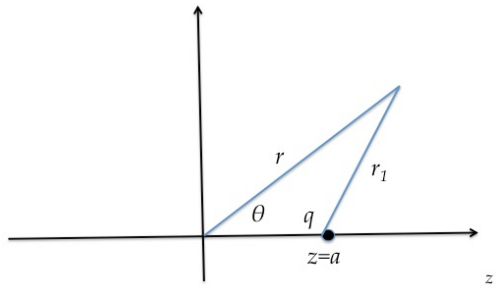
PHYS 404
HANDOUT 4- *Applications of Legendre Functions in Physics*

1. Consider an electric charge q placed on the z-axis at $z=a$ as shown in figure. Express the potential at a point in terms of the spherical polar coordinates r and θ (consider $r > a$).



2. Repeat the previous problem having two charges $\pm q$ at the positions $\pm a$ respectively.

3. Solve the differential equation $\nabla^2\psi = 0$ in spherical coordinates for a physical system symmetric with respect to ϕ.

4. Calculate the electric potential (a) in the interior and (b) in the exterior of a hollow sphere, if the upper hemisphere is kept at a constant potential V_0 and the lower potential is at potential zero.

5. Calculate the electric potential when we place a neutral conducting sphere inside a uniform electric field.

6. Calculate the electrostatic potential produced by a ring carrying a total electric charge q.

7. The amplitude of a scattered wave is given by

$$f(\theta) = \lambda \sum_{\ell=0}^{\infty} (2\ell+1)\exp(i\delta_\ell)\sin(\delta_\ell)P_\ell(\cos\theta).$$

Here θ is the angle of scattering, ℓ the angular momentum and δ_ℓ the phase shift produced by the central potential that is doing the scattering. The total cross section is $\sigma_{tot} = \int f^*(\theta) f(\theta) d\Omega$. Show that:

$$\sigma_{tot} = 4\pi\lambda^2 \sum_{\ell=0}^{\infty} (2\ell+1)\sin^2(\delta_\ell).$$

8. (i) Show that if we consider $x = \cos\theta$ the associated Legendre differential equation

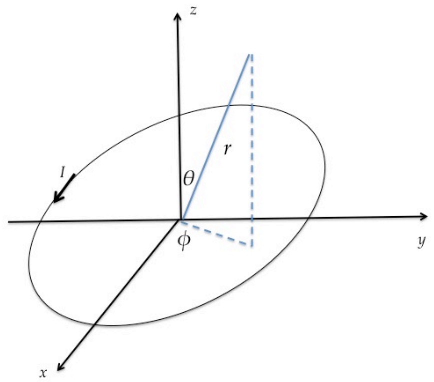
$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]u=0,$$

takes the form:

$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0$$

where $u(x)=\Theta(\theta)$.

(ii) A steady current flows through the loop shown in the picture.



The magnetic vector potential is given by: relation: $\mathbf{A}=\hat{\phi}A_\phi(r,\theta)$. Find the quantity $A_\phi(r,\theta)$ if it satisfies the differential equation:

$$\frac{\partial^2 A_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial A_\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\cot\theta A_\phi\right)=0.$$

(Hint: use separation of variables and consider $A_\phi(r,\theta)=R(r)\Theta(\theta)$)

9. A function $f(r,\theta,\phi)$ may be expressed as a Laplace series

$$f(r,\theta,\phi)=\sum_{\ell=0}^{\infty}a_{\ell m}r^\ell Y_\ell^m(\theta,\phi)$$

show that the average over a sphere centered at the origin is:

$$\left\langle f(r,\theta,\phi)\right\rangle_{sphere}=f(0,0,0).$$

10. Show that $\sin\theta P_n^{'}(x)=P_n^1(\cos\theta)$.

11. Use the generating function for the associated Legendre functions to show that:

$$P_{2n}^0(0)=0,\quad P_{2n+1}^0(0)=(-1)^n\frac{(2n+1)!}{\left(2^n n!\right)^2}.$$