Mid 2 / actu 464

Question 1. (3+2+3=8 marks)
Let S be a compound random variable with claim X and claim number N. Suppose further that $f_X(1) = 0.5$, $f_X(2) = 0.1$, $f_X(3) = 0.4$ and $N \sim \text{Poisson}(4)$.

a) Compute the mean, variance and mgf of S.
b) Compute, using Sparse algorithm, the mass function of S at $s = 0,1,2,3$.
c) Conclude the stop-loss premium $\pi(3) = E(S-3)_+$.

a) $E(X) = (1)(0.5) + (2)(0.1) + (3)(0.4) = 0.5 + 0.2 + 1.2 = \boxed{1.9}$

$E(X^2) = (1)^2(0.5) + (2)^2(0.1) + (3)^2(0.4) = 0.5 + 0.4 + 3.6 = \boxed{4.5}$

$Var(X) = 4.5 - (1.9)^2 = \boxed{0.89}$

- $E(S) = E(X)E(N) = (1.9)(4) = \boxed{7.6}$
- $Var(S) = Var(X)E(N) + Var(N)(E(X))^2$
  $= (0.89)(4) + (4)(1.9)^2 =$
  $= 3.56 + 14.44 = \boxed{18}$
- $m_X(t) = E\, e^{tX} = 0.5e^t + 0.1e^{2t} + 0.4e^{3t}$

$M_S(t) = m_N(\ln(m_X(t)))$
$= e^{4(0.5e^t + 0.1e^{2t} + 0.4e^{3t} - 1)}$.

b) $S = N_1 + 2N_2 + 3N_3$. $N_1 \sim Poi(2)$, $N_2 \sim Poi(0.4)$, $N_3 \sim Poi(1.6)$

$f(0) = P((0,0,0)) = \boxed{e^{-4}}$.

$f(1) = P((1,0,0)) = \boxed{2e^{-4}}$

$f(2) = P((2,0,0)) + P((0,1,0))$
$= (2 + 0.4)e^{-4} = \boxed{2.4e^{-4}}$

$f(3) = P((3,0,0)) + P((1,1,0)) + P((0,0,1))$
$= \left(\frac{2^3}{3!} + \frac{2^1}{1!}\frac{0.4^1}{1!} + \frac{1.6^1}{1!}\right)e^{-4} = \boxed{3.73e^{-4}}$
$= \left(\frac{4}{3} + 0.8 + 1.6\right)e^{-4}$

c)

s	$f(s)$	$F(s)$	$\pi(s)$
0	e^{-4}	e^{-4}	7.6
1	$2e^{-4}$	$3e^{-4}$	$6.6 + e^{-4}$
2	$2.4e^{-4}$	$5.4e^{-4}$	$5.6 + 4e^{-4}$
			$4.6 + 9.4e^{-4}$

$\pi(3) = 4.6 + 9.4e^{-4}$

Question 2. (4 marks)
Let S be an aggregate loss random variable with a discrete frequency distribution N (number of claims) defined by the table below.

n	0	1	2
P(N = n)	0.3	0.5	0.2

The severity claim X has a geometric distribution with parameter p = 0.2. Find the mass function of S using the convolution formula.

$$f_S(s) = \sum_{n=0}^{\infty} f_X^{*n}(s) f_N(n)$$
$$= 0.3 f_X^{*0}(s) + 0.5 f_X^{*1}(s) + 0.2 f_X^{*2}(s).$$

$$f_X^{*0}(s) = \begin{cases} 1, & s=0 \\ 0, & s\neq 0 \end{cases}, \quad f_X^{*1}(s) = p(1-p)^s, \quad s=0,1,\dots$$

$$f_X^{*2}(s) = (s+1)p^2(1-p)^s.$$

- $f(0) = 0.3 + 0.1 + (0.2)^3 = 0.408$
- $s=1,\dots \quad f(s) = 0.5\, p(1-p)^s + 0.2\,(s+1)p^2(1-p)^s$
$= (0.1 + 0.008(s+1))(0.8)^s$
$= (0.108 + 0.008s)(0.8)^s$

Question 3. (4 marks)
Let S a compound random variable with claim X and claim number N. Suppose further that $f_X(2) = 0.3 = 1 - f_X(3)$ and $f_N(j) = j/3$ for j = 1,2. Compute the mass function of S.

	2	3
2	4	5
3	5	6

s	f_X^{*0}	f_X^{*1}	f_X^{*2}	
0	1			0
1				
2		0.3	-	0.3/3 = 0.1
3		0.7		0.7/3 = 0.23
4			0.09	0.06
5			0.42	0.28
6			0.49	0.32
n	0	1	2	
		1/3	2/3	

Question 4. (3+3=6 marks)

The frequency distribution of an aggregate loss S follows a binomial distribution with parameters $n = 4$ and $p = 0.5$. Loss amount has the following mass distribution function: $f_X(1) = 0.2$ and $f_X(3) = 0.8$. Using Panjer's recursive formula for the binomial distribution with $a = -1$ and $b = 5$,

a) Compute the mass function f of S at $s = 1,2,3$.

b) Compute the recursive formula for the mass function f of S for all $s \geq 4$.

$a=-1,\ b=5 \qquad p(1)=0.2,\ p(3)=0.8$

a) $f(s) = \sum_{h=1}^{s} \left(-1 + \frac{5h}{s}\right) p(h) f(s-h)$

$f(1) = (-1+5)\,p(1) f(0)$; $f(0) = q_0 = (0.5)^4 = 0.0625$

$= 4(0.2)(0.5)^4 = \boxed{0.05}$

$f(2) = \left(-1+\frac{5}{2}\right) p(1) f(1) + (-1+5)\,p(2) f(0)$.

$= \frac{3}{2}(0.2)(4)(0.2)(0.5)^4 = \boxed{0.015}$

$f(3) = \left(-1+\frac{5}{3}\right) p(1) f(2) + \left(-1+\frac{10}{3}\right) p(2) f(1)$

$+ (-1+5)\,p(3) f(0)$

$= \frac{2}{3}(0.2)(0.015) + 4(0.8)(0.5)^4$.

$= 0.002 + 0.2 = \boxed{0.202}$

b) $f(s) = \left(-1+\frac{5}{s}\right) p(1) f(s-1) + \left(-1+\frac{15}{s}\right) p(3) f(s-3)$

$= 0.2\left(-1+\frac{5}{s}\right) f(s-1) + 0.8\left(-1+\frac{15}{s}\right) f(s-3)$

Theorem 3.5.1 (Panjer's recursion)

Consider a compound distribution with integer-valued non-negative claims with pdf $p(x)$, $x = 0,1,2,\ldots$, for which, for some real a and b, the probability q_n of having n claims satisfies the following recursion relation

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1}, \quad n = 1,2,\ldots \tag{3.26}$$

Then the following relations for the probability of a total claim equal to s hold:

$$f(0) = \begin{cases} \Pr[N=0] & \text{if} \quad p(0) = 0; \\ m_N(\log p(0)) & \text{if} \quad p(0) > 0; \end{cases}$$
$$f(s) = \frac{1}{1 - ap(0)} \sum_{h=1}^{s} \left(a + \frac{bh}{s}\right) p(h) f(s-h), \quad s = 1,2,\ldots \tag{3.27}$$

Question 5. (3 marks)
Let S a compound random variable with claim X~Uniform(c, d) and claim number N~Geometric(p). Compute the mean, variance and mgf of S.

- $E(S) = E(N)\,E(X) = \frac{1-p}{p} \cdot \frac{c+d}{2}$.
- $var(S) = var(X)\,E(N) + var(N)\,(E(X))^2$
  $= \frac{(d-c)^2}{12}\,\frac{1-p}{p} + \frac{1-p}{p^2}\,\frac{(d+c)^2}{4}$.
- $m_S(t) = \dfrac{p}{1-(1-p)\dfrac{e^{dt}-e^{ct}}{(d-c)t}}$.

Bonus Question. (3 marks)
Assume that S is compound Poisson distributed with parameter $\lambda = 12$ and $\chi^2(4)$ distributed claims. Using the normal approximation, we can find that $P(S < 85) \approx \phi(k)$, where ϕ is the cdf of a standard normal distribution. Find k.

$E(S) = 12(4) = 48$.
$var(S) = 12(16+8) = 12(24) = 288$
$\sigma = \sqrt{288} = 16.97$
$P(S < 85) = P\left(\frac{S-48}{16.97} < \frac{85-48}{16.97}\right)$
$= \phi(k)$
with $k = \frac{85-48}{16.97} = \frac{37}{16.97} = 2.18$