PHYS 404
HANDOUT 1-*Legendre Functions*

1. You are given the equation

$$\left\{\nabla^2 + k^2 - U(r)\right\}\Psi\left(\mathbf{r}\right) = 0, \quad k = const$$

Show that, by using the method of separating variables, the differential equation is equivalent to the following three partial differential equations:

$$\frac{1}{r^2}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left[k^2 - U(r) - \frac{\lambda}{r^2}\right]R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{\mu^2}{\sin^2\theta}\right)\Theta = 0$$

$$\frac{d^2\Phi}{d^2\phi} + \mu^2\Phi = 0$$

2. Let the Laplace equation $\nabla^2\Psi\left(\mathbf{r}\right) = 0$ and a solution of it given as $\Psi\left(\mathbf{r}\right) = R(r)\Theta(\theta)$. Show the functions R, Θ satisfy the equations:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda R = 0$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \lambda\left(\sin\theta\right)\Theta = 0.$$

3. Show that the differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{\mu^2}{\sin^2\theta}\right)\Theta = 0$$

leads to the associated Legendre equation if we consider the substitutions: $x = \cos\theta$, $\lambda = \nu\left(\nu+1\right)$, $u(x) = \Theta\left(\theta\right)$.

4. Produce the Legendre polynomials using the generating function.

5. Show that for the Legendre polynomials we have:

$$\left(n+1\right)P_{n+1}(x) = \left(2n+1\right)xP_n(x) - nP_{n-1}(x).$$

6. Show that for the Legendre polynomials we have:

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x).$$

7. Show that for the Legendre polynomials we have:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

8. Show that for the Legendre polynomials we have:

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)$$
$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x)$$
$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$

9. Show that $P_{2n}(0) = \dfrac{(-1)^n (2n)!}{2^{2n}(n!)^2}$ and $P_{2n+1}(0) = 0$. Also show that $P_n(1) = 1$ and that $P_n(-1) = (-1)^n$.

10. Show the parity property: $P_n(x) = (-1)^n P_n(-x)$. *This plays an important role in quantum mechanics. For central forces the index n is a measure of the orbital angular momentum, thus linking parity and orbital angular momentum.*

11. Show that, if $m \neq n$, then $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$.

12. Show that $\int_{-1}^{1}\left[P_n(x)\right]^2 dx = \dfrac{2}{2n+1}$.

13. If $P_0(x) = 1$, $P_1(x) = x$, calculate $P_2(x)$, $P_3(x)$.

14. Use any of the recurrence relations to calculate the integral: $I_n = \int_0^1 P_{2n+1}(x)\,dx$.

15. Derive the Schaefli integral and show that it satisfies Legendre's equation.

16. Show by mathematical induction that $P'_{n+1}(1) = \dfrac{1}{2}(n+1)(n+2)$

17. Show by direct integration of the Schaefli integral that $P_n(1) = 1$.

18. Show how we can use Legendre polynomials in calculating the potential created by a point-like electric charge and by an electric dipole.