(A) = \nu(\mathbb{R}) - \nu(A) = \nu(A^c)$.
2. If $A, B \in \mathscr{F}$ and $A \subset B$, then $B \cap A^c \in \mathscr{F}$:
   $\mu(B) = \mu(A) + \mu(B \cap A^c) = \nu(B) = \nu(A) + \nu(B \cap A^c)$. Therefore $\mu(B \cap A^c) = \nu(B \cap A^c)$

3. If $(A_n)_n$ is a monotone sequence in $\mathscr{F}$, then $\lim\limits_{n\to+\infty} A_n \in \mathscr{F}$.

**Theorem 2.15**

Let $A \in \mathscr{F}$, the set : $\tilde{A} = \{B \in \mathscr{B}_{\mathbb{R}} : \ A \cup B, B \cap A^c, A \cap B^c \in \mathscr{F}\}$ is a $\sigma-$algebra.

**Proof** .
We have $\emptyset \in \tilde{A}$. Moreover from the definition of $\tilde{A}$, we have $B \in \tilde{A} \iff A \in \tilde{B}$. Also if $A \in \mathscr{F}$ and $B \in \tilde{A}$, then $A \cap B \in \mathscr{F}$. Therefore $\tilde{A} \subset \mathscr{F}$.
We want to prove first that $\mathbb{R} \in \tilde{A}$. We have
$\mu(\mathbb{R} \cup A) = \mu(\mathbb{R}) = \nu(\mathbb{R}) = \nu(\mathbb{R} \cup A)$, $\mu(\mathbb{R} \cap A^c) = \mu(A^c) = \nu(A^c) = \nu(\mathbb{R} \cap A^c)$ and $\mu(\mathbb{R}^c \cap A) = \mu(\emptyset) = \nu(\emptyset) = 0 = \nu(\mathbb{R}^c \cap A)$. Then $\mathbb{R} \in \tilde{A}$.
In this step we want to prove that $A^c \in \tilde{A}$.
$\mu(A \cup A^c) = \mu(\mathbb{R}) = \nu(\mathbb{R}) = \nu(A \cup A^c)$, $\mu(A \cap (A^c)^c) = \mu(A) = \nu(A) = \nu(A \cap (A^c)^c)$, $\mu(A^c \cap A^c) = \mu(A^c) = \nu(A^c) = \nu(A^c \cap A^c)$. Then $A^c \in \tilde{A}$.
Let $B \in \tilde{A}$. We want to prove that $B^c \in \tilde{A}$

$$\begin{aligned} \mu(A \cup B^c) &= \mu\left((A \cap B) \cup B^c\right) = \mu(A \cap B) + \mu(B^c) \\ &= \nu(A \cap B) + \nu(B^c) = \nu(A \cup B^c) \end{aligned}$$

$\mu(B^c \cap A^c) = \mu(A \cup B)^c = \nu(B \cup A)^c$. Since $B \in \tilde{A}$, then $A \cap B \in \mathscr{F}$. Then $B^c \in \tilde{A}$.
If $(B_n)_n$ is an increasing sequence in $\tilde{A}$ and $B = \lim\limits_{n\to+\infty} B_n$, the sequences $(B_n \cup A)_n$ and $(B_n \cap A^c)_n$ are increasing, so $A \cup B$ and $B \cap A^c$ are elements of $\mathscr{F}$. But the sequence $(A \cap B_n^c)_n$ is decreasing and since $\mu(\mathbb{R}) = \nu(\mathbb{R}) < +\infty$, then $A \cap B^c \in \mathscr{F}$. $\square$

**Corollary 2.16**

For every $A \in \mathscr{C}$, $\tilde{A} = \mathscr{B}_{\mathbb{R}}$.

**Proof** .
If $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$. Therefore $\mu(A \cap B) = \nu(A \cap B)$. On the other hand, since $\mu(A) = \nu(A)$, then $\mu(A \cap B^c) = \nu(A \cap B^c)$ and so $\mu(A^c \cap B) = \nu(A^c \cap B)$. Therefore $\mu(A \cup B) = \nu(A \cup B)$. Since $\tilde{A}$ is a $\sigma-$algebra and since it contains $\mathscr{C}$ then $\tilde{A} = \mathscr{B}_{\mathbb{R}}$. $\square$

**Proof of the theorem (2.3.2)** .

If $A \in \mathscr{B}_{\mathbb{R}}$, then $A \in \tilde{\mathbb{R}}$. Therefore $A \in \mathscr{F}$. $\square$

**Theorem 2.17**

Let $\mu$ and $\nu$ be two measures on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and suppose there is a class $\mathscr{C}$ of measurable sets verifying the following properties:

1. If $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$.
2. $\mathscr{C}$ generates the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$.
3. $\mu(C) = \nu(C) < +\infty$ for every $C \in \mathscr{C}$.
4. There is an increasing sequence $(X_n)_n$ in $\mathscr{C}$ such that $\mathbb{R} = \lim\limits_{n \to +\infty} X_n$.

Then $\mu = \nu$.

**Proof** .
Define $\mu_n$ and $\nu_n$ the measures $\mathscr{B}_{\mathbb{R}}$ as follows: $\mu_n(A) = \mu(A \cap X_n)$ and $\nu_n(A) = \nu(A \cap X_n)$. We deduce from the theorem (2.3.2) that $\mu_n = \nu_n$ and since the measures $(\mu_n)_n$ and $(\nu_n)_n$ are increasing, then $\mu = \nu$, where $\mu$ and $\nu$ are the limits respectively of $(\mu_n)_n$ and $(\nu_n)_n$. $\square$

### 2.3.3 The Lebesgue Measure

**Theorem 2.18**

The restriction of the outer measure $\lambda^*$ on the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ is a measure. We denote this measure by $\lambda$ and called the Lebesgue measure on $\mathbb{R}$.
$\lambda$ is the unique measure on $\mathscr{B}_{\mathbb{R}}$ which verifies the following properties:

1. $\lambda([0, 1]) = 1$
2. $\lambda(A + x) = \lambda(A)$, for all $x \in \mathbb{R}$ and for all $A \in \mathscr{B}_{\mathbb{R}}$. (we say that $\lambda$ is invariant by translation)

**Proof** .
The restriction of the outer measure $\lambda^*$ on the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ is a measure results from the inequality (2.2) if we take the set $X = \cup_{n=1}^{+\infty} A_n$.

**The uniqueness:** Suppose there are two measures $\mu$ and $\nu$ on $\mathscr{B}_{\mathbb{R}}$ that they achieve the proof.
As $\nu[0, \frac{1}{n}[\leq \frac{1}{n}$, then $\nu\{0\} = 0$ and any finite or countable set is a zero set. Also the intervals $[a, b]$, $]a, b]$, $[a, b[$ and $]a, b[$ has the same measure $b - a$.
Let $\mathscr{C}$ be set of finite union of intervals $[a, b[$, where $a, b \in \mathbb{R}$.
The set $\mathscr{C}$ closed under finite intersection and $\mathbb{R} = \bigcup_{n=1}^{+\infty}[-n, n[$. Then $\mu = \nu$ on $\mathscr{C}$ and using the theorem (2.3.2), we have $\mu = \nu$ on $\mathscr{B}_{\mathbb{R}}$. □

**Remark 24 :**
The Lebesgue measure $\lambda$ can be defined on the $\sigma-$algebra $\mathscr{B}^* = \mathscr{B} \cup \mathscr{N}$, where $\mathscr{N}$ is the set null sets. We proved that $\mathscr{B}_{\mathbb{R}} \subset \mathscr{B} \subset \mathscr{B}^*$.

## 2.4 Measurable Functions

In which follow, $\Omega$ is a measurable set in $\mathbb{R}$.

**Definition 2.19**

We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is measurable if $f^{-1}(A) \in \mathscr{B}$ for any Borel set $A$, $(A \in \mathscr{B}_{\mathbb{R}})$.
The of measurable functions on $\Omega$ will be denoted by $\mathscr{M}(\Omega)$ and the set of non negative measurable functions on $\Omega$ will be denoted by $\mathscr{M}^+(\Omega)$.

**Theorem 2.20**

Let $f : \Omega \longrightarrow \mathbb{R}$ be a function. The following properties are equivalent:

1. The function $f$ is measurable,
2. $f^{-1}[a, +\infty[ \in \mathscr{B}$ for every $a \in \mathbb{R}$,
3. $f^{-1}]-\infty, a[ \in \mathscr{B}$, for every $a \in \mathbb{R}$,
4. $f^{-1}]-\infty, a] \in \mathscr{B}$, for every $a \in \mathbb{R}$,
5. $f^{-1}]a, b[ \in \mathscr{B}$, for every $a, b \in \mathbb{R}$,
6. $f^{-1}[a, b[ \in \mathscr{B}$, for every $a, b \in \mathbb{R}$.

This theorem results from the definition of the Borel $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ which generated by any of the following family of sets:

1. $\{[a, +\infty[: \ a \in \mathbb{R}\}$,
2. $\{]a, +\infty[: \ a \in \mathbb{R}\}$,
3. $\{]-\infty, a[: \ a \in \mathbb{R}\}$,
4. $\{]-\infty, a] : \ a \in \mathbb{R}\}$,

5. $\{]a,b[:\ a,b\in\mathbb{R}\}$,
6. $\{[a,b[:\ a,b\in\mathbb{R}\}$,
7. $\{]a,b]:\ a,b\in\mathbb{R}\}$,
8. $\{[a,b]:\ a,b\in\mathbb{R}\}$.

$\square$

**Remark 25 :**
Let $\Omega$ be an open set. Any continuous function $f\colon\Omega\longrightarrow\mathbb{R}$ is measurable.

**Theorem 2.21**

1. If $f\in\mathscr{M}(\Omega)$, then the function $|f|\in\mathscr{M}(\Omega)$.
2. If $(f_n)_n$ is a sequence in $\mathscr{M}(\Omega)$, then the following functions are measurable
   (a) $g=\sup\limits_{n\in\mathbb{N}} f_n$,
   (b) $h=\overline{\lim}_{n\to+\infty} f_n$,
   (c) $k=\underline{\lim}_{n\to+\infty} f_n$.

**Proof** .

1. If $a<0$, then $\Omega=\{x\in\Omega:\ |f(x)|>a\}$
   If $\geq 0$, then
$$\begin{aligned}\{x\in\Omega:\ |f(x)|>a\} &= \{x\in\Omega:\ f(x)>a\}\cup\{x\in\Omega:\ f(x)<-a\}\\ &= f^{-1}(]a,+\infty])\cup f^{-1}([-\infty,-a[)\in\mathscr{B}.\end{aligned}$$
2. $h(x)=\inf\limits_{n\in\mathbb{N}}(\sup\limits_{j\geq n} f_j(x))$
   $\{x\in\Omega:\ g(x)>a\}=\bigcup_{n\in\mathbb{N}}\{x\in\Omega:\ f_n(x)>a\}\in\ BBB$,
$$\{x\in\Omega:\ h(x)>a\}=\bigcap_{n=1}^{+\infty}\bigcup_{j=n}^{\infty}\{x\in\Omega:\ f_j(x)>a\}\in\mathscr{B}$$
3. $k(x)=\sup\limits_{n\in\mathbb{N}}(\inf\limits_{j\geq n} f_j(x))$.
$$\{x\in\Omega:\ k(x)>a\}=\bigcup_{n=1}^{+\infty}\bigcap_{j=n}^{\infty}\{x\in\Omega:\ f_j(x)>a\}\in\mathscr{B}$$

$\square$

**Corollary 2.22**

1. If $f \in \mathscr{M}(\Omega)$, then the functions $f^+ = \sup(f, 0)$ and $f^- = \inf(f, 0)$ are measurable.
2. If $(f_n)_n$ is a pointwise convergent sequence of measurable functions. The limit function $f$, is measurable.
3. Let $(f_n)_n$ be a sequence of measurable functions. The set $C$ of points $x \in \Omega$ where the sequence $(f_n)_n(x)$ has a limit in $\overline{\mathbb{R}}$ is measurable.

**Proof** .

1. The proof results from the theorem (2.4).
2. The function $f = \underline{\lim}_{n\to+\infty} f_n$ is measurable.
3. Let $g = \underline{\lim}_{n\to+\infty} f_n$ and $h = \overline{\lim}_{n\to+\infty} f_n$. The set $D = C^c = \{x \in \Omega :$ $\underline{\lim}_{n\to+\infty} f_n(x) < \overline{\lim}_{n\to+\infty} f_n(x)\}$. For every number $r$, the set

$$D_r = \{x \in \Omega : \; g(x) < r < h(x)\} = \{g(x) < r\} \cap \{h(x) > r\}$$

is measurable, so the set $D = \bigcup_{r\in\mathbb{Q}} D_r$ is also measurable.

$\square$

## 2.5 Exercises

**7-2-1** Let $\mu$ be a measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$$

for every $A, B \in \mathscr{B}_{\mathbb{R}}$.

**7-2-2** Give an example of measure $\mu$ on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and a decreasing sequence $(A_n)_n$ such that $\lim\limits_{n\to+\infty} \mu(A_n) \neq \mu(\lim\limits_{n\to+\infty} A_n)$.

**7-2-3** Let $\varepsilon > 0$. Give a dense open subset of $\mathbb{R}$ and its measure is less than $\varepsilon$.

**7-2-4** Let $A$ be a measurable set in $\mathbb{R}$ of finite measure.
Prove that the function $f(x) = \lambda(A \cap ]-\infty, x])$ is continuous.

**7-2-5** Prove that for each increasing function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is measurable.

**7-2-6** Let $f\colon \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a measurable function.
Prove that the set $\{x \in \mathbb{R} :\ f(x) \neq 0\}$ is measurable.

**7-2-7** Let $(\mathbb{R}, \mathscr{B}, \lambda)$ be the measure space where $\lambda$ is the Lebesgue measure and $\mathscr{B}$ the Lebesgue $\sigma-$algebra.
For every measurable set $A$, we define the function $\mu$ as follows:

$$\mu(A) = \int_A \frac{1}{1+x^2} d\lambda(x).$$

Prove that $\mu$ is a measure.

**7-2-8** Let $f$ be an integrable function on the measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \lambda)$.
Prove that the set $\{x \in \mathbb{R} :\ f(x) = \pm\infty\}$ is a null set.

**7-2-9** Let $f$ be an integrable function such that $\displaystyle\int_E f(x)d\mu(x) \ = 0$ for all measurable set $E$.
Prove that $f = 0$ a.e.

**7-2-10** Prove that the two functions $\sin(x^2)$ and $\cos(x^2)$ are not integrable on $[0, +\infty[$.

# 3 The Lebesgue Integration

## 3.1 Simple Functions

**Definition 3.1**

A function $f\colon \Omega \longrightarrow \mathbb{R}$ is called simple if it is measurable and takes a infinite number of values.

If $f\colon \Omega \longrightarrow \bar{\mathbb{R}}$ is a simple function and if $\{c_1, \dots, c_m\}$ are the different values of $f$, then $f = \sum_{j=1}^{m} c_j \chi_{A_j}$, where $A_j = f^{-1}\{c_j\}$ and the function $f$ is measurable if and only if the sets $A_j$ are measurable for each $j = 1, \dots, m$.

**Theorem 3.2**

Let $f\colon \Omega \longrightarrow \overline{\mathbb{R}}$

1. If $f$ is a bounded measurable function, there exists a sequence of simple functions which converges uniformly on $\Omega$ to $f$.
2. If $f$ is a non-negative measurable function, there exists a sequence of non-negative simple functions which increases to $f$.

**Proof** .

1. Let $M > 0$ such that $|f(x)| < M$ for every $x \in \Omega$. For $(n,k) \in \mathbb{N}_0 \times \mathbb{Z}$ and $-2^n \leq k \leq 2^n - 1$, consider the measurable subsets
$$A_{n,k} = \{x \in \Omega :\ \frac{kM}{2^n} \leq f(x) < \frac{(k+1)M}{2^n}\}$$
and the measurable functions $f_n = \sum_{k=-2^n}^{2^n-1} \frac{kM}{2^n} \chi_{A_{n,k}}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $x_0 \in \Omega$, there exists $k_0$ such that $x_0 \in A_{n,k_0}$. Then $f_n(x_0) = \frac{Mk_0}{2^n}$ and $|f(x_0) - f_n(x_0)| < \frac{M}{2^n}$. Hence, the sequence $(f_n)_n$ converges uniformly on $\Omega$ to $f$.
2. For $n \in \mathbb{N}$, the function $g_n = \inf(f, n) - \frac{1}{n}$ is bounded and measurable, then from the first case there exists a sequence $(f_m)_m$ of simple functions such that $||f_n - g_n||_\infty < \frac{1}{2^n}$. Therefore
$$\lim_{n\to+\infty} f_n = \lim_{n\to+\infty} g_n = \lim_{n\to+\infty} \inf(f, n) = f.$$

$$f_n \le g_n + \frac{1}{2^n} = \inf(f,n) - \frac{1}{n} + \frac{1}{2^n} \le \inf(f,n+1) - \frac{1}{n+1} + \frac{1}{2^{n+1}} \le f_{n+1}.$$

(It suffices to prove that for $n$ big enough $-\frac{1}{n} + \frac{1}{2^n} < -\frac{1}{n+1} + \frac{1}{2^{n+1}}$.)
So the sequence $(f_n)_n$ increasing.

□

## 3.2 The Lebesgue Integration

To define the Lebesgue integral of measurable functions, we first define the integral of non negative positive simple functions. Then we define the integral of non-negative measurable functions using the increasing limit. For arbitrary measurable functions $f$, we use the decomposition $f = f^+ - f^-$ as the difference of two non-negative measurable functions and we extend the definition of the integral to the measurable functions only if one of the integral of $f^+$ or $f^-$ is finite.

**Definition 3.3**

If $f = \sum_{k=1}^{N} c_k \chi_{\{f=c_k\}}$ is a non negative simple function, we define the integral of the function $f$ as follows:

$$\int_\Omega f(x)d\,\lambda(x) = \sum_{k=1}^{N} c_k \lambda(\{f = c_k\}). \tag{3.3}$$

If $A = \{x \in \Omega : \ f(x) = 0\}$ and $\lambda(A) = +\infty$ or if $A = \{x \in \Omega : \ f(x) = +\infty\}$ and $\lambda(A) = 0$, we assume that $0.\infty = 0$.

**Theorem 3.4**

Let $\mathscr{E}^+$ be the set of non negative simple functions defined on $\Omega$. The integral defined on $\mathscr{E}^+$ fulfills the following properties:

1. $\int_\Omega \alpha\ f(x)d\,\lambda(x) = \alpha \int_\Omega f(x)d\,\lambda(x)$ for every $\alpha \in \mathbb{R}^+$ and for each $f \in \ \mathscr{E}^+$.

2. $\int_\Omega (f+g)(x)d\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x)d\lambda(x)$ for every $f, g \in \mathscr{E}^+$.

3. $\displaystyle\int_\Omega f(x)d\,\lambda(x) \leq \int_\Omega g(x)d\,\lambda(x)$ for every $f, g \in \mathscr{E}^+$ such that $f \leq g$.

4. If $(f_n)_n$ is an increasing sequence in $\mathscr{E}^+$ and if $\lim_{n\to+\infty} f_n = f \in \mathscr{E}^+$, then $\displaystyle\int_\Omega f(x)d\,\lambda(x) = \lim_{n\to+\infty}\int_\Omega f_n(x)d\,\lambda(x)$.

**Proof** .
It is obvious that if $\alpha \geq 0$ and $f$ and $g$ are in $\mathscr{E}^+$ then $\alpha f \in \mathscr{E}^+$ and $f+g \in \mathscr{E}^+$.

1. The first property is evident.

2. Let $f$ and $g$ be two elements of $\mathscr{E}^+$ and let $F$ (resp $G$) be the set of values of $f$ (resp of $g$). We have:

$$f = \sum_{a\in F} a\chi_{\{f=a\}}, \quad g = \sum_{b\in G} b\chi_{\{g=b\}}.$$

$$\{f = a\} = \bigcup_{b\in G}\{f = a, g = b\}, \qquad \forall\, a \in F$$

$$\{g = b\} = \bigcup_{a\in F}\{f = a, g = b\}, \qquad \forall\, b \in G$$

$$\int_\Omega f(x)d\,\lambda(x) = \sum_{a\in F} a\lambda\{f = a\} = \sum_{(a,b)\in F\ timesG} a\lambda\{f = a, g = b\}$$

$$\int_\Omega g(x)d\,\lambda(x) = \sum_{b\in G} a\lambda\{g = b\} = \sum_{(a,b)\in F\ timesG} b\lambda\{f = a, g = b\}$$

$$\int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x)d\,\lambda(x) = \sum_{(a,b)\in F\times G}(a+b)\lambda\{f = a, g = b\}$$

$\{f + g = u\} = \bigcup_{(a,b)\in F\times G, a+b=u}\{f = a, g = b\}$. Therefore

$$\lambda\{f + g = u\} = \sum_{(a,b)\in F\times G, a+b=u}\lambda\{f = a, g = b\}.$$

Then

$$\int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x)d\,\lambda(x) = \sum_u u\lambda\{f+g=u\}$$
$$= \int_\Omega (f+g)(x)d\,\lambda(x).$$

3. If $\int_\Omega f(x)d\,\lambda(x) = +\infty$, then $\int_\Omega g(x)d\,\lambda(x) = +\infty$.

The result is evident if $\int_\Omega f(x)d\,\lambda(x) < +\infty$ and the $\int_\Omega g(x)d\,\lambda(x) = +\infty$ .

Suppose that $\int_\Omega f(x)d\,\lambda(x) < +\infty$ and $\int_\Omega g(x)d\,\lambda(x) < +\infty$.
So the sets $\{x \in \Omega : \ f(x) = +\infty\}$ and $\{x \in \Omega : \ g(x) = +\infty\}$ are zero sets. Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ the sets of finite values of $f$ respectively of $g$.

$\tilde{f} = \sum_{j=1}^{n} a_j \chi_{\{x\in\Omega:\ f(x)=a_j\}}$ and $\tilde{g} = \sum_{j=1}^{m} b_j \chi_{\{x\in\Omega:\ g(x)=b_j\}}$. Therefore $\int_\Omega f(x)d\,\lambda(x) = \int_\Omega \tilde{f}(x)d\,\lambda(x)$ and $\int_\Omega g(x)d\,\lambda(x) = \int_\Omega \tilde{g}(x)d\,\lambda(x)$ and $h = \tilde{g} - \tilde{f} \in \mathscr{E}^+$.
We deduce from 2. that

$$\int_\Omega g(x)d\,\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega h(x)d\,\lambda(x) \geq \int_\Omega f(x)d\,\lambda(x)).$$

□

**Lemma 3.5**

Let $(f_n)_n$ be an increasing sequence in $\mathscr{E}^+$. if there exists $g \in \mathscr{E}^+$ such that $g \leq \lim_{n\to+\infty} f_n$, then $\int_\Omega g(x)d\,\lambda(x) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x)$.

**Proof** .
Let $E_y = \{x \in \Omega : \ g(x) = y\}$ for every $y \in g(\Omega)$. To prove the lemma it therefore suffices to prove that for all $y \in g(X)$

$$\int_\Omega g(x)\chi_{E_y}(x)d\,\lambda(x) = y\lambda(E_y) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x).$$

The result is obvious if $y = 0$.
Now suppose that $y > 0$, for every $0 < t < y$, define the sets $A_n = E_y \cap \{x \in$

$\Omega: \ f_n(x) \geq t\}$.
The sequence $(A_n)_n$ is increasing and measurable and $E_y = \lim\limits_{n\to+\infty} A_n$ because for $x \in E_y$, $f_n(x) > t$ for every $n$ big enough.

$$\begin{aligned} t\lambda\{E_y \cap \{x\in\Omega: \ f_n(x) > t\}\} &= \int_\Omega t\chi_{E_y\cap\{x\in\Omega: \ f_n(x)>t\}}(x)d\,\lambda(x) \\ &\leq \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x). \end{aligned}$$

$t\lambda(E_y) \leq \lim\limits_{n\to+\infty} \displaystyle\int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x)$.
This is for every $0 < t < y$. Therefore

$$y\lambda(E_y) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x).$$

To prove (4), we define the function $g = \lim\limits_{n\to+\infty} f_n$.
$f_n \leq g$, for $n \in \mathbb{N}$ and the sequence $\left(\displaystyle\int_\Omega f_n(x)d\,\lambda(x)\right)_n$ is increasing and bounded above by the number $\displaystyle\int_\Omega g(x).\lambda(x)$.
To prove the other inequality, we apply the lemma (3.2). $\square$

**Definition 3.6**

Let $f$ be a non negative measurable function, we define the integral of $f$ by:

$$\int_\Omega f(x)d\,\lambda(x) = \sup\{\int_\Omega g(x)d\,\lambda(x) : g \leq f, \ g \in \mathscr{E}^+\}.$$

This is a non negative real number or $+\infty$.

**Remark 26 :**
If $f$ is a non negative measurable function, by theorem (3.1) there exists an increasing sequence $(f_n)_n$ in $\mathscr{E}^+$ which converges to $f$. We conclude from which above that $\lim\limits_{n\to+\infty} \displaystyle\int_\Omega f_n(x)d\,\lambda(x) \leq \int_\Omega f(x)d\,\lambda(x)$. On the other hand, according to the lemma (3.2) for any function $g \in \mathscr{E}^+$ such that $g \leq f = \lim_{n\to+\infty} f_n$, we have $\displaystyle\int_\Omega g(x)d\,\lambda(x) \leq \lim_{n\to+\infty}\int_\Omega f_n(x)d\,\lambda(x)$. So by definition

( 3.2) $\int_\Omega f(x)d\,\lambda(x) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)d\lambda(x)$. Therefore

$$\int_\Omega f(x)d\,\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x).$$

. This result is not related to the sequence $(f_n)_n$ in $\mathscr{E}^+$ which converges to $f$.

**Theorem 3.7**

If $f$ and $g$ are in $\mathscr{M}^+(\Omega)$ and $\alpha \geq 0$, then

1. $\int_\Omega \alpha f(x)d\,\lambda(x) = \alpha \int_\Omega f(x)d\,\lambda(x)$
2. $\int_\Omega (f+g)(x)d\,\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x))d\,\lambda(x)$
3. If $f \leq g$, then $\int_\Omega f(x)d\,\lambda(x) \leq \int_\Omega g(x)d\,\lambda(x)$.

**Proof** .
For proof, it suffices to take two increasing sequences $(f_n)_n$ and $(g_n)_n$ in $\mathscr{E}^+$ which converge respectively to $f$ and $g$, and we apply the theorem (3.2). $\square$

**Definition 3.8**

Let $f, g$ two functions. We say that $f = g$ outside a zero set or $f = g$ a.e. If the set $\{x \in \Omega : \ f(x) \neq g(x)\}$ is a null set.
Let $A$ be a measurable set. The function $\chi_A = 0$ a.e. if and only if $\lambda(A) = 0$.

**Definition 3.9**

We say that a function $f$ is defined a.e. on $\Omega$, if there exists a null set $N$ so that the function $f$ is defined on $\Omega \setminus N$.

**Definition 3.10**

We say that sequence of functions $(f_n)_n$ on $\Omega$ is convergent a.e. if there exists a function $f$ such that $\{x \in \Omega : f_n(x) \not\longrightarrow f(x)\}$ is a null set.

**Theorem 3.11**

Let $f, g$ be two functions in $\mathscr{M}^+(\Omega)$.

1. $\displaystyle\int_\Omega f(x) d\,\lambda(x) = 0$ If and only if $f = 0$ a.e.
2. If $f = g$ a.e then $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \int_\Omega g(x).\lambda(x)$.

**Proof** .

1. Suppose that $\displaystyle\int_\Omega f(x) d\,\lambda(x) = 0$. Then for every $n \in \mathbb{N}$, the subsets $A_n = \{x \in \Omega : \ f(x) \geq \frac{1}{n}\}$ are measurable and $\chi_{A_n} \leq nf$. Then

$$\int_\Omega \chi_{A_n}(x) d\,\lambda(x) = \lambda(A_n) \leq n \int_\Omega f(x) d\,\lambda(x) = 0$$

   and $\lambda(A_n) = 0$, for every $n \in \mathbb{N}$. Therefore $\{x : \ f(x) \neq 0\} = \bigcup_{n=1}^{+\infty} A_n$ is a null set.

   If $f = 0$ a.e, the set $A = \{x \in \Omega : \ f(x) \neq 0\}$ is a null set and the function $g = \infty.\chi_A$ is a simple and $f \leq g$. As $\displaystyle\int_\Omega g(x) d\lambda(x) = 0$, then $\displaystyle\int_\Omega f(x) d\,\lambda(x) = 0$.

2. suppose that $f \leq g$. the function $h = g - f$ is defined a.e and equal to 0 a.e.

   If $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x) = +\infty$, the result is correct.

   If $\displaystyle\int_\Omega f(x) d\,\lambda(x) < +\infty$, and $\displaystyle\int_\Omega g(x) d\,\lambda(x) < +\infty$, then

   $\displaystyle 0 = \int_\Omega h(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x) - \int_\Omega f(x) d\,\lambda(x).$

The function $h = \inf(f,g)$ is non negative and measurable and $h = f = g$ a.e. As $h \leq f$ then $\int_\Omega f(x)d\,\lambda(x) = \int_\Omega f(x)d\,\lambda(x)$. Also as $h \leq g$, then $\int_\Omega h(x)d\,\lambda(x) = \int_\Omega g(x)d\,\lambda(x)$. We conclude that $\int_\Omega f(x)d\,\lambda(x) = \int_\Omega g(x)d\,\lambda(x)$.

$\square$

**Definition 3.12**

We say that a function $f\colon \Omega \longrightarrow \bar{\mathbb{R}}$ is integrable if the functions $f^+$ and $f^-$ are integrable, where $f^+ = \sup(f,0)$ and $f^- = \sup(-f,0)$. In this case we define the integral of $f$ as:

$$\int_\Omega f(x)d\,\lambda(x) = \int_\Omega f^+(x)d\,\lambda(x) - \int_\Omega f^-(x)d\,\lambda(x).$$

Also if the function $f$ is measurable and $\int_\Omega f^+(x)d\,\lambda(x) < \infty$ or $\int_\Omega f^-(x)d\,\lambda(x) < \infty$ We define the integral of the function $f$ on $\Omega$ by:

$$\int_\Omega f(x)d\,\lambda(x) = \int_\Omega f^+(x)d\,\lambda(x) - \int_\Omega f^-(x)d\,\lambda(x).$$

The set of integrable functions on $\Omega$ is denoted by $\mathcal{L}^1(\Omega)$.

**Theorem 3.13**

The set $\mathcal{L}^1(\Omega)$ is a vector space on $\mathbb{R}$ and the map $f \longmapsto \int_\Omega f(x)d\,\lambda(x)$ is linear on the space $\mathcal{L}^1(\Omega)$ and

$$\left| \int_\Omega f(x)d\,\lambda(x) \right| \leq \int_\Omega \left| f(x) \right| d\,\lambda(x),$$

for every $f \in \mathcal{L}^1(\Omega)$.

**Proof** .

As $\left|f+g\right| \leq |f|+|g|$, for every $f, g \in \mathscr{M}(\Omega)$, then

$$\int_\Omega \left|f(x)+g(x)\right| d\,\lambda(x)| \leq \int_\Omega \left|f(x)\right| d\,\lambda(x) + \int_\Omega \left|g(x)\right|\; d\,\lambda(x).$$

If $f+g \in \mathcal{L}^1(\Omega)$.
$f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$.
Then $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$, and

$$\begin{aligned}\int_\Omega (f+g)^+(x) d\,\lambda(x) \quad &+ \quad \int_\Omega f^-(x) d\,\lambda(x) + \int_\Omega g^-(x)\; d\,\lambda(x) \\ &= \int_\Omega (f+g)^-(x) d\,\lambda(x) + \int_\Omega f^+(x)\; d\,\lambda(x) \\ &\quad + \int_\Omega g^+(x) d\,\lambda(x)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (f+g)(x) d\,\lambda(x) &= \int_\Omega (f+g)^+(x) d\,\lambda(x)) - \int_\Omega (f+g)^-(x)\; d\,\lambda(x) \\ &= \int_\Omega f^+(x) d\,\lambda(x) - \int_\Omega f^-(x)\; d\,\lambda(x) \\ &\quad + \int_\Omega g^+(x) d\,\lambda(x) - \int_\Omega g^-(x) d\,\lambda(x) \\ &= \int_\Omega f(x) d\,\lambda(x) + \int_\Omega g(x) d\,\lambda(x).\end{aligned}$$

The other properties are evident. $\square$

**Corollary 3.14**

1. If the function is $f$ measurable and $a \leq f \leq b$ and $\lambda(\Omega) < +\infty$, then $f \in \mathcal{L}^1(\Omega)$ and $a\lambda(\Omega) \leq \displaystyle\int_\Omega f(x)\; d\,\lambda(x) \leq b\lambda(\Omega)$.

2. If $f \leq g$, where $f \in \mathscr{M}(\Omega)$ and $g \in \mathcal{L}^1(\Omega)$, then $\displaystyle\int_\Omega f(x) d\,\lambda(x) \leq \int_\Omega g(x) d\,\lambda(x)$.

3. If $E$ is a measurable null set, $\displaystyle\int_E f(x) d\,\lambda(x) = 0$ for every measur-

able function $f$.

**Remarks 27 :**

1. If $f$ is an integrable function, then the set $\{x \in \Omega : \ f(x) = \pm\infty\}$ is a null set.

2. We introduce the equivalence relation $\sim$ on $\mathcal{L}^1(X, \mathscr{A}, \mu)$ by setting $f \sim g \iff f = g$ a.e. Thus we may consider the quotient space $L^1(X, \mathscr{A}, \mu) = \mathcal{L}^1(X, \mathscr{A}, \mu)/_\sim$. This space is often abbreviated to $L^1(\mu)$.

## 3.3 The Monotone Convergence Theorem

**Theorem 3.15**

[Monotone Convergence Theorem]
(The theorem is called also the Beppo-Levi's Theorem)
Let $(f_n)_n$ be an increasing sequence of non-negative measurable functions on $\Omega$, then

$$\int_\Omega \lim_{n\to+\infty} f_n(x)d\,\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x).$$

**Proof** .
For every $n \in \mathbb{N}$, there exists a non-negative increasing sequence $(\varphi_{n,j})_j$ in $\mathscr{E}^+$ which converge to $f_n$. For every $j$, define the function $\psi_j = \sup\limits_{1\le n\le j} \varphi_{n,j}$. The sequence $(\psi_j)_j \in \mathscr{E}^+$ is increasing because

$$\psi_j = \sup_{1\le n\le j} \varphi_{n,j} \le \sup_{1\le n\le j} \varphi_{n,j+1} \le \sup_{1\le n\le j+1} \varphi_{n,j+1} = \psi_{j+1}.$$

for every $j \ge n$, $\varphi_{n,j} \le \psi_j$, therefore $f_n = \lim\limits_{j\to+\infty} \varphi_{n,j} \le \lim\limits_{j\to+\infty} \psi_j$. Then $f = \lim\limits_{n\to+\infty} f_n \le \lim\limits_{j\to+\infty} \psi_j$. on the other side inequalities $\varphi_{n,j} \le f_n \le f$ prove that $\psi_j \le f$ and $\lim\limits_{j\to+\infty} \psi_j \le f$. The sequence $(\psi_j)_j$ is increasing in $\mathscr{E}^+$ with limit $f$. Then $\displaystyle\int_\Omega f(x)d\,\lambda(x) = \lim_{j\to+\infty} \int_\Omega \psi_j(x)d\,\lambda(x)$. Moreover $\psi_j \le f_j$, then

$$\lim_{j\to+\infty} \int_\Omega \psi_j(x)d\,\lambda(x) \le \lim_{j\to+\infty} \int_\Omega f_j(x)\ d\,\lambda(x) \le \int_\Omega f(x)d\,\lambda(x).$$

$\square$

### Corollary 3.16

Let $(f_n)_n \in \mathscr{M}^+(\Omega)$ be a sequence, then

$$\int_\Omega \sum_{n=1}^{+\infty} f_n(x) d\lambda(x) = \sum_{n=1}^{+\infty} \int_\Omega f_n(x) d\lambda(x).$$

### Corollary 3.17

Let $f \in \mathscr{M}^+(\Omega)$, then for every $A \in \mathscr{B}_{\mathbb{R}}$, the function

$$\mu(A) = \int_\Omega f(x)\chi_A(x) d\lambda(x)$$

is a measure on $\mathscr{B}_{\mathbb{R}}$.

**Proof** .
Let $(A_n)_n$ be a disjoint sequence of measurable sets ($A_j \cap A_k = \emptyset$ for every $j \neq k$). Then $f\chi_{\cup_n A_n} = \sum_{n=1}^{+\infty} f\chi_{A_n}$ and

$$\begin{aligned}
\mu\left(\bigcup_n A_n\right) &= \int_\Omega f(x)\chi_{\cup_n A_n}(x) d\lambda(x) \\
&= \int_\Omega \sum_{n=1}^{+\infty} f(x)\chi_{A_n}(x) d\lambda(x) \\
&= \sum_{n=1}^{+\infty} \int_\Omega f(x)\chi_{A_n}(x) d\lambda(x).
\end{aligned}$$

The second part of the result is true if the function $f$ is the characteristic function of a measurable set, and therefore is true for every simple function. So if $f$ is a non negative measurable function, there exists an increasing sequence of simple functions which increases to $f$. We get the result using the monotone convergence theorem. $\square$

## 3.4 Fatou's Lemma

**Lemma 3.18**

[Fatou's Lemma]
If $(f_n)_n \in \mathscr{M}^+(\Omega)$, then

$$\int_\Omega \underline{\lim}_{n\to+\infty} f_n(x)d\,\lambda(x) \leq \underline{\lim}_{n\to+\ infty} \int_\Omega f_n(x)d\,\lambda(x).$$

**Proof** .
$\underline{\lim}_{n\to+\infty} f_n \ = \lim_{n\to+\infty}(\inf_{j\geq n} f_j)$. Therefore $\displaystyle\int_\Omega \inf_{j\geq n} f_j(x)\,d\,\lambda(x) \leq \inf_{j\geq n} \int_\Omega f_j(x)d\,\lambda(x)$ and we get the result using the monotone convergence theorem. $\square$

**Example 27 :**
Let $f_n = n^2\chi_{[0,\frac{1}{n}]}$, $\displaystyle\int_{\mathbb{R}} \underline{\lim}_{n\to+\infty} f_n(x)d\lambda(x) = 0$ and $\displaystyle\underline{\lim}_{n\to+\infty} \int_{\mathbb{R}} f_n(x)d\lambda(x) = +\infty$

## 3.5 Dominate Convergence Theorem

**Theorem 3.19**

[Dominate Convergence Theorem or Lebesgue's theorem]
Let $(f_n)_n \in \mathscr{M}(\Omega)$ such that

1. $(f_n)_n$ converges a.e. to a function $f$ defined a.e.
2. There exists a non negative integrable function $g$ so that: $|f_n| \leq g$ a.e. for every $n$.
   Then the sequence $(f_n)_n$ and the function $f$ is integrable and

$$\int_\Omega f(x)\;d\,\lambda(x) \;\; = \;\; \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x).$$

.

**Theorem 3.20**

Let $(f_n)_n \in \mathscr{M}(\Omega)$. Assume that there is a non negative integrable function $g$ such that for every $n$, $|f_n| \leq g$ a.e. Then

$$\int_{\Omega} \underline{\lim} f_n(x) d\lambda(x) \leq \underline{\lim} \int_{\Omega} f_n(x) d\lambda(x) \tag{3.4}$$

$$\int_{\Omega} \overline{\lim} f_n d\lambda(x) \geq \overline{\lim} \int_{\Omega} f_n(x) d\lambda(x) \tag{3.5}$$

If the sequence $(f_n)_n$ converges a.e. on $\Omega$ and its limit is a measurable function $f$ defined a.e., then $f \in L^1(\Omega)$ and

$$\int_{\Omega} f(x) d\lambda(x) \; = \lim_{n \to +\infty} \int_{\Omega} f_n(x) d\lambda(x). \tag{3.6}$$

**Proof** .
As the function $g$ is Integral, the set $\{x \in \Omega : \; |f(x)| = +\infty\}$ is a null set. So we can be substitute the function $g$ by the function $g\chi_{\{x:\; g(x)<+\infty\}}$. This substitution does not change anything about the inequality: $|f_n| \leq g$ a.e.. The sequence $(f_n)_n$ can also be substituted by the sequence $f_n\chi_{\{|f_n|\leq g\}}$. This substitution does not change the value of the integral $\int_{\Omega} f_n(x) d\lambda(x)$ and not in the limit $\lim\limits_{n\to+\infty} f_n$ a.e. So we can assume that $|f_n| \leq g$ on $\Omega$. So the functions $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are integrable on $\Omega$. Using Fatou's lemma on the sequence $f_n + g$, we get

$$\int_{\Omega} \underline{\lim}(f_n + g)(x) d\lambda(x) \leq \underline{\lim} \int_{\Omega} (f_n + g)(x) d\;, \lambda(x).$$

As $\underline{\lim}_{n\to+\infty}(f_n + g) = (\underline{\lim}_{n\to+\infty} f_n\;\;) + g$ on $\Omega$, then

$$\int_{\Omega} \underline{\lim}_{n\to+\infty} f_n(x) d\lambda(x) \leq \underline{\lim}_{n\to+\infty} \int_{\Omega} f_n(x) d\lambda(x).$$

.
and using Fatou's lemma on the sequence $(-f_n + g)_n$, we get

$$\int_{\Omega} \underline{\lim}_{n\to+\infty}(-f_n)(x) d\lambda(x) \leq \underline{\lim}_{n\to+\infty} \int_{\Omega} -f_n(x) d\lambda(x).$$

Then

$$\int_{\Omega} \overline{\lim}_{n\to+\infty} f_n(x) d\lambda(x) \geq \overline{\lim}_{n\to+\infty} \int_{\Omega} f_n(x) d\lambda(x).$$

$\square$

**Example 28 :**
Let $f$ be an Integrable function on $[0, +\infty[$. We want to prove that

$$\lim_{n\to+\infty} \int_0^{+\infty} e^{-n\sin^2 x} f(x) dx = 0.$$

Consider the sequence $(f_n)_n$ defined on $[0, \infty[$ by: $f_n(x) = e^{-n\sin^2 x} f(x)$. Let $A = \{x : \ f(x) = \pm\infty\} \cup \mathbb{N}_0$. For every $x \notin A$, $\lim\limits_{n\to+\infty} f_n(x) = 0$ and $|f_n| \leq |f|$ and the function $f$ is Integrable. Then

$$\lim_{n\to+\infty} \int_0^{+\infty} e^{-n\sin^2 x} f(x)dx = 0.$$

## 3.6 Exercises

**7-3-1** Find the following limits:

(a) $\lim\limits_{n\longrightarrow+\infty}\int_0^n \sqrt{x}\ln x(1-\frac{x}{n})^n dx$,

(b) $\lim\limits_{n\longrightarrow+\infty}\int_{\mathbb{R}} \frac{|\sin x|^{\frac{2}{n}}}{1+x^2}dx$,

(c) $\lim\limits_{n\longrightarrow+\infty}\int_0^{+\infty} \frac{dx}{(1+x^p)^n}, p>0$,

(d) $\lim\limits_{n\longrightarrow+\infty}\int_0^{+\infty} e^{-n\sin^2 x}f(x)\ dx,\ f\in L^1([0,+\infty[)$,

(e) $\lim\limits_{h\longrightarrow 0^+}\int_\alpha^{+\infty} \frac{hf(x)}{h^2+x^2}dx$, and $\lim\limits_{h\longrightarrow 0^+}\int_0^{+\infty} \frac{hf(x)}{h^2+x^2}dx$, Where $f$ is an integrable function on the interval $[0,+\infty[$ and continuous at 0 and $\alpha>0$.

(f) $\lim\limits_{n\to+\infty}\int_0^{+\infty} \frac{\sin(e^x)}{1+nx^2}dx$,

(g) $\lim\limits_{n\to+\infty}\int_0^n (1+\frac{x}{n})^{-n}\cos x dx$,

(h) $\lim\limits_{n\to+\infty}\int_0^n (1+\frac{x}{n})^n e^{-2x}dx$,

(i) $\lim\limits_{n\to+\infty}\int_0^n (1-\frac{x}{n})^{-n} e^{\frac{x}{2}}dx$,

(j) $\lim\limits_{n\to+\infty}\int_0^n (1-\frac{x}{n})^n \frac{1+nx}{n+x}\cos x dx$,

(k) $\lim\limits_{n\to+\infty}\int_0^{+\infty} (1+\frac{x}{n})^{n^2} e^{-nx}dx$.

**7-3-2** Prove that

$$\int_0^{+\infty}\frac{e^{-2x}dx}{1+e^x} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{3+n}$$

and find the value of the series.

**7-3-3** (a) Let $f\in L^1(\mathbb{R})$ and $\alpha>0$.

Prove that $\lim\limits_{n\to+\infty}\frac{f(nx)}{n^\alpha}=0$ a.e. $x\in\mathbb{R}$. (We can integrate the series $\sum\limits_{n=1}^{+\infty}\frac{f(nx)}{n^\alpha}$ on $\mathbb{R}$.)

(b) Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a measurable function and $T-$periodic and $\displaystyle\int_0^T |f(t)|\, dt < +\infty$.

i. Prove that $\displaystyle\lim_{n\to+\infty} \frac{f(nx)}{n^2} = 0$ a.e.

ii. Prove that $\displaystyle\lim_{n\to+\infty} (|\cos nx|)^{\frac{1}{n}} = 1$ a.e. (We can use the function $(\ln|\cos x|)^2$.)

7-3-4 Consider the sequence $(I_n)_n$ defined on $]1, +\infty[$ as follows:

$$I_n(x) = \int_x^{+\infty} \frac{dt}{t^2 \ln^n(t)}.$$

Prove that the sequence $(I_n)_n$ is well defined and find its limit.

7-3-5 Let $f(x) = \dfrac{xe^{-ax}}{1-e^{-bx}}$, with $a > 0$ and $b > 0$.

Prove that the function $f$ is integrable on $[0, +\infty[$ and $\displaystyle\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \frac{1}{(a+nb)^2}$.

7-3-6 Consider the sequence $(I_n)_n$ where $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n(x)\, dx$.

Find the limit of the sequence $(I_n)_n$ and deduce the sum of the following sequence: $U_n = \dfrac{(-1)^n}{2n+1}$ and $U_n = \dfrac{(-1)^n}{n}$.

# 4 Riemann Integral and Lebesgue Integral

## 4.1 The Riemann and Lebesgue Integral

Let $\lambda$ be the Lebesgue measure to on the $\sigma-$algebra $\mathscr{B}$ of measurable functions on the interval $[a, b]$.

If $f\colon [a, b] \longrightarrow \mathbb{R}$ is a Riemann integrable function, then $\displaystyle\int_a^b f(x)dx$ symbolizes the Riemann integral of $f$ on the interval $[a, b]$, and if the function is Lebesgue integrable on $[a, b]$, then $\displaystyle\int_{[a,b]} f(x)d\,\lambda(x)$ symbolizes the Lebesgue integral of $f$ on the interval $[a, b]$.

**Theorem 4.1**

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a Riemann integrable function, then $f$ Lebesgue integrable on $[a,b]$ and

$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

**Proof** .
As the function $f$ is Riemann integrable on $[a,b]$, there exists a sequence $(\sigma_n = \{x_0 = a, \ldots, x_{p_n} = b\})_n$ of partitions of $[a,b]$ such that

$$U(f) = \lim_{n\to+\infty} U(\sigma_n, f) = L(f) = \lim_{n\to+\infty} L(\sigma_n, f).$$

We define two sequences $(g_n)_n$ and $(h_n)_n$ of simple functions as follows:

$$g_n(x) = \begin{cases} m_k = \inf_{t\in[x_k,x_{k+1}[} f(t) & x_k \leq x < x_{k+1} \\ g_n(b) = f(b) \end{cases}$$

$$h_n(x) = \begin{cases} M_k = \sup_{t\in[x_k,x_{k+1}[} f(t) & x_k \leq x < x_{k+1} \\ h_n(b) = f(b) \end{cases}$$

The sequence $(g_n)_n$ is increasing and the sequence $(h_n)_n$ is decreasing. Let $g = \lim_{n\to+\infty} g_n$ and $h = \lim_{n\to+\infty} h_n$. Then

$$U(\sigma_n, f) = \int_a^b h_n(x)dx = \int_{[a,b]} h_n(x)d\,\lambda(x).$$

.

$$L(\sigma_n, f) = \int_a^b g_n(x)dx = \int_{[a,b]} g_n(x)d\,\lambda(x).$$

Since the functions $g$ and $h$ are measurable, using the monotone convergence theorem, we get

$$\lim_{n\to+\infty} \int_a^b g_n(x)dx = L(f) = \int_{[a,b]} g(x)d\,\lambda(x) \tag{4.7}$$

$$\lim_{n\to+\infty} \int_a^b h_n(x)dx = U(f) = \int_{[a,b]} h(x)d\,\lambda(x). \tag{4.8}$$

De deduce from (4.7) and (4.8) that $\displaystyle\int_{[a,b]} h(x)d\lambda(x) = \int_{[a,b]} g(x)d\lambda(x)$.
Then $\displaystyle\int_{[a,b]} (h(x) - g(x))d\lambda(x) = 0$. and since the function $h - g$ is non negative

and integrable, then $h = g$ a.e. and $f = g$ a.e. So the function $f$ is measurable and

$$\int_a^b f(x)dx = \int_{[a,b]} f(x)d\,\lambda(x).$$

$\square$

**Theorem 4.2**

Let $f$ be a bounded function on an interval $[a, b]$.

1. The function $f$ is Riemann integral on $[a, b]$ if and only if the set of points where the function $f$ is not continuous is a null set.

2. Inversely, if the set of points where the function $f$ is not continuous is a null set, $f$ is integrable and

$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

For the proof, we keep the same notations as in theorem (4.1) and we need the following lemma:

**Lemma 4.3**

For every $x \in [a, b] \setminus \left( \bigcup_{n=1}^{+\infty} \sigma_n \right)$, $g(x) = h(x)$ if and only if the function $f$ is continuous at $x$.

**Proof** .
Let $x \in [a, b] \setminus \left(\cup_{n=1}^{+\infty} \sigma_n\right)$ and $\delta_n = ||\sigma_n||$. If the function $f$ is continuous at $x$, for each $\varepsilon > 0$, there exists $\eta > 0$ such that $|f(x) - f(t)| < \varepsilon$ for every $t \in [a, b]$ and $|t - x| < \eta$. Since the sequence $(\delta_n)_n$ converges to 0, there exists $n_0$ such that $\delta_{n_0} < \eta$ for every $n \geq n_0$.
For each partition $\sigma_n$, with $n > n_0$, there exists $k \in \{0, \ldots, p_n - 1\}$ such that $x_k < x < x_{k+1}$.
Then $|f(x) - f(t)| < \varepsilon$ for every $t \in [x_k, x_{k+1}]$. Therefore $h_n(x) = M_k \leq f(x) + \varepsilon$, $g_n(x) = m_k \geq f(x) - \varepsilon$ and $h_n(x) - g_n(x) \leq \varepsilon$. and since this is for each $n \geq n_0$ then $h(x) - g(x) \leq \varepsilon$ for every $\varepsilon > 0$. Then $g(x) = h(x)$.
Inversely: let $x \notin (\bigcup_{n=1}^{\infty} \sigma_n)$, where $g(x) = h(x)$. as $g(x) \leq f(x) \leq h(x)$, then $f(x) = g(x) = h(x)$. So the two sequences $(g_n(x))_n$ and $(h_n(x))_n$ converge and have the same limit $f(x)$.

Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 \leq f(x) - g_n(x) < \varepsilon$ and $0 \leq h_n(x) - f(x) < \varepsilon$, for every $n \geq n_0$. Since $\sigma_{n_0}$ is a partition of the interval $[a,b]$, there exists $k \in \{0, \ldots, p_{n_0} - 1\}$ such that $x \in [x_k, x_{k+1}[$ and

$$h_{n_0}(x) - \varepsilon < f(x) < g_{n_0}(x) + \varepsilon.$$

On the other hend $h_{n_0}(x) = \sup\limits_{t \in [x_k, x_{k+1}]} f(t)$ and $g_{n_0}(x) = \inf\limits_{t \in [x_k, x_{k+1}]} f(t)$. Then $f(t) - \varepsilon < f(x) < f(t) + \varepsilon$ for every $t \in ]x_k, x_{k+1}[$. So the function $f$ is continuous at $x$. □

**Proof of Theorem (4.1)** .

1. The function $f$ is Riemann integral if and only if $U(f) = L(f)$ and this is equivalent to $h = g$ a.e and we deduce the result from the previous lemma.

   The function $f$ is Riemann integral if and only if $h = g$ a.e and this is equivalent to the set $\{x : \ h(x) \neq g(x)\} \cup (\bigcup_{n=1}^{\infty} \sigma_n)$ is a null set, which is equivalent to the function $f$ is continuous a.e on the interval $[a,b]$.

2. If the set where $f$ is not continuous is a null set, then $\lim\limits_{n \to +\infty} g_n(x) = \lim\limits_{n \to +\infty} h_n(x) = f(x)$ at each point of continuity of the function $f$. So the function $f$ is measurable and we can deduce the result from the dominated convergence theorem.

$$\lim_{n \to +\infty} \int_{[a,b]} g_n(x) d\lambda(x) = \int_{[a,b]} f(x) d\lambda(x)$$

$$\lim_{n \to +\infty} \int_{[a,b]} h_n(x) d\lambda(x) = \int_{[a,b]} f(x) d\lambda(x).$$

   So the function $f$ is Riemann integrable and

$$\int_{[a,b]} f(x) d\,\lambda(x) = \int_a^b f(x) dx.$$

□

We now give another proof of the following theorem:

**Theorem 4.4**

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. The function $f$ is Riemann integral if and only if $f$ is continuous a.e. on the interval $[a,b]$.

**Proof** .

1. Assume that the function $f$ is Riemann integral. For any $x \in [a, b]$, define the functions

$$g(x) = \sup_{\delta > 0} \inf_{y \in [a,b], |y-x| \leq \delta} f(y) = \liminf_{y \to x} f(y),$$

$$h(x) = \inf_{\delta > 0} \sup_{y \in [a,b], |y-x| \leq \delta} f(y) = \limsup_{y \to x} f(y).$$

The function $f$ is continuous at $x$ if and only if $g(x) = h(x)$. We have $g \leq f \leq h$. If $\sigma$ is a partition of interval $[a, b]$, then $U(\sigma, g) \leq U(\sigma, f) \leq U(\sigma, h)$ and $L(\sigma, g) \leq s(\sigma, f) \leq s( \; sigma, h)$. But $U(\sigma, f) = U(\sigma, h)$ and $L(\sigma, g) = s(\sigma, f)$. Because for every open interval $]c, d[ \subset [a, b]$,

$$\inf_{x \in ]c,d[} g(x) = \inf_{x \in ]c,d[} f(x), \qquad \sup_{x \in ]c,d[} f(x) = \sup_{x \in ]c,d[} h(x).$$

Therefore

$$L(f) = L(g) \leq U(g) \leq U(f), \quad L(f) \leq L(h) \leq U(h) = U(f).$$

As the function $f$ is Riemann integrable, the two functions $g$ and $h$ are Riemann integrable, and their integral is $\displaystyle\int_a^b f(x)dx$.

If the functions $g$ and $h$ are Lebesgue integrable and have the same integral. But $g \leq h$, therefore $g = h$ a.e. As the function $f$ is continuous at every point where the two functions $g$ and $h$ are equal, the function $f$ is continuous a.e.

2. Assume that the function $f$ is continuous a.e.then for every $n \in \mathbb{N}$, let $\sigma_n$ be the uniform partition of the interval $[a, b]$ and the number of points of $\sigma_n$ is $2^n$.
Let

$$h_n(x) = \sup_{y \in ]c,d[} f(y), \quad g_n(x) = \inf_{y \in ]c,d[} f(y).$$

If there is an open interval $]c, d[$ of the partition $\sigma_n$ and contains the point $x$ and $h_n(x) = g_n(x) = f(x)$ if $x \in \sigma_n$. So the sequences $(g_n)_n$ and $(h_n)_n$ are respectively increasing and decreasing and

$$L_(\sigma_n, f) = \int_a^b g_n(x)dx \qquad U(\sigma_n, f) = \int_a^b h_n(x)dx.$$

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$ at every point $x$ where the function $f$ is continuous, so

$$f = \lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n \quad a.e.$$

Using the dominated convergence theorem

$$\lim_{n\to\infty} \int_a^b g_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty} \int h_n(x)dx.$$

and this proves that $L(f) \geq \int_a^b f(x)dx \geq U(f)$. So the function $f$ is Riemann integrable.

□

## 4.2 Improper Integral and Lebesgue Integral

**Theorem 4.5**

Let $f\colon ]a,b[ \longrightarrow \mathbb{R}$ be a function Lebesgue integrable on every closed and bounded interval of $]a,b[$.
The function $f$ is Lebesgue integrable on $]a,b[$ if and only if the iproper integral $\displaystyle\int_a^b |f(x)|dx$ is convergent. In this case, the Lebesgue and the Riemann integral of $f$ are equal:

$$\int_a^b f(x)dx = \int_{]a,b[} f(x)d\,\lambda(x).$$

**Proof** .
Suppose that the integral $\displaystyle\int_a^b |f(x)|dx$ is convergent. Let $(a_n)_n$ and $(b_n)_n$ two sequences in $]a,b[$ so that the sequence $(a_n)_n$ is decreasing and tends to $a$ and the sequence $(b_n)_n$ is increasing and tends to $b$. We define the sequence of functions $(F_n)_n$ as follows:

$$F_n(x) = |f(x)|\chi_{[a_n,b_n]}.$$

. The sequence $(F_n)_n$ is increasing, measurable. Its limit isthe function $|f|\chi_{]a,b[}$. So the function $f$ is measurable and by using the dominate convergence theorem we get:

$$\lim_{n\to+\infty} \int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_{]a,b[} |f(x)|d\,\lambda(x).$$

On the other hand, using the previous theorem $\displaystyle\int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$.
Using the previous definition, we get:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_a^b |f(x)|dx.$$

So the function $f$ is Lebesgue integrable. To prove that the two integrals are equal, we define the sequence of functions $(g_n)_n$ as follows: $g_n = f\chi_{[a_n,b_n]}$. The sequence $(g_n)_n$ is convergent and its limit is the function $f\chi_{]a,b[}$. The functions $g_n$ are integrable and $|g_n| \leq |f|\chi_{[a,b]}$. Using the dominate convergence theorem

$$\lim_{n\to+\infty}\int_{]a,b[} g_n(x)d\,\lambda(x) = \int_{]a,b[} f(x)\ d\,\lambda(x).$$

**Inversely:** If the function $f$ is Lebesgue integrable on the interval $]a,b[$, the the function $|f|$ is also Lebesgue integrable on the interval $]a,b[$.
Let $(a_n)_n$ and $(b_n)_n$ two sequences in $]a,b[$ as previous. Using the dominate convergence theorem

$$\lim_{n\to+\infty}\int_{]a,b[} F_n(x)d\,\lambda(x) = \int_{]a,b[} |f(x)|d\,\lambda(x) < +\infty.$$

On the other hand $\displaystyle\int_{]a,b[} F_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$, So the limit $\displaystyle\lim_{n\to+\infty}\int_{a_n}^{b_n} |f(x)|dx$ in $\mathbb{R}$ and $\displaystyle\int_a^b |f(x)|dx < +\infty$. $\square$

## 4.3 Exercises

7-4-1 (a) Calculate the integral of the following functions on $[0,1]$.

$$f(x) = \frac{1}{\sqrt{x}} + \chi_{\mathbb{Q}}(x) \qquad \begin{aligned} g(x) &= \sin x;\ x \in \mathbb{Q} \\ g(x) &= \cos x;\ x \in \mathbb{R} \setminus \mathbb{Q} \end{aligned}$$

(b) Find whether the following functions are integrable on $]0,+\infty[$?

$$f(x) = \frac{\sin x}{x} \qquad h(x) = \frac{1}{x(1+|\ln x|)^2}$$

$$g(x) = \frac{1}{(1+x^2)\sqrt{|\sin x|}}$$

7-4-2 Calculate the following integrals:

(a) $\displaystyle\int_{[0,+\infty[} e^{-[x]} d\lambda(x)$, Where $[x]$ is the entire part of the real number $x$.

(b) $\displaystyle\int_{[0,\pi]} f(x) d\lambda(x)$, where $f(x) = \sin x$ if $x \in \mathbb{Q} \cap [0,\pi]$ and $f(x) = \cos x$ Otherwise.

(c) $\displaystyle\int_{[0,1]} \chi_{\mathbb{Q}}(x) d\lambda(x)$.

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