Tut. Session (7)

Pb*

Let $\{X_n\}$ be a Markov chain with state space $S=\{1,2\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\|\begin{array}{cc} 0.5 & 0.5 \\ 1 & 0 \end{array}\right\| \end{array}$$

Find $pr\{X_5 = 2 | X_3 = 1\}$

Ans:

$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix}$$

$$pr\{X_5 = 2 | X_3 = 1\} = p_{12}^2 = 0.25$$

$$= 25\%$$

Pb 3.2.2 p. 85 Textbook

Given in Pb.

$$\mathbf{P=} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array}\right\| \end{array}$$

Calculate $pr\{X_n = 0 | X_0 = 0\}$ for $n = 0,1,2,3,4$

Ans:

At $n = 0$

$$pr\{X_0 = 0 | X_0 = 0\} = P_{00}^0 = 1$$

At $n = 1$

$$pr\{X_1 = 0 | X_0 = 0\} = P_{00}^1 = 0$$

At $n = 2$

$$pr\{X_2 = 0 | X_0 = 0\} = P_{00}^2$$

$$P^2 = P.P$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$\therefore pr\{X_2 = 0 | X_0 = 0\} = P_{00}^2 = 0.5$$

At $n = 3$

$$pr\{X_3 = 0 | X_0 = 0\} = P_{00}^3$$

$$P^3 = P.P^2$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.375 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.375 & 0.25 \end{bmatrix}$$

$$\therefore pr\{X_3 = 0 | X_0 = 0\} = P_{00}^3 = 0.25$$

Finally, at $n = 4$

$$pr\{X_4 = 0 | X_0 = 0\} = P_{00}^4$$

$$P^4 = P^2.P^2$$

$$= \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3750 & 0.3125 & 0.3125 \\ 0.3125 & 0.3750 & 0.3125 \\ 0.3125 & 0.3125 & 0.3750 \end{bmatrix}$$

$\therefore pr\{X_4 = 0 | X_0 = 0\} = P_{00}^4 = 0.375$

Pb 3.2.5 p. 85 Textbook

3.2.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array}\right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1 | X_0 = 0\}.$$

Solution:

$P_r(X_3 = 1 | X_1 = 0) = P_{01}^2 = 0.27$

$$P^2 = \left\|\begin{array}{ccc} 0.27 & 0.27 & 0.46 \\ 0.24 & 0.24 & 0.52 \\ 0.21 & 0.21 & 0.58 \end{array}\right\|$$

$P_r(X_2 = 1 | X_0 = 0) = P_{01}^2 = 0.27$

Pb 3.2.6 p. 86 Textbook

A Markov chain $X_0,\ X_1,\ X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array}\right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the following probabilities

i) $pr\{X_2 = 0\}$ ii) $pr\{X_3 = 0\}$

Ans:

i) $\because\ pr\{X_2 = 0\} = pr\{X_2 = 0 | X_0 = 0\}\, pr\{X_0 = 0\}$

$+ pr\{X_2 = 0 | X_0 = 1\}\, pr\{X_0 = 1\}$

$\therefore\ pr\{X_2 = 0\} = P_{00}^2 P_0 + P_{10}^2 P_1\ ,\quad P_0 = 0.5,\ P_1 = 0.5$

$$P^2 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.40 & 0.19 & 0.41 \\ 0.40 & 0.18 & 0.42 \end{bmatrix}$$

$$\therefore \; pr\{X_2 = 0\} = (0.44)(0.5) + (0.4)(0.5)$$

$$= 0.42$$

ii) $\because \; pr\{X_3 = 0\} = pr\{X_3 = 0 | X_0 = 0\} \, pr\{X_0 = 0\}$

$$+ pr\{X_3 = 0 | X_0 = 1\} \, pr\{X_0 = 1\}$$

$$= P_{00}^3 P_0 + P_{10}^3 P_1 \;, \quad P_0 = 0.5, \; P_1 = 0.5$$

$$P^3 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.40 & 0.19 & 0.41 \\ 0.40 & 0.18 & 0.42 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4120 & 0.1820 & 0.4060 \\ 0.4200 & 0.1810 & 0.3990 \\ 0.4200 & 0.1820 & 0.3980 \end{bmatrix}$$

$$\therefore \; pr\{X_3 = 0\} = (0.412)(0.5) + (0.42)(0.5)$$

$$= 0.416$$

Pb 3.3.3 p. 93 Textbook

Given in Pb

$\Pr\{\xi_n = 0\} = 0.5, \;\; \Pr\{\xi_n = 1\} = 0.4, \;\; \Pr\{\xi_n = 2\} = 0.1.$ and suppose s=0 and $S = 3$.

Determine the transition probability matrix for the end-of-period inventory level X_n.

Ans:

The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|ccccc|} & -1 & 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0 & 0 & 0.1 & 0.4 & 0.5 \\ 1 & 0.1 & 0.4 & 0.5 & 0 & 0 \\ 2 & 0 & 0.1 & 0.4 & 0.5 & 0 \\ 3 & 0 & 0 & 0.1 & 0.4 & 0.5 \end{array}$$

where,

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\}$$
$$= \begin{cases} \Pr(\xi_{n+1} = 3 - j), \ i \le 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), \ 0 < i \le 3 & \text{without replenishment} \end{cases}$$

Pb 3.3.4 p. 93 Textbook

Given in Pb

Suppose that $s = 0$ and $S = 3$ and that the probability distribution for the demand is $\Pr\{\xi_n = 0\} = 0.1$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.3$ and $\Pr\{\xi_n = 3\} = 0.2$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n.

Ans:

The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccccc|} & -2 & -1 & 0 & 1 & 2 & 3 \\ \hline -2 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ -1 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 1 & 0.2 & 0.3 & 0.4 & 0.1 & 0 & 0 \\ 2 & 0 & 0.2 & 0.3 & 0.4 & 0.1 & 0 \\ 3 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \end{array}$$

where,

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\}$$
$$= \begin{cases} \Pr(\xi_{n+1} = 3 - j), \ i \le 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), \ 0 < i \le 3 & \text{without replenishment} \end{cases}$$