1

poisson processes

* pb 5.1.3 p. 228

$X \sim \text{poisson}(\alpha)$, $Y \sim \text{poisson}(\beta)$

المطلوب

$$pr\{X=k \mid N=n\}$$

$$= pr\{X=k \mid X+Y=n\}$$

Ans: $X \sim \text{Poisson}(\alpha)$, $Y \sim \text{poisson}(\beta)$

$\therefore X+Y \sim \text{poisson}(\alpha+\beta)$

$$pr\{X=k \mid N=n\} = pr\{X=k \mid X+Y=n\}$$

$$= \frac{pr\{X=k\} \cap pr\{X+Y=n\}}{pr\{X+Y=n\}}$$

$$= \frac{pr\{X=k\} \cap pr\{Y=n-k\}}{pr\{X+Y=n\}}$$

$$= \frac{\cancel{e^{-\alpha}} \alpha^k / k! \cdot \cancel{e^{-\beta}} \beta^{n-k} / (n-k)!}{\cancel{e^{-(\alpha+\beta)}} (\alpha+\beta)^n / n!}$$

$$= \alpha^k \beta^{n-k} \left(\frac{1}{\alpha+\beta}\right)^n \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \left(\frac{\beta}{\alpha+\beta}\right)^{n-k}$$

$\therefore pr\{X=k \mid N=n\} = \binom{n}{k} p^k (1-p)^{n-k}$, $p = \frac{\alpha}{\alpha+\beta}$ which is Binomial distn

Note that $1-p = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$

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pb 5.1.2 p. 228

Given $P_k = Pr\{X=k\}$ for poisson (λ)

verify
$$\begin{cases} P_0 = e^{-\lambda} \\ P_k = \left(\frac{\lambda}{k}\right) P_{k-1} \end{cases}$$
علاقة تكرارية
recurrence relation

Ans: $\because P_k = \dfrac{e^{-\lambda}\lambda^k}{k!} \quad , k = 0, 1, 2, \ldots$

$$\therefore P_0 = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} \quad , k = 0$$

and $\because P_k = \dfrac{\lambda}{k} \cdot \dfrac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}$

$$\therefore P_k = \left(\frac{\lambda}{k}\right) P_{k-1}$$

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* pb 5.1.7 p.229

$X(t)$ is the # of customers (arrivals), $\lambda = 2$

$X(0)=0$ — 2 Customers — $X(1)=2$

$t=0$ — $t=1$

$\lambda t = 2(1) = 2$

a) $pr\{X(1)=2\}$

$= pr\{X(1)-X(0)=2\}$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{2^2 e^{-2}}{2!} = 2e^{-2}$$

b) $pr\{X(1)=2 \text{ and } X(3)=6\}$

$X(0)=0$ — 2 Cust. — $X(1)=2$ — 4 Cust. — $X(3)=6$

$t=0$ — 1 — 3

$\lambda t = 2(1) = 2$ $\quad$ $\lambda t = 2(2) = 4$

$$= pr\{\underbrace{X(1)-X(0)}=\underbrace{2}_{k},\ \underbrace{X(3)-X(1)}=\underbrace{4}_{k}\}$$

indep. r.v.s

$$= \frac{2^2 e^{-2}}{2!} \cdot \frac{4^4 e^{-4}}{4!}$$

$$= 2e^{-2} \cdot \frac{64}{6} e^{-4} = \frac{64}{3} e^{-6}$$

$pr(x,y) = pr(x)\,pr(y)$ if X and Y are indep. r.v.s

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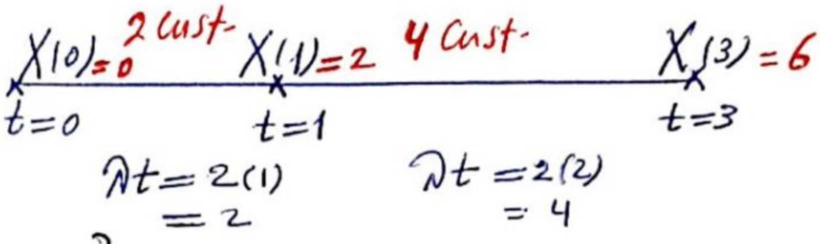
d) $pr\{X(3)=6 \mid X(1)=2\}$

$$= pr\{X(3)-X(1)=4 \mid X(1)-X(0)=2\}$$



$$= pr\{X(3)-X(1)=4\}$$

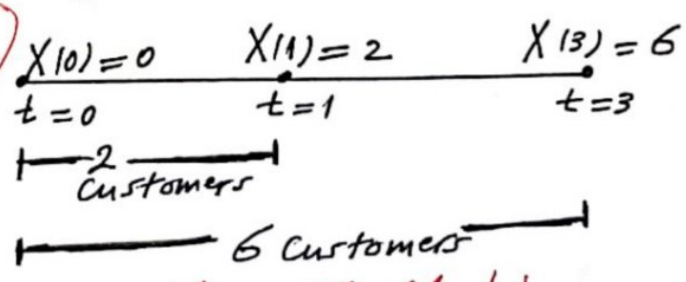
$$= \frac{4^4 e^{-4}}{4!}$$

$$= \frac{64}{6} e^{-4} = \frac{32}{3} e^{-4}$$

$X(3)-X(1)$ and $X(1)-X(0)$ are 2 indep. r.v

Note that $pr(A|B) = pr(A)$ where A and B are independent events.

c) $pr\{X(1)=2 \mid X(3)=6\}$



كأنما التجربة أجريت 6 مرات، وكان احتمال وصول الزبائن عند $t=1$ هو

$$p = \frac{2}{6} = \frac{1}{3}$$

و احتمال عدم الوصول هو

$$q = \frac{4}{6} = \frac{2}{3}$$

$X \sim Bin(6, \frac{1}{3})$ (n = 6, p = $\frac{1}{3}$)

$$\Rightarrow X \sim Bin(6, \tfrac{1}{3})$$

$$\therefore pr\{X(1)=2 \mid X(3)=6\} = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$$

$$= \frac{80}{243} \simeq 0.3292$$

* حل آخر للفقرة (c)

$$pr\{X(1)=2 \mid X(3)=6\}$$

$$= \frac{pr\{X(1)=2 \text{ and } X(3)=6\}}{pr\{X(3)=6\}}$$

$$= \frac{(2e^{-2}/2!)\cdot(4^4 e^{-4}/4!)}{6^6 e^{-6}/6!}$$

$$= \frac{64/3 \ \cancel{e^{-6}}}{64.8 \ \cancel{e^{-6}}}$$

$\simeq 0.3292$ #

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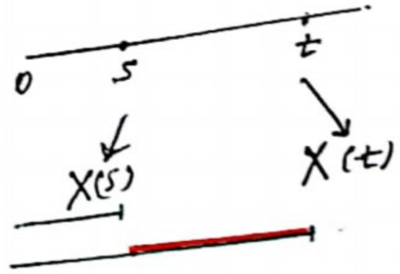
* Pb 5.1.4 p. 228

$X(t)$ represents the # of customers that have arrived up to time t.

(a) $$\Pr\{X(t)=k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k=0,1,2,\dots$$

(b) The conditional probability



$$\Pr\{X(t)=n+k \mid X(s)=n\}$$
$$= \Pr\{X(t)-X(s)=k\}$$
$$= \Pr\{X(t-s)=k\}$$
$$= \frac{[\lambda(t-s)]^k e^{-\lambda(t-s)}}{k!}$$

and the expected value

$$E[X(t)X(s)] = E[X(t)]\,E[X(s)] + E[X(s)]$$
$$= \lambda t \cdot (\lambda s) + \lambda s$$
$$= \lambda^2 t s + \lambda s$$

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* Pb 5.1.5 p. 229

The conditional probability function of $X|\lambda$ is

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

and the prob. density fn of λ is

$$f(\lambda) = \theta e^{-\theta\lambda}$$

The prob. fn of X is

$$f(x) = \int_0^\infty f(x|\lambda) f(\lambda)\, d\lambda$$
$$f(x) = \frac{\theta}{x!}\int_0^\infty \lambda^x e^{-(1+\theta)\lambda}\, d\lambda$$

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let $\boxed{(1+\theta)\lambda = u}$

$\Rightarrow \lambda = \frac{u}{1+\theta}, \quad d\lambda = \frac{du}{1+\theta}$

Then $f(x) = \frac{\theta}{x!}\int_0^\infty \left(\frac{u}{1+\theta}\right)^x e^{-u}\,\frac{du}{1+\theta}$

$= \frac{\theta}{x!}\cdot\frac{1}{(1+\theta)^{x+1}}\int_0^\infty u^x e^{-u}\,du$

$= \frac{\theta}{x!}\,\frac{1}{(1+\theta)^{x+1}}\,\Gamma(x+1)$

$= \frac{\theta}{\cancel{x!}}\,\frac{1}{(1+\theta)^{x+1}}\,\cancel{x!}$

$\int_0^\infty t^\alpha e^{-t}\,dt = \Gamma(\alpha+1) = \alpha!, \quad \alpha = 0, 1, \ldots$

Gamma function

$\therefore \boxed{f(x) = \frac{\theta}{(1+\theta)^{x+1}}} = \frac{\theta}{1+\theta}\cdot\frac{1}{(1+\theta)^x}$

$\therefore f(x) = (1-p)p^x, \quad x = 0, 1, \ldots, \quad p = \frac{1}{1+\theta}$

which is the prob. mass fn for X.

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* pb 5.1.6 p. 229

(a) $X(t)$ represents the # messages that arrive at the telegraph office at anytime.

$\therefore Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$

$\therefore Pr\{X(12) - X(8) = 0\} = \frac{(3\times 4)^0 e^{-3(4)}}{0!} = e^{-12}$

$= 6.1442 \times 10^{-6}$

where $\lambda = 3$, $t = 12 - 8 = 4$ and $k = 0$.

(b) Consider T is the r.v that represents the time at which the first afternoon message arrives.

So, we can write

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$Pr\{T>t\}$

$= Pr\{$The first afternoon message arrives after t units of time$\}$

$= Pr\{X(t) - X(12) = 0\}$

$= \dfrac{[3(t-12)]^0 e^{-3(t-12)}}{0!}$

12:00 P.M. — t — 12:00 A.M.

$= e^{-3(t-12)}$, which is the survival/reliability function for T.

Also, $Pr\{T \leq t\} = 1 - Pr\{T > t\}$

$= 1 - e^{-3(t-12)}$

$= 1 - e^{-3x}, \quad x = t - 12$

which is the cumulative distribution function for T.

$\therefore T \sim exp(3)$

i.e. $T \sim$ exponential dist$^{\underline{n}}$ with parameter equals 3.

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*pb 5.1.9 p. 229

For the poisson process $\{X(t);\ t \geq 0\}$

$E[X(t)] = \lambda t$ and $Var[X(t)] = \lambda t$

(a) $E[X(1)] = 2(1) = 2$,

$E[X(2)] = 2(2) = 4$, where $\lambda = 2$.

(b) To get $E[\{X(1)\}^2]$

$\because Var[X(1)] = E[\{X(1)\}^2] - [E\{X(1)\}]^2$

$2 = E[\{X(1)\}^2] - 2^2$

$\therefore E[\{X(1)\}^2] = 2 + 4 = 6$

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* Another solution for (b)

$$E[\{X(1)\}^2] = E[X(1)]E[X(1)] + E[X(1)]$$

$$= 2(2) + 2 = 6$$

, See pb 5.1.4 p.228

(c) $E[X(1)X(2)]$

$$= E[X(1)]E[X(2)] + E[X(1)]$$

$$= 2(4) + 2 = 10.$$

, See pb 5.1.4 p.228