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Tut. Session (11)

Brownian Motion (Wiener process)

- Pb 8.1.5 p. 402 Textbook

Determine the covariance functions for the stochastic processes

(a) $U(t) = e^{-t} B(e^{2t})$ for $t \geq 0$

(b) $V(t) = (1-t) B(t/(1-t))$, for $0 < t < 1$

(c) $W(t) = t\, B(1/t)$, with $W(0) = 0$

where $B(t)$ is a standard Brownian motion.

Ans:

(a)

$$Cov[U(s), U(t)]$$

$$= Cov[e^{-s} B(e^{2s}), e^{-t} B(e^{2t})]$$

$$= e^{-(s+t)} Cov[B(e^{2s}) B(e^{2t})]$$

$$= e^{-(s+t)} \min\{e^{2s}, e^{2t}\} \quad \text{Cov. prop. for S.B.M.}$$

$$= e^{-(s+t)} e^{2s} = e^{s-t} \quad \text{if } s<t$$

$$= e^{-(s+t)} e^{2t} = e^{t-s} \quad \text{if } t<s$$

Thus, $Cov[U(s), U(t)] = e^{-|s-t|}$ #

(b)

$$Cov[V(s), V(t)]$$

$$= Cov[(1-s) B(s/(1-s)), (1-t) B(t/(1-t))]$$

$$= (1-s)(1-t) Cov[B(s/(1-s)), B(t/(1-t))]$$

$$= (1-s)(1-t) \min\{s/(1-s), t/(1-t)\}$$

$$= \cancel{(1-s)}(1-t)\left(\frac{s}{\cancel{1-s}}\right) \quad \text{if } 0<s<t<1$$

$$= s(1-t), \quad 0<s<t<1$$

$$= (1-s)\cancel{(1-t)}\left(\frac{t}{\cancel{1-t}}\right) \quad \text{if } 0<t<s<1$$

$$= t(1-s), \quad 0<t<s<1$$ #

(c)

$$Cov[W(s), W(t)]$$

$$= Cov[s\,B(1/s), t\,B(1/t)]$$

$$= st\, Cov[B(1/s), B(1/t)]$$

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$\therefore \operatorname{Cov}[W(s), W(t)]$

$= st \min\{1/s, 1/t\}$

$= \cancel{s}t(\frac{1}{\cancel{s}})$, if $\frac{1}{s} < \frac{1}{t}$

$= t$, $0 \le t < s$

$= s\cancel{t}(\frac{1}{\cancel{t}})$, if $\frac{1}{t} < \frac{1}{s}$

$= s$, $0 \le s < t$

$= \min\{s,t\}$ for $s,t \geq 0$ #

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Evaluate $E[e^{\lambda B(t)}]$ for an arbitrary constant λ and standard Brownian motion $B(t)$.

Ans:

For $X(t) = e^{\lambda B(t)}$, $\ln X(t) = \lambda B(t)$

$\Rightarrow \ln X(t) \sim N(0, \lambda^2 t)$ scale invariance prop.

which is a Brownian motion with mean 0 and Variance $\lambda^2 t$

$\Rightarrow X(t) \sim \text{lognormal}(0, \lambda^2 t)$

If $B(t) \sim N(0,t)$ then $\lambda B(t) \sim N(0, \lambda^2 t)$, $B(t)$ is called scale-invariant

$\Rightarrow E[X(t)] = e^{(\mu + \sigma^2/2)t}$

$= e^{0 + \frac{1}{2}\lambda^2 t}$

$\therefore E[e^{\lambda B(t)}] = \exp[\frac{1}{2}\lambda^2 t]$. #

Also, we can find the Variance as

$Var[X(t)] = e^{(\cancel{2\mu}_0 + \sigma^2)t}(e^{\sigma^2 t} - 1)$

$\therefore Var[e^{\lambda B(t)}] = e^{\lambda^2 t}[e^{\lambda^2 t} - 1]$ #

Note:

The process of the form $X(t) = e^{\mu t + \sigma B(t)}$ where $B(t)$ is a standard Brownian motion (i.e $B(t) \sim N(0,t)$) is called a Geometric Brownian motion, also, it can be written as

$X(t) = e^{\mu t + \sigma W(t)}$

where $W(t)$ is referred to Wiener process ($W(t) \sim N(0,t)$).

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For $n = 0, 1, \dots$ Show that

(a) $B(n)$ and (b) $[B(n)]^2 - n$ are martingales

Ans:

(a)

(1) $E[|B(n)|] = E[B(n)], \quad n = 0, 1, \dots$

$= 0 < \infty$, where $B(n) \sim N(0, n)$

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(2) $E[B(n+1) \mid B(0), \dots, B(n)]$

$= E[B(n+1) - B(n) \mid B(0), \dots, B(n)] + B(n)$

$= 0 + B(n)$

$= B(n)$

$\therefore$ From (1) and (2), we obtain that $B(n)$, $n = 0, 1, \dots$ is a martingale.

(b)

(1) $E\left[\left|[B(n)]^2 - n\right|\right] = E\left[[B(n)]^2 - n\right], \quad n = 0, 1, \dots$

$= E\left[[B(n)]^2\right] - E(n)$

$= Var[B(n)] - n$

$= n - n = 0 < \infty$

$Var[B(n)] = E\left[[B(n)]^2\right] - \cancel{[E[B(n)]]^2}$ 0

(2) $E\left[[B(n+1)]^2 - (n+1) \mid [B(n)]^2 - n\right]$

$= E\left[[B(n+1)]^2 - [B(n)]^2 - 1 + [B(n)]^2 - n \mid [B(n)]^2 - n\right]$

$= [B(n)]^2 - n + E\left[[B(n+1)]^2 - [B(n)]^2\right] - 1$

$= [B(n)]^2 - n + E[B(n+1)]^2 - E[B(n)]^2 - 1$

$= [B(n)]^2 - n + Var[B(n+1)] - Var[B(n)] - 1$

$= [B(n)]^2 - n + \cancel{n+1} - \cancel{n} - \cancel{1}$

$= [B(n)]^2 - n$

Remember

$E[g(X) \mid Y = y] = E[g(X)]$ where X and Y are indep.

$E[h(Y) \mid Y = y] = h(y)$

Thus, from (1) and (2), we get $[B(n)]^2 - n$, $n = 0, 1, \dots$ is a martingale. #