Chapter ①: Bayesian estimation.

① Continuous mixing of distributions:

* Discrete mixing:

X, Y random variables, $q \in (0,1)$.

we say that Z is a q-mixture of X and Y

if $F_Z(t) = q\, F_X(t) + (1-q)\, F_Y(t)$.

$\rightarrow \quad E(Z^m) = q\, E(X^m) + (1-q)\, E(Y^m)$:

$\rightarrow \quad m_Z(t) = q\, m_X(t) + (1-q)\, m_Y(t)$

* Continuous mixing:

X random variable depending on a parameter λ. We suppose that Λ is a random variable.

We say that X is a Λ-mixture distribution if:

$$f_X(x) = \int f_{X|\Lambda}(x|\lambda)\, f_\Lambda(\lambda)\, d\lambda.$$

Example ①: Suppose: $\Lambda \sim Exp(1)$, $X|\Lambda=\lambda \sim Exp(\lambda)$.

Compute $E(X)$, $E(X^2)$, $f_X(x)$?

$$\boxed{E(X) = E[E(X|Y)] = \int E(X|Y=y)\, f_Y(y)\, dy}$$

* $E(X) = \int E(X|\Lambda=\lambda)\, f_\Lambda(\lambda)\, d\lambda = E\frac{1}{\Lambda}$

$$= \int_0^\infty \frac{1}{\lambda}\, e^{-\lambda}\, d\lambda$$

$Y \sim \text{Gamma}(\alpha, \beta)$.

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad y \geq 0.$$

$$\int_0^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\Gamma(\alpha)}{\beta^\alpha}.$$

$$\alpha = 0, \beta = 1 \to \int_0^\infty \frac{1}{y} e^{-y} dy = \frac{\Gamma(0)}{1} = \Gamma(0).$$

$*$ $$E(X^2) = E\,E(X^2|\Lambda) = E\,\frac{2}{\Lambda^2}.$$

$$= \int_0^\infty \frac{2}{\lambda^2} e^{-\lambda} d\lambda = 2\,\Gamma(-1).$$

$*$ $$f_X(x) = \int \underbrace{f_{X|\Lambda}(x|\lambda)}\, \underbrace{f_\Lambda(\lambda)}\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x} \quad e^{-\lambda} d\lambda$$

$$= \int_0^\infty \lambda e^{-(x+1)\lambda} d\lambda = \frac{\Gamma(2)}{(x+1)^2}$$

$$f_X(x) = \frac{\Gamma(2)}{(x+1)^2} \quad ; \quad x > -1.$$

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let X the Λ-mixture distribution:

- $$F_X(x) = \int F_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda.$$
- $$E(X^m) = E\,E(X^m|\Lambda) = \int E(X^m|\Lambda=\lambda) f_\Lambda(\lambda) d\lambda.$$
  $$= \iint x^m f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda\, dx.$$
- $$Var(X) = E(Var(X|\Lambda)) + Var(E(X|\Lambda)).$$

Exaple (1):

$$Var(X) = E\left(\frac{1}{\Lambda^2}\right) + Var\left(\frac{1}{\Lambda}\right)$$

$$= E\left(\frac{2}{\Lambda^2}\right) - \left(E\left(\frac{1}{\Lambda}\right)\right)^2.$$

$$= EX^2 - E(X)^2.$$

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Example ① : $Y \sim N(0,1)$; $X|Y=y \sim N(y,1)$.

Compute $E(X)$, $E(X^2)$, $m_X(t)$, $f_X(t)$?

- $E(X) = E\,E(X|Y) = E(Y) = 0$.
- $E(X^2) = E\,\underline{E(X^2|Y)} = E(1+Y^2) = 1+1 = 2$.
- $m_X(t) = E\,e^{tX} = E\,\underline{E(e^{tX}|Y)} = E\left(e^{tY+t^2/2}\right)$

$$\boxed{Z \sim N(\mu,\sigma^2), \quad m_Z(t) = e^{\mu t + \sigma^2 t^2/2}}$$

$= e^{t^2/2}\, e^{t^2/2} = e^{t^2} \longrightarrow X \sim N(0,2)$.

- $f_X(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$.

Example ③ : $Y \sim \text{Uniform}(0,1)$; $X|Y=y \sim \text{Geo}(y)$.

Compute $E(X)$, $E(X^2)$, $m_X(t)$; $f_X(t)$?

- $E(X) = E\,E(X|Y) = E\left(\frac{1-Y}{Y}\right)$

$= \int_0^1 \frac{1-y}{y}\, 1\, dy = \cancel{y - y^2/2 \Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}}$

$= \infty$.

- $E(X^2) = \infty$.
- $m_X(t) = E\,E(e^{tX}|Y) = E\,\frac{Y}{Y-(1-Y)e^t}$

$= \int_0^1 \frac{y}{y-(1-y)e^t}\, 1\, dy = \int_0^1 \frac{y}{(1+e^t)y - e^t}\, dy$.

$= \frac{1}{1+e^t} \int_0^1 \frac{((1+e^t)y - e^t) + e^t}{(1+e^t)y - e^t}\, dy$.

$= \frac{1}{1+e^t} \int_0^1 \left(1 + \frac{e^t}{(1+e^t)y - e^t}\right) dy$

$= \frac{1}{1+e^t}\left(y + \frac{e^t}{(1+e^t)} \ln|(1+e^t)y - e^t| \right)\Big|_0^1$

$$= \frac{1}{1+e^t}(1+0) - \frac{1}{1+e^t}\left(0 + \frac{te^t}{1+e^t}\right)$$

$$= \frac{1}{1+e^t} - \frac{te^t}{(1+e^t)^2}.$$

- $\cancel{f_X(x) = \int_0^1 f_{X|P}(x|y)\, 1\, dy}$

$$=$$

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② Basics of Bayesian inference:

Ex: Risk $X \sim Exp(\lambda)$.

$x_1, x_2, \dots, x_n$ n observations (exposures).

in Bayesian inference, we suppose that the Coefficient λ is a random variable.

- We suppose n observations $X_1, X_2, \dots, X_n$. We suppose a parameter θ to estimate and θ the random variable.

* Prior distribution of θ : $\pi(\theta)$.

[Coin $\leftrightarrow$ p , $\overset{T}{0}, 0, \overset{H}{1}, 1, 0, 1, 1, \dots, 1$

$P \approx [0.49, 0.52]$], $P = \frac{1}{2}$ with prob 1.

* model distribution:

$$f_{X|\theta}(x|\theta) = f_{X_1|\theta}(x_1|\theta) \cdot f_{X_2|\theta}(x_2|\theta) \cdots f_{X_n|\theta}(x_n,\theta).$$

* Joint distribution:

$$f_{X,\theta}(x,\theta) = f_{X|\theta}(x|\theta)\, \pi(\theta).$$

* Posterior distribution of θ:

$$\pi_{\theta|X}(\theta|x) = \frac{f_{X,\theta}(x,\theta)}{f_X(x)}.$$

* marginal distribution of X:

$$f_X(x) = \int f_{X|\theta}(x|\theta)\,\pi(\theta)\,d\theta.$$

* predictive distribution of a new observation y, given data x.

$$f_{Y|X}(y|x) = \int f_{Y|\theta}(y|\theta)\,\pi_{\theta|X}(\theta|x)\,d\theta.$$

Example 1: A Coin is tossed 5 times: T, H, H, T, T.

let $X = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$ and $P \sim unif(0.4, 0.6)$.

H → 1

T → 0

$X|_{P=p} \sim Ber(p)$.

a) what is the prior distribution of P.

b) Find the model distribution of X.

c) —— joint distribution of (X, P).

d) —— marginal —— of X.

e) —— posterior —— of P.

f) —— predictive —— of Y.

$x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0,\ x_5 = 0.$

a) $\pi(p) = \frac{1}{0.2} = 5 \quad, p \in (0.4, 0.6)$

b) $f_{X|P}(x, p) = f_{X_1|P}(x_1|p) \cdots f_{X_5|P}(x_5|p)$

$= (1-p)^3 p^2.$

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c) $f_{\underline{X},P}(\underline{x}, p) = 5p^2(1-p)^3, \quad p \in (0.4, 0.6)$

d) $f_{\underline{X}}(\underline{x}) = \int_{0.4}^{0.6} 5p^2(1-p)^3 \, dp$. $\quad u = 1-p$.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= \int_{0.4}^{0.6} 5p^2(1-3p+3p^2-p^3) \, dp$.

$= 0.03$ . .

e) $\pi_{P|X}(p|\underline{x}) = \dfrac{5p^2(1-p)^3}{0.03}, \quad p \in (0.4, 0.6)$.

$\sim$ Beta $(3, 4)$.

f) $f_{Y|\underline{X}}(y|\underline{x}) = \int f_{Y|P}(y|p) \, \pi_{P|\underline{X}}(p|\underline{x}) \, dp$.

$f_{Y|\underline{X}}(1|\underline{x}) = \int_{0.4}^{0.6} p \, \dfrac{5p^2(1-p)^3}{0.03} \, dp = \dfrac{1}{2}$.

$X_1 \quad X_2 \quad \ldots \quad X_5 \quad \boxed{Y_1 = X_6} \quad \boxed{Y_2 = X_7}$

$f_{Y|\underline{X}}(0|\underline{x}) = \ldots\ldots = \dfrac{1}{2}$.

Example ②: (i) we have a portfolio of 100 iid risks.

(ii) The number of claims for each risk, follows a Poisson distribution with parameter λ.

(iii) $\pi(\lambda) = \dfrac{(50\lambda)^4 e^{-50\lambda}}{6\lambda}; \quad \lambda > 0$. $\sim$ Gamma $(4, 50)$.

During one year, we have:

# of claims	# of risks
0	90
1	7
2	2
3	1
Total	100

Data: $N_1, N_2, \ldots\ldots, N_{100}$

$\underbrace{0,0,\ldots\ldots,0}_{90}, \underbrace{1,\ldots,1}_{7}, 2, 2, 3.$

a) model distribution:

$$f_{\underline{N}|\Lambda}(\underline{n}|\lambda) = f_{N_1|\Lambda}(n_1|\lambda)\cdots\cdots f_{N_{100}|\Lambda}(n_{100}|\lambda)$$

$$= \left(\frac{e^{-\lambda}\lambda^0}{0!}\right)^{90}\left(e^{-\lambda}\frac{\lambda^1}{1!}\right)^7\left(e^{-\lambda}\frac{\lambda^2}{2!}\right)^2\left(e^{-\lambda}\frac{\lambda^3}{3!}\right)^1 = \frac{e^{-100\lambda}\lambda^{14}}{24}$$

b) $$f_{\underline{N},\Lambda}(\underline{n},\lambda) = f_{\underline{N}|\Lambda}(\underline{n}|\lambda)\cdot\pi(\lambda)$$

$$= \frac{(50\lambda)^4}{6\lambda}e^{-50\lambda}\left(e^{-\lambda}\right)^{90}\left(e^{-\lambda}\lambda\right)^7\left(e^{-\lambda}\frac{\lambda^2}{2}\right)^2\left(e^{-\lambda}\frac{\lambda^3}{6}\right)$$

$$= \frac{390625}{9}e^{-150\lambda}\lambda^{17}.$$

c) $$f_{\underline{N}}(\underline{n}) = \int f_{\underline{N},\Lambda}(\underline{n},\lambda)\,d\lambda = \frac{390625}{9}\cdot\frac{\Gamma(18)}{150^{18}}$$

d) $$\pi_{\Lambda|\underline{N}}(\lambda|\underline{n}) = \frac{150^{18}}{\Gamma(18)}e^{-150\lambda}\lambda^{17} \sim Gamma(18,150).$$

e) $$f_{Y|\underline{N}}(y|\underline{n}) = \int f_{Y|\Lambda}(y|\lambda)\,\pi_{\Lambda|\underline{N}}(\lambda,\underline{n})\,d\lambda.$$

$$= \int e^{-\lambda}\frac{\lambda^y}{y!}\frac{150^{18}}{\Gamma(18)}\lambda^{17}e^{-150\lambda}\,d\lambda.$$

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$$= \frac{150^{18}}{\Gamma(18)} \int \frac{1}{y!} \lambda^{y+17} e^{-151\lambda} d\lambda .$$

$$= \frac{150^{18}}{\Gamma(18)} \frac{1}{y!} \frac{\Gamma(y+18)}{(151)^{y+18}} .$$

$$\boxed{\int_0^{\infty} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda = \frac{\Gamma(\alpha)}{\beta^{\alpha}} .}$$

$$\Gamma(n) = (n-1)!$$

f) $E(Y \mid \underline{N} = \underline{n}) = \sum_{y=0}^{\infty} y \, f_{Y \mid \underline{N}=\underline{n}}(y \mid \underline{n}) .$

Let a function g:

* $E(g(Y) \mid \underline{X} = \underline{x}) = \int g(y) \int f_{Y \mid \theta}(y \mid \theta) \, \pi_{\theta \mid \underline{X}}(\theta \mid \underline{x}) \, d\theta \, dy .$

$$= \int E \, g(Y \mid \theta = \theta) \, \pi_{\theta \mid \underline{X}}(\theta \mid \underline{x}) \, d\theta .$$

Example : * $X \mid_{\theta=\theta} \sim$ Pareto( ) , $\qquad \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} ; x \geq x_0 .$

$f_{X \mid \theta}(x \mid \theta) = \frac{\theta}{(x+\theta)^2} ; \quad x > 0 . \qquad x_0 -$

* $\pi_{\theta}(\cancel{1}) = \pi_{\theta}(3) = \frac{1}{2} .$

* $x_1 = 5 , \quad P(X_2 \geq 8 \mid X_1 = 5) = ?$

$g(y) = \mathbb{1}(y \geq 8)$

$P(X_2 \geq 8 \mid X_1 = 5) = \int P(X_2 \cancel{\geq} 8 \mid \theta = \theta) \, \pi_{\theta \mid X_1}(\theta \mid x_1 = 5) \, d\theta .$

$\rightarrow P(X_2 \geq 8 \mid \theta = \theta) = \int_8^{\infty} \frac{\theta}{(\theta + x)^2} dx = - \frac{\theta}{\theta + x} \Big|_8^{\infty} = \frac{\theta}{\theta + 8} .$

$\rightarrow \cancel{\pi_{\theta \mid X_1}(\theta \mid x_1) = \frac{\theta}{(5+\theta)^2}} \pi(\theta)$

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$\rightarrow$ $f_{\underline{X}|\theta}(\underline{x}|\theta) = f_{X_1|\theta}(x_1|\theta) = \dfrac{\theta}{(5+\theta)^2}$;

$\rightarrow$ $f_{\underline{X};\theta}(\underline{x},\theta) = \dfrac{\theta}{(5+\theta)^2}\pi(\theta)$.

$\rightarrow$ $f_{X_1}(x_1) = \displaystyle\int \frac{\theta}{(5+\theta)^2}\pi(\theta)\,d\theta = \sum_{\theta=1,3}\frac{\theta}{(5+\theta)^2}\pi(\theta)$

$= \dfrac{1}{72} + \dfrac{3}{128} = 0.0373$.

$\rightarrow$ $\pi_{\theta|X_1}(\theta|x_1) = 26.79\,\dfrac{\theta}{(5+\theta)^2}\pi(\theta)$.

$$P(X_2 \geq 8 | X_1 = 5) = \sum_{\theta=1,3}\frac{\theta}{\theta+8}\,26.79\,\frac{\theta}{(5+\theta)^2}\pi(\theta)$$

$$= 26.79\left(\frac{1}{9}\cdot\frac{1}{72} + \frac{3}{11}\cdot\frac{3}{128}\right) = 0.21$$

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③ Bayesian Parameter estimation:

* $X_1, X_2, \ldots, X_n \,|\, \Theta = \theta$ are iid.
* Prior distribution $\pi(\theta)$ of Θ.

• A Bayes estimate of a parameter θ is the value that minimizes the expected loss function $l(\theta)$, given the posterior distribution of θ.

• We have 3 loss function:
- $\rightarrow$ Squared-error loss: $l(\theta) = (\hat{\theta} - \theta)^2$.
- $\rightarrow$ Absolute loss: $l(\theta) = |\hat{\theta} - \theta|$.
- $\rightarrow$ Zero-one loss: $l(\theta) = \begin{cases} 1 & ; \theta = \hat{\theta} \\ 0 & ; \theta \neq \hat{\theta} \end{cases}$.

• Theorem:

loss function	estimator
SEL	$\hat{\theta} = E(\theta \| \underline{X})$.
AL	$\hat{\theta}$ = median of $\pi_{\theta\|\underline{X}}$.
ZOL	$\hat{\theta}$ = mode of $\pi_{\theta\|\underline{X}}$.

⑧

Example: Suppose: $\pi_{\theta|\underline{x}}(\theta|\underline{x}) = \cancel{\frac{1}{\tilde{x}} e^{-\theta/\tilde{x}}} \sim Exp\left(\frac{1}{\tilde{x}}\right)$.

$$= \frac{1}{\tilde{x}} e^{-\theta/\tilde{x}} \; ; \; \theta \geq 0.$$

Compute the Bayes estimate using SEL, AL, ZOL.

a) $$\hat{\theta} = E(\theta | \underline{X} = \underline{x}) = \int \theta \, \pi_{\theta|\underline{X}}(\theta|\underline{x}) \, d\theta.$$

$$= \int_0^\infty \theta \frac{1}{\tilde{x}} e^{-\theta/\tilde{x}} d\theta = \tilde{x}.$$

b) $$\int_M^\infty \frac{1}{\tilde{x}} e^{-\theta/\tilde{x}} d\theta = \frac{1}{2} = S(M) = e^{-\frac{M}{\tilde{x}}} = 1 - F(M)$$

$$M = \hat{\theta} = \tilde{x} \ln(2) = -\tilde{x} \ln\left(\frac{1}{2}\right)$$

c) $$\hat{\theta} = 0.$$

Example: * X_1 = number of claims:

$X|Q = q \sim Bin(2, q)$.

* $\pi(q) = 4q^3; \quad 0 \leq q \leq 1.$

* $x_1 = 1 \; ; \; x_2 = 1.$

? Bayes estimate of X_3?

$\longrightarrow$ $$E(X_3 | x_1 = 1, x_2 = 1) = \int E(X_3 | Q = q) \, \pi_{Q|\underline{X}}(q|\underline{x}) \, dq.$$

* $E(X_3 | Q = q) = 2q.$

* $$f_{\underline{X},Q}(\underline{x}, q) = 4q^3 f_{X_1|Q=q}(1) f_{X_2|Q=q}(1)$$

$$= 4q^3 * 2q(1-q) * 2q(1-q)$$

$$= 16q^5 (1-q)^2.$$

* $$f_{\underline{X}}(\underline{x}) = \int_0^1 16 q^5 (1-q)^2 dq = \frac{16}{168}.$$

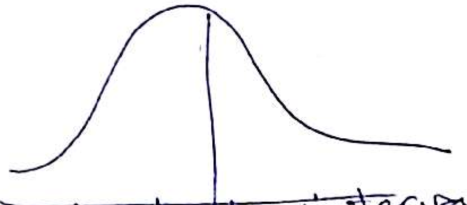
* $$\pi_{Q|\underline{X}}(q|\underline{x}) = 168 q^5 (1-q)^2.$$

$\longrightarrow$ $$E(X_3 | X_1 = 1, X_2 = 1) = \int_0^1 336 q^6 (1-q)^2 dq = \frac{4}{3}.$$

- A $100(1-\alpha)\%$ credibility interval $[a,b]$ for the parameter θ if:

$$P\left(a \leq \theta \leq b \mid \underline{X} = \underline{x} \right) \geq 1-\alpha .$$

- Theorem: $\pi_{\theta|\underline{X}}$ is continuous and <u>Unimodal</u>.



Then the shortest interval is:

$$\begin{cases} \int_a^b \pi_{\theta|\underline{X}}(\theta|\underline{x})\, d\theta = 1-\alpha \\ \pi_{\theta|\underline{X}}(a|\underline{x}) = \pi_{\theta|\underline{X}}(b|\underline{x}) . \end{cases}$$

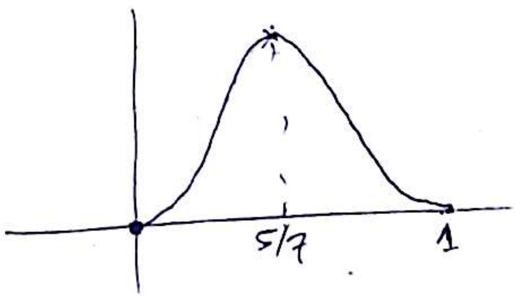
Example:

$$\pi_{Q|\underline{X}}(q|\underline{x}) = 168\, q^5 (1-q)^2 .$$

$$\frac{d\pi_{\theta|\underline{X}}(q|\underline{x})}{dq} = 168 \left(5q^4(1-q)^2 - 2q^5(1-q) \right)$$

$$= 168\, q^4 (1-q)\left(5(1-q) - 2q\right)$$

$$= \_\_\_\_\_\_ (5-7q) = 0 .$$



Example:

- Y = the Probability of having at least one loss during a year.

  N = number of claims.

- $\boxed{Y|_{P=p} = P(N \geq 1 | P=p) = p \rightarrow P(Y=p|P=p) = 1,\ P(Y \neq p | P=p) = 0.}$

$X|_{\theta=\theta} \sim$ —

- $P \sim \text{Unif}(0, 0.5)$.
- $Y_1 = 1,\ Y_2 = 1,\ \ldots,\ Y_8 = 1$.
- $Y|_{P=p} \sim \text{Bernoulli}(p)$.

Find a non-zero width 95% credibility interval for P.

$$\pi(p) = 2 \longrightarrow \quad f_{Y,P}(y, p) = 2p^8. \qquad 0 \leq p \leq \tfrac{1}{2}.$$

$$f_Y(y) = \int_0^{1/2} 2p^8\, dp = \frac{2}{9} p^9 \Big|_0^{1/2} = \frac{1}{9 * 2^8}.$$

$$\pi_{P|Y}(p|y) = 9 * 2^9 p^8.$$

$$\begin{cases} 9 * 2^9 a^8 = 9 * 2^9 b^8 \Leftrightarrow b = -a. \\ \displaystyle\int_a^b 9 * 2^9 p^8\, dp = 0.95 \qquad (\alpha = 0.05). \end{cases}$$

$$2^9 p^9 \Big|_a^b = 0.95.$$

$$b^9 - a^9 = 0.95 * 2^{-9}.$$

$$2b^9 = 0.95 * 2^{-9}.$$

$$b = \sqrt[9]{0.95 * 2^{-10}} = 0.46$$

$$a = -0.46.$$

(4) Bayesian central limit theorem:

$\pi(\theta)$ prior distribution.

$f_{\underline{x}|\theta}(\underline{x}|\theta)$ model distribution.

Then : $\pi_{\theta|\underline{x}}(\theta|\underline{x}) \sim$ Normal$(-,-)$.

Example: ~~θ~~ $X|_q \sim$ Geometric(q),

$Q \sim$ Unif$(0,1)$.

Find a .95% credibility interval for the posterior distribution, using data: $\underline{x} = (2, 1, 1, 0, 0, 1, 2, 2, 1, 4)$.

$\rightarrow$ $f_{\underline{x},Q}(\underline{x}, q) = f_{x_1|q}(x_1|q) \cdots f_{x_{10}|q}(x_{10}|q)\, \pi(q)$

$= q(1-q)^{x_1} \cdots q(1-q)^{x_{10}}\, 1$.

$= q^{10}(1-q)^{14}$.

$f_{\underline{x}}(\underline{x}) = C = \int_0^1 q^{10}(1-q)^{14}\, dq$.

$\pi_{Q|\underline{x}}(q|\underline{x}) = \dfrac{q^{10}(1-q)^{14}}{C} \sim \text{Beta}(11, 15)$

$\rightarrow$ (BCLT) $\Rightarrow$ $\pi_{Q|\underline{x}}(q|\underline{x}) \sim N\left(\underset{\mu}{\frac{11}{26}}; \underset{\sigma^2}{\frac{\frac{11}{26} * \frac{15}{26}}{27}}\right)$

Credibility interval $= (\mu - c, \mu + c)$.

$c = \phi^{-1}(0.975)\, \sigma$.

$d = 1 - \frac{1-0.95}{2}$??

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(5) Conjugate prior distribution:

Prior $\Rightarrow$ Model distribution $\Rightarrow$ Posterior.

A Prior distribution is called Conjugate prior distribution if the posterior distribution is of the same family as the prior.

Ex: $\theta \sim Gamma(\alpha, \beta)$.
$X | \theta = \theta \sim Exp(\theta)$, $X_1, \dots, X_n$.
Show that θ has a conjugate prior distribution?

$\rightarrow f_{\underline{X}, \theta}(\underline{x}, \theta) = \theta e^{-\theta x_1} \cdots \theta e^{-\theta x_n} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$.

$= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha+n-1} e^{-(\beta + \sum x_i)\theta}$

$\rightarrow \pi_{\theta|\underline{X}}(\theta|\underline{x}) = C \, \theta^{\alpha+n-1} e^{-(\beta+\sum x_i)\theta}$

$\sim Gamma(\alpha+n, \beta + \sum_{i=1}^{n} x_i)$.

Ex: Gamma $\rightarrow$ Poisson $\xrightarrow{?}$ Gamma